Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 1, 181–190
DOI: 10.7546/nntdm.2025.31.1.181-190

Diophantine equations with Lucas and Fibonacci number coefficients

Cemil Karaçam [1], Alper Vural [2], Bilal Aytepe [3], and Ferhat Yıldız [4]

[1] Department of Mathematics, Muğla Science and Art Center
Muğla, Türkiye
e-mail: cemil-karacam@hotmail.com

[2] Department of Computer Engineering, Boğaziçi University
Istanbul, Türkiye
e-mails: vuralalperalper2005@gmail.com

[3] Department of Mathematical Engineering, Yıldız Technical University
Istanbul, Türkiye
e-mail: aytepebilal@gmail.com

[4] Department of Economics, Boğaziçi University
Istanbul, Türkiye
e-mail: ferhatyildiz1304@gmail.com

Received: 19 March 2024 **Revised:** 13 April 2025
Accepted: 14 April 2025 **Online First:** 30 April 2025

Abstract: Fibonacci and Lucas numbers are special number sequences that have been the subject of many studies throughout history due to the relations they provide. The studies are continuing today, and findings about these number sequences are constantly increasing. The relations between the Fibonacci and Lucas numbers, which were found during the proof of the prime between two consecutive numbers belonging to the Fibonacci or Lucas number sequence with the Euclidean algorithm, started our project. In the project, Diophantine equations whose coefficients are Lucas or Fibonacci numbers have been studied, various relations have been found, and their proofs have been made.

$$F_n x - F_{n+1} y = (-1)^n,$$
$$L_n x - L_{n-1} y = 1.$$

As in the above example, the equivalents of x and y values were found in the Diophantine equations with Fibonacci and Lucas number coefficients; and based on this example, different variations of the Diophantine equations whose coefficients were selected from the Fibonacci and Lucas number sequences were created, and their proofs were made.

Secondly, the geometric shapes consisting of vertices determined by pair of numbers selected from the Fibonacci or Lucas number sequence were considered, and their properties were examined. Various relations were found between them, and generalizations were made.
Keywords: Fibonacci numbers, Diophantine equations, Polygon areas.
2020 Mathematics Subject Classification: 11B39, 11D04.

1 Introduction

Leonardo Fibonacci was born in the 12-th century in Pisa, Italy. Since his father Guglielmo, whose nickname is Bonaccio, was a merchant, he had the opportunity to take lessons from Islamic mathematicians in North Africa and Algeria. Traveling to the countries on the Mediterranean coast with his father, Leonardo was called “Fibonacci” by shortening “Fillus Bonacci” meaning “son of Bonacci”. He returned to Italy in the 1200s and collected what he learned in his book Liber Abaci (Book of Calculation). With this work, he introduced the modern decimal number system to Europe. This book became the handbook of European scientists who calculated with the Roman number system until that time. Thus, Fibonacci contributed to the development of science in Europe. In his book, Fibonacci included a rabbit problem as an exercise question to set an example for the modern number system. The numbers obtained from the solution of the problem were called Fibonacci numbers in the periods after Fibonacci. The fact that Fibonacci numbers can appear in many different areas, even in unexpected situations, causes these numbers to attract a lot of attention.

Each term of the Fibonacci sequence is obtained by adding the two terms immediately before it, i.e., $F_n = F_{n-1} + F_{n-2}$, with initial terms being $F_0 = 0$ and $F_1 = 1$. It is known that the limit of the sequence obtained from the ratio of the consecutive terms of this sequence of numbers is related to the irrational number $(1+\sqrt{5})/2$, which is called the Golden ratio [9].

A similar sequence is the Lucas numbers, created by the French mathematician Edward Lucas. Unlike the Fibonacci numbers, the Lucas numbers continue by starting the sequence with the numbers 2 and 1. The number sequence continues as $2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots$ [3].

A lot of work has been done on these number sequences and their polynomials [1, 5, 8]. For example, the general way of obtaining any term in the Fibonacci numbers without the need to calculate all the terms in between was given by Jacques Philippe Marie Binet, and over time, for other number sequences and polynomials [2], the formulas used to represent the general term, which is expressed in the literature as Binet's formula or Binet-like formulas, are given in [6, 7]. Thus, for these number sequences and polynomials, many new identities are obtained by means of Binet formulas. The most canonical form of Binet formula is $F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$.

Diophantine equations are a subsection of Number Theory. Finding integer solutions to algebraic equations with integers and one or more unknowns is one of the most difficult problems in Number Theory. Diophantine equation solutions, on the other hand, aim to find all the integer solutions of such equations depending on the parameters [4, 11].

In the project, special solutions of Diophantine equations with coefficients of Fibonacci and Lucas numbers were examined, and identities of Fibonacci and Lucas numbers were formed based on these solutions, and the proofs of these identities were made with the help of Binet's formulas.

After making area calculations of the polygons in the analytical plane, whose coordinate components of vertices were selected from the Fibonacci and Lucas numbers, the findings were examined and identities of the Fibonacci and Lucas numbers were created. The identities were proved by the determinant method.

