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Padovan and Perrin numbers of the form $7^t - 5^z - 3^y - 2^x$

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Abstract: Consider the Padovan sequence $(p_n)_{n\geq 0}$ given by $p_{n+3} = p_{n+1} + p_n$ with $p_0 = p_1 = p_2 = 1$. Its companion sequence, the Perrin sequence $(\wp_n)_{n\geq 0}$, follows the same recursive formula as the Padovan numbers, but with different initial values: $p_0 = 3$, $p_1 = 0$ and $p_2 = 2$. In this paper, we leverage Baker's theory concerning nonzero linear forms in logarithms of algebraic numbers along with a reduction procedure that employs the theory of continued fractions. This enables us to explicitly identify all Padovan and Perrin numbers that conform to the representation $7^t - 5^z - 3^y - 2^x$, where x, y, z and t are positive integers with $0 \leq x, y, z \leq t$.

Keywords: Exponential Diophantine equations, Padovan numbers, Perrin numbers, Linear forms in logarithms.
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1 Introduction

The exploration of representing recurrence sequences and special numbers in various forms has a long and fascinating history. Notably, in 2006, Bugeaud, Mignotte, and Siksek [3] demonstrated that the set of perfect power Fibonacci numbers consists solely of 0, 1, 8, and 144, while perfect powers among Lucas numbers are limited to 1 and 4. Luca and Szalay [6] established that only finitely many Fibonacci numbers adhere to the form $p^a \pm p^b + 1$, where p is a prime number and a and b are positive integers with $\max\{a, b\} \geq 2$. Building on this, in 2013, Marques and Togbé [8] characterized all Fibonacci and Lucas numbers expressible as $2^x + 3^y + 5^z$.

One of the useful generalizations of the Fibonacci sequence, which is called *k-generalized Fibonacci sequence* $\left(F_n^{(k)}\right)$ with $n \geq -(k-2)$ is given by the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \text{ for } n \geq 2$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$.

Subsequently, in 2014, Marques [7] resolved the Diophantine equation $F_n^{(k)} = 2^x + 3^y + 5^z$ with the constraint $\max(x, y) \leq z$. Recently, Irmak and Alp [5] delved into the k-generalized Fibonacci numbers in proximity to the form $2^x + 3^y + 5^z$. Moreover, other scholarly works have pursued Fibonacci and Lucas numbers following patterns like $2^x + 3^y + 5^z + 7^t$ (refer to [10] for comprehensive insights).

Moving forward, it is imperative to define Padovan and Perrin numbers, which hold significant prominence in the literature. We begin with the Padovan sequence $(P_n)_{n\geq 0}$, characterized by the ternary recurrence relation $P_{n+3} = P_{n+1} + P_n$ for $n \geq 0$, where the initial values are $P_0 = P_1 = P_2 = 1$. The first few terms of this sequence are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, \ldots$$

Next, consider the sequence of Perrin numbers, denoted as $(\wp_n)_{n\geq 0}$, which follows the identical recursive pattern of the sequence of Padovan numbers, namely, $\wp_{n+3} = \wp_{n+1} + \wp_n$ for $n \geq 0$, but with initial values $\wp_0 = 3$, $\wp_1 = 0$ and $\wp_2 = 2$. The first few terms of this sequence are

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, 367, 486, 644, 853, \ldots.$$

In this paper, we extend the preceding discourse by elucidating the comprehensive identification of Padovan numbers and Perrin numbers expressible as the sums and differences of four distinct prescribed bases raised to perfect powers. Specifically, we rigorously establish all Padovan and Perrin numbers conforming to the format $7^t - 5^z - 3^y - 2^x$, where $0 \leq x, y, z \leq t$. Our study culminates in the validation of the following two pivotal results:

Theorem 1.1. *The only solutions* (n, x, y, z, t) *of the Diophantine equation:*

$$P_n = 7^t - 5^z - 3^y - 2^x \tag{1}$$

with $0 \leq \max(x, y, z) \leq t$ *is* $(14, 2, 1, 1, 2)$.

Theorem 1.2. *The only solutions* (n, x, y, z, t) *of the Diophantine equation:*

$$\wp_n = 7^t - 5^z - 3^y - 2^x \tag{2}$$

with $0 \leq \max(x, y, z) \leq t$ *are* $(10, 2, 1, 2, 2)$ *and* $(13, 1, 1, 1, 2)$.

2 Preliminaries and known results

In this section, we gather pertinent insights concerning Padovan and Perrin numbers along with several preliminary lemmas that serve as fundamental components of our primary argument. For more exhaustive discussions, readers are referred to [1, 2, 11]. It is worth recalling that Padovan numbers and Perrin numbers exhibit several analogous properties. Notably, they share an identical recurrence relation, implying that both sequences are governed by the same characteristic equation: $x^3 - x - 1 = 0$. This equation has the roots α, β and γ, where

$$\alpha = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}},\ \beta = \omega\sqrt[3]{\frac{9+\sqrt{69}}{18}} + \overline{\omega}\sqrt[3]{\frac{9-\sqrt{69}}{18}}$$

with $\gamma = \overline{\beta}$. Here, ω is a cubic root of 1. It is well-known that for all $n \geq 0$ the Binet's formula for the n-th term of Padovan sequence is given by

$$P_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n, \tag{3}$$

where

$$\begin{cases} c_\alpha = \dfrac{1+\alpha}{-\alpha^2+3\alpha+1} \\ c_\beta = \dfrac{1+\beta}{-\beta^2+3\beta+1} \\ c_\gamma = \dfrac{1+\gamma}{-\gamma^2+3\gamma+1} \end{cases}.$$

The Binet's formula for $\wp_n$ is given by

$$\wp_n = \alpha^n + \beta^n + \gamma^n, \text{ for all } n \geq 0. \tag{4}$$

In numerical terms, the ensuing approximations encapsulate the magnitudes of the quantities α, β, γ, c_α, c_β and c_γ:

$$\begin{cases} 1.32 < \alpha < 1.33 \\ 0.86 < |\beta| = |\gamma| < 0.87 \\ 0.72 < c_\alpha < 0.73 \\ 0.24 < |c_\beta| = |c_\gamma| < 0.25 \\ |\beta| = |\gamma| = \alpha^{-1/2} \end{cases}. \tag{5}$$

Further, using induction, one can prove that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \text{ holds for all } n \geq 4 \tag{6}$$

and

$$\alpha^{n-2} \leq \wp_n \leq \alpha^{n+1} \text{ holds for all } n \geq 2. \tag{7}$$

Certainly, our inquiry necessitates leveraging insights from the theory concerning lower bounds for nonzero linear forms in the logarithms of algebraic numbers.

Suppose we have a set of real algebraic numbers $\eta_1, \eta_2, \ldots, \eta_l$, each distinct from 0 and 1 ($1 \leq i \leq l$), along with corresponding nonzero integers $b_1, \ldots, b_l$. Let D denote the degree of the number field $\mathbb{Q}(\eta_1, \eta_2, \ldots, \eta_l)$ over $\mathbb{Q}$. We define

$$B = \max\{|b_1|, |b_2|, \ldots, |b_l|\},$$

and

$$\Gamma = \eta_1^{b_1}\eta_2^{b_2}\cdots\eta_l^{b_l} - 1.$$

Remember that the logarithmic height of a k-degree algebraic number η is defined as follows:

$$h(\eta) = \frac{1}{k}\left(\log a + \sum_{j=1}^{k}\max\{0, \log|\eta^{(j)}|\}\right),$$

where $a > 0$ is the leading coefficient of the minimal polynomial of η (over $\mathbb{Z}$) and $\eta^{(j)}$ $(1 \leq j \leq k)$ are the conjugates of η (over $\mathbb{Q}$). Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq \max\{Dh(\eta_j), |\log\eta_j|, 0.16\}, 1 \leq j \leq l.$$

We will use the following deep Lemma proved by Matveev [9].

Lemma 2.1. *If* $\eta_1^{b_1}\eta_2^{b_2}\cdots\eta_l^{b_l} \neq 1$, *then*

$$\log\left|\eta_1^{b_1}\eta_2^{b_2}\cdots\eta_l^{b_l} - 1\right| \geq -1.4 \times 30^{l+3} \times l^{4.5} \times D^2(1+\log D)(1+\log B)A_1\cdots A_l.$$

Once an initial upper bound on n in each of our equations has been obtained, typically it tends to be excessively large. The subsequent pivotal step involves its reduction. To achieve this reduction, we rely on a variant of the renowned Baker–Davenport lemma credited to Dujella and Pethő (refer to [4]). For any real number x, we denote the distance from x to the nearest integer by $\|x\| = \min\{|x-n|;\ n \in \mathbb{Z}\}$.

Lemma 2.2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number κ such that $q > 6M$, and let A, B, τ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon = \|\tau q\| - M\|\kappa q\|$. If $\varepsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\kappa - v + \tau| < AB^{-\omega}$$

in positive integers u, v, and ω with $u \leq M$ and

$$\omega \geq \frac{\log\left(\frac{Aq}{\varepsilon}\right)}{\log B}.$$

Lemma 2.3. *Let κ be a real number and x, y be integers such that*

$$\left|\kappa - \frac{x}{y}\right| < \frac{1}{2y^2}.$$

Then $x/y = p_k/q_k$ is a convergent of κ. Furthermore, let M and N be a non-negative integers such that $q_N > M$ and putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$. Then the inequaliy

$$\left|\kappa - \frac{x}{y}\right| > \frac{1}{(a(M)+2)\,y^2} \tag{8}$$

holds for all pairs (x, y) of positive integers with $0 < y < M$.

Now, we are ready to present the proofs of our results.

3 Proofs of main results

This section is dedicated to prove the two principal outcomes, outlined in this paper.

Proof of Theorem 1.1. We divide the proof into two parts.

Part I (Deriving an absolute upper bound for n). Combining the Binet's formula (3) and equation (1), we see that

$$\left|c_\alpha\alpha^n - 7^t\right| = \left|2^x + 3^y + 5^z + c_\beta\beta^n + c_\gamma\gamma^n\right| \leq 2^x + 3^y + 5^z + \left|c_\beta\beta^n\right| + \left|c_\gamma\gamma^n\right|.$$

On the other hand, since $t \geq \max\{x, y, z\}$, it follows that $2^x \leq 7^{0.4t}$, $3^y \leq 7^{0.6t}$ and $5^z \leq 7^{0.9t}$. By (5), we conclude that

$$\left|c_\alpha\alpha^n 7^{-t} - 1\right| < \frac{5}{7^{0.1t}}. \tag{9}$$

We first observe that $c_\alpha\alpha^n 7^{-t} - 1 \neq 0$. Otherwise, $c_\alpha\alpha^n = 7^t$. Conjugating the last relation and taking the absolute value, we obtain

$$1 < 7^t = \left|c_\beta\beta^n\right| < 1.$$

However, this assertion leads to a contradiction. Hence, it follows that we must have

$$0 < \left|c_\alpha\alpha^n 7^{-t} - 1\right| < \frac{5}{7^{0.1t}}. \tag{10}$$

In the case when $n = 0$, by (1), this yields $7^t = 2^x + 3^y + 5^z - 1 \leq 2^x + 3^y + 5^z$, implying that t is either 0 or 1. For $n > 0$, considering the left-hand side of (6), we obtain $\alpha^{n-2} \leq 7^t$, which consequently leads to

$$n < 6.93t + 2. \tag{11}$$

If $t \leq 12$, then $n \leq 85$. A brute force search with *Pari/GP* in the range $0 \leq t, n \leq 85$ turned up that the only solution of (1) is $(n, x, y, z, t) = (14, 2, 1, 1, 2)$.

Now, we assume that $n \geq 86$ and so $t > 12$. Since $x, y, z \leq t$, there exist three real numbers $r_1, r_2, r_3 \in\ ]0, 1[$ such that $2^x < 7^{r_1 t}$, $3^y < 7^{r_2 t}$ and $5^z < 7^{r_3 t}$. Indeed, it suffices to choose that $r_1 = 0.36$, $r_2 = 0.57$ and $r_3 = 0.83$. By (1) and the right-hand side of (6) we see that

$$\begin{aligned}\alpha^{n-1} &\geq 7^t - 5^z - 3^y - 2^x > 7^t - 7^{0.83t} - 7^{0.57t} - 7^{0.36t}\\ &= 7^x\left(1 - \frac{1}{7^{0.17t}} - \frac{1}{7^{0.43t}} - \frac{1}{7^{0.64t}}\right) > 0.98 \times 7^t,\end{aligned}$$

which gives $t \leq 0.15n - 0.13 < n$.

Applying Matveev's outcome (refer to Lemma 2.1), we employ it on the left-hand side of (9). With reference to (10), we have demonstrated that the expression on the left-hand side of (9) is nonzero. We take $l = 3$, $\eta_1 = c_\alpha$, $\eta_2 = \alpha$, $\eta_3 = 7$ and $b_1 = 1$, $b_2 = n$, $b_3 = -t$. For this choice, we have $D = [\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. Since $t < n$, therefore we can take $B = n = \max\{1, t, n\}$. Note that the minimal polynomial of c_α over $\mathbb{Z}$ is given by $23X^3 - 23X^2 + 6X - 1$, which has roots c_α, c_β and c_γ. Since $|c_\alpha| < 1$ and $|c_\beta| = |c_\gamma| < 1$, we conclude that

$$h(\eta_1) = h(c_\alpha) = \frac{\log 23}{3}.$$

In addition, we have $h(\eta_2) = h(\alpha) = \frac{\log \alpha}{3}$ and $h(\eta_3) = h(7) = \log 7$. Thus, by (5), we can take

$$\begin{cases} \max\{3h(\eta_1), |\log \eta_1|, 0.16\} \leq 3.2 = A_1, \\ \max\{3h(\eta_2), |\log \eta_2|, 0.16\} \leq 0.29 = A_2, \\ \max\{3h(\eta_3), |\log \eta_2|, 0.16\} \leq 5.84 = A_3. \end{cases}$$

By Matveev's result stated in Lemma 2.1, after straightforward calculation, we get

$$\left|c_\alpha \alpha^n 7^{-t} - 1\right| > \exp(-c_1(1 + \log n)), \tag{12}$$

where $c_1 = 1.47 \times 10^{13}$. It follows from (10) and (12) that

$$e^{c_1(1+\log n)} > \frac{7^{0.1t}}{5},$$

where by (11), $t > \frac{n-2}{6.93}$. Therefore, we obtain

$$0.14n - 1.90 < 1.20 \times 10^{13}(1 + \log n),$$

which gives $n < 3.1 \times 10^{15}$. Thus, we conclude that if (n, x, y, z) is a solution in positive integers of (1) with the condition $0 \leq \max(x, y, z) \leq t$, then $n < 3.10 \times 10^{15}$.

Part II (Reducing the bound on n). Let us put

$$\Lambda_1 = \log c_\alpha + n \log \alpha - z \log 7.$$

Clearly, $\Lambda_1 \neq 0$ since $\alpha c_\alpha \notin \mathbb{Z}$. Hence, we must consider the following two possibilities.

If $\Lambda_1 > 0$, then $e^{\Lambda_1} > 1$, so from (10) we obtain $0 < \Lambda_1 < e^{\Lambda_1} - 1 < \frac{5}{7^{0.1t}}$.

Suppose now that $\Lambda_1 < 0$. It is easy to check that $\frac{5}{7^{0.1t}} < \frac{1}{2}$ for all $t > 4$. Then, from (10), we have that $\left|e^{\Lambda_1} - 1\right| < \frac{1}{2}$ and, therefore, $e^{|\Lambda_1|} < 2$. Now, since $\Lambda_1 < 0$, we have

$$0 < |\Lambda_1| < e^{|\Lambda_1|} - 1 \leq e^{|\Lambda_1|}\left(e^{|\Lambda_1|} - 1\right) < \frac{10}{7^{0.1t}}.$$

In both cases, the inequality $0 < |\Lambda_1| < \frac{10}{7^{0.1t}}$ holds for all $t > 5$. Replacing Λ_1 in the previous inequality by its formula and dividing by $\log 7$, we conclude that

$$0 < \left|n\frac{\log \alpha}{\log 7} - t + \frac{\log c_\alpha}{\log 7}\right| < \frac{10}{7^{0.1t} \times \log 7} < 5.14 \times (2.65)^{-t}.$$

We are now ready to apply Lemma 2.2 with the following parameters

$$\kappa = \frac{\log \alpha}{\log 7}, \quad \tau = \frac{\log c_\alpha}{\log 7}, \quad A = 5.14, \quad B = 2.65.$$

Clearly, κ is an irrational number. Let p_k/q_k represent the k-th convergent of κ's continued fraction. To narrow down the bound on n, we set $M = 3.1 \times 10^{15}$ as an upper limit for n. Our objective now is to identify a convergent of κ with a denominator exceeding $6M$. Specifically, we find that $q_{27} = 38595361995753261$ satisfies this condition, using the following *Pari/GP* program (Listing 1).

```
memo=Map();
memoize(f,A[..]) =
{
my(res);
if(!mapisdefined(memo, [f,A], &res),
res = call(f,A);
mapput(memo, [f,A], res));
res;
}
\p 500
padovan(n) = if(n <= 2,1,memoize(padovan, n-2)+memoize(padovan, n-3));
alpha = ((9 + sqrt(69))/18)^(1/3) + ((9 - sqrt(69))/18)^(1/3);
c_ {alpha} = (1+alpha)/(-( alpha^2) + 3*alpha + 1);
kappa = log(alpha)/log(7); tau = log(c_ {alpha})/log(7);
M = 3.10*10^15;
cf = contfrac(kappa,600);
dis(x) = abs(x-round(x));
cvg = contfracpnqn(cf,#cf)[2,27];
eps = 10;
s = 0;
b = dis(tau *cvg) - M*dis(kappa*cvg); if(b < 0, s = s+1);
eps = min(eps,b);eps

% 14 = 0.29175...
```

Listing 1. *Pari/GP* code for computing ε (Lemma 2.2).

Thus, by Lemma 2.2 with $\varepsilon = \|\tau q\| - M\|\kappa q\| \geq 0.29$, we deduce that if (n,x,y,z,t) constitutes a solution in positive integers to Equation (1), then

$$t < \frac{\log\left(\frac{Aq}{\varepsilon}\right)}{\log B} = \frac{\log\left(\dfrac{5.14 \times 38595361995753261}{0.29}\right)}{\log 2.65} = 42.139,$$

which gives $n < 293$ by (11). Applying Lemma 2.2 once again and performing the same calculations with $M = 293$ and $q_7 = 8394$. Here, $\varepsilon \geq 0.31$. So, if (n,x,y,z,t) is a solution in positive integers of equation (1), then $t \leq 12$, which implies that $n \leq 85$. This contradicts the hypothesis that $n \geq 86$. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. From (2) and (4), we obtain

$$\left|\alpha^n - 7^t\right| = \left|2^x + 3^y + 5^z - \beta^n - \gamma^n\right|,$$

or, equivalently,

$$\left|\alpha^n 7^{-t} - 1\right| = \left|\frac{2^x}{7^t} + \frac{3^y}{7^t} + \frac{5^z}{7^t} - \frac{\beta^n}{7^t} - \frac{\gamma^n}{7^t}\right| < \frac{5}{7^{0.1t}}. \tag{13}$$

From the left-hand side of (7) and (2), we obtain the estimate $\alpha^{n-2} \leq 7^t$, this yields

$$(n-2) < \frac{t\log 7}{\log\alpha} + 2,$$

i.e., $n < 6.93t + 2$.

If $t \le 19$, then $n \le 135$. A brute force search with *Pari/GP* in the range $0 \le t \le 19$ and $0 \le n \le 135$ revealed that the only solutions (n, x, y, z, t) of (2) are $(10, 2, 1, 2, 2)$ and $(13, 1, 1, 1, 2)$. Thus, we assume that $t \ge 20$. By (2) and the second inequality of (7), it follows that

$$\begin{aligned} \alpha^{n+1} &\ge 7^t - 5^z - 3^y - 2^x > 7^t - 7^{0.83t} - 7^{0.57t} - 7^{0.36t} \\ &= 7^t\left(1 - \frac{1}{7^{0.17t}} - \frac{1}{7^{0.43t}} - \frac{1}{7^{0.64t}}\right) > 0.98 \cdot 7^t, \end{aligned}$$

which implies that $6.91t - 1.072 < n$, and this also yields $t < n$.

We further utilize Matveev's result as outlined in Lemma 2.1 on the left-hand side of (13). It is noteworthy that the expression on the left-hand side of (13) is non-zero; its vanishing would imply $\alpha^n = 7^t \in \mathbb{Z}$, so $\alpha^n \in \mathbb{Z}$. This is impossible. We take $l = 2$, $\eta_1 = \alpha$, $\eta_2 = 7$ and $b_1 = n$, $b_2 = -t$. For this choice, we have $D = [\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. Note that $h(\eta_1) = \frac{\log \alpha}{3}$, $h(\eta_2) = \log 7$. Thus, we can choose $A_1 = 0.28$, $A_2 = 5.84$. Note that $B = \max\{|b_1|, |b_2|\} = \max\{n, t\} = n$. Applying Lemma 2.1, we get

$$\left|\alpha^n 7^{-t} - 1\right| > \exp\left(-c_2\left(1 + \log n\right)\right), \tag{14}$$

where $c_2 = 2.36 \times 10^{10}$. Thus from (13), (14) and the fact that $t > \frac{n-2}{6.93}$, taking logarithms in inequalities (13), (14) and comparing the resulting inequalities, we get that

$$0.972n - 1.61 < 2.36 \times 10^{10}\left(1 + \log n\right),$$

giving $n < 6.8 \times 10^{11}$.

We proceed to diminish the bound concerning n by leveraging the extremal characteristic of continued fractions. Considering (13), we introduce

$$\Lambda_2 = n \log \alpha - t \log 7.$$

Note that $\Lambda_2 \ne 0$, thus, we distinguish the following cases. If $\Lambda_2 > 0$, then $e^{\Lambda_2} > 1$, so from (13) we obtain $0 < \Lambda_2 < e^{\Lambda_2} - 1 < \frac{5}{7^{0.1t}}$.

Suppose now that $\Lambda_2 < 0$. It is easy to check that $\frac{5}{5^{0.1t}} < \frac{1}{2}$ for all $t > 5$. Then, from (13), we have that $\left|e^{\Lambda_2} - 1\right| < \frac{1}{2}$ and so $e^{|\Lambda_2|} < 2$. Since $\Lambda_2 < 0$, we also have

$$0 < |\Lambda_2| < e^{|\Lambda_2|} - 1 \le e^{|\Lambda_2|}\left(e^{|\Lambda_2|} - 1\right) < \frac{10}{7^{0.1t}}.$$

In both cases, the inequality $0 < |\Lambda_2| < \frac{10}{7^{0.1t}}$ holds for all $t > 5$. Replacing Λ_2 in the above inequality by its formula and dividing by $\log 7$, we see that

$$0 < \left|n\frac{\log \alpha}{\log 7} - t\right| < \frac{10}{\log 7 \times 7^{0.1t}}. \tag{15}$$

Let $[a_0, a_1, a_2, a_3, a_4, \ldots] = [0, 6, 1, 11, 1, \ldots]$ be the continued fraction of the ratio $\frac{\log \alpha}{\log 7}$, and let p_k/q_k be its k-th convergent. Recall that $n < 6.8 \times 10^{11}$. A quick inspection using *Pari/GP* reveals that $q_{19} < 6.8\times10^{11} < q_{20}$. Furthermore, $a_M = \max\{a_i : i = 0, 1, \ldots, 20\} = a_{10} = 146$. Applying the continued fraction inequality (see Lemma 2.3), we also see that

$$\left|n\frac{\log\alpha}{\log 7} - t\right| > \frac{1}{(a_M+2)\,n} = \frac{1}{148n}. \tag{16}$$

Comparing inequalities (15) and (16), we obtain

$$\frac{1}{148n} < \frac{10}{\log 7\times 7^{0.1t}}.$$

Since $t > \frac{n-2}{6.93}$, we conclude that $0.072\,n < 4.85 + \log n$, and hence $0 \le n < 135.54$. This implies that $t < \frac{n+1.072}{6.91} < 19.69$. A contradiction since we have assumed before that $t \ge 20$. Thus, the proof of Theorem 1.2 is finished. □

For further research, we close this paper by the following conjecture.

Conjecture 3.1. *Each of the Diophantine equations* $P_n = q_4^t - q_3^z - q_2^y - q_1^x$ *and* $\wp_n = q_4^t - q_3^z - q_2^y - q_1^x$ *has a unique solution for infinitely many prime numbers* q_1, q_2, q_3 *and* q_4 *with* $2 \le q_1 < q_2 < q_3 < q_4$ *and* $0 \le x, y, z \le t$.

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