2 Main results

In this section, we chose Lucas numbers as parameters for finding solutions to first-order two variable Diophantine equations. There can be a relation between solutions and Lucas numbers.

Lemma 2.1. *Take the Diophantine equation* $47x + 29y = 5$. *Searching solutions with Euclidian algorithm,*

$$\begin{aligned}
47 &= 29 \cdot 1 + 18 = (11 - 7) \cdot 3 - 7 \cdot 1 = 11 \cdot 3 - 4 \cdot 7 \\
18 &= 11 \cdot 1 + 7 = 11 \cdot 3 - 4 \cdot (18 - 11) = 7 \cdot 11 - 4 \cdot 18 \\
7 &= 4 \cdot 1 + 3 = (29 - 18) \cdot 7 - 4 \cdot 18 = -11 \cdot 18 + 7 \cdot 29 \\
3 &= 1 \cdot 1 + 2 = -11(47 - 29) + 7 \cdot 29 \\
2 &= -1 \cdot 1 + 3 = 18 \cdot 29 - 11 \cdot 47 \\
3 &= -2 \cdot 1 + 5 = -11 \cdot 47 + 29 \cdot 18 = 5
\end{aligned}$$

This equation can be defined with $L_n L_{n+3} - L_{n+1} L_{n+2} = 5(-1)^n$.

Proof. Let us take a look at Binet's formulas.

$$\begin{aligned}
&(\alpha^n + \beta^n)(\alpha^{n+3} + \beta^{n+3}) - (\alpha^{n+1} + \beta^{n+1})(\alpha^{n+2} + \beta^{n+2}) = 5(-1)^n \\
&\alpha^{2n+3} + \alpha^n\beta^{n+3} + \alpha^{n+3}\beta^n + \beta^{2n+3} - \alpha^{2n+3} - \alpha^{n+1}\beta^{n+2} - \alpha^{n+2}\beta^{n+1} - \beta^{2n+3} = 5(-1)^n \\
&\alpha^n\beta^{n+3} + \alpha^{n+3}\beta^n - \alpha^{n+1}\beta^{n+2} - \alpha^{n+2}\beta^{n+1} = 5(-1)^n \\
&(\alpha^n\beta^n)(\alpha^3 + \beta^3 - \alpha\beta^2 - \alpha^2\beta) = 5(-1)^n \\
&(-1)^n(2(\alpha + \beta) + 3) = 5(-1)^n.
\end{aligned}$$

□

Lemma 2.2. *It holds that:*

$$L_{n-k}L_{n+k} - L_nL_n = 5(-1)^{n-k}{F_k}^2.$$

Proof.

$$L_{n+k} = \alpha^{n+k} + \beta^{n+k} \qquad L_{n-k} = \alpha^{n-k} + \beta^{n-k} \qquad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$$L_{n-k}L_{n+k} - L_nL_n = ?$$

$$\begin{aligned}
&(\alpha^{n-k} + \beta^{n-k})(\alpha^{n+k} + \beta^{n+k}) - (\alpha^n + \beta^n)(\alpha^n + \beta^n) \\
&= (\alpha^{2n} + \alpha^{n-k}\beta^{n+k} + \alpha^{n+k}\beta^{n-k} + \beta^{2n}) - (\alpha^{2n} + 2\alpha^n\beta^n + \beta^{2n}) \\
&= (\alpha^{n-k}\beta^{n-k})(\alpha^{2k} + \beta^{2k}) - 2\alpha^n\beta^n \\
&= (-1)^{n-k}(\alpha^{2k} + \beta^{2k}) - 2(-1)^n \\
&= (-1)^{n-k}((\alpha^k - \beta^k)^2 + 2\alpha^k\beta^k) - 2(-1)^n \\
&= (-1)^{n-k}(5F_k^2 + 2(-1)^k) - 2(-1)^n \\
&= (-1)^{n-k}5F_k^2 + 2(-1)^n - 2(-1)^n) \\
&= 5(-1)^{n-k}{F_k}^2.
\end{aligned}$$

□

Lemma 2.3. *It holds that:*

$$F_nL_{n-2} - F_{n-1}L_{n-1} = (-1)^n.$$

Proof. Using Binet's formulas we can transform the equation to

$$(\alpha^{n-2} + \beta^{n-2})(\frac{\alpha^n - \beta^n}{\alpha - \beta}) - (\alpha^{n-1} + \beta^{n-1})(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}) = (-1)^n.$$

$$\frac{(\alpha^{n-2} + \beta^{n-2})(\alpha^n - \beta^n) - (\alpha^{n-1} + \beta^{n-1})(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta} = \frac{\alpha^n\beta^{n-2} - \alpha^{n-2}\beta^n}{\alpha - \beta}$$

Since $\alpha^n\beta^n = (-1)^n$,

$$\frac{\alpha^n\beta^{n-2} - \alpha^{n-2}\beta^n}{\alpha - \beta} = \frac{\alpha^n\beta^n(\alpha^{-2} - \beta^{-2})}{\alpha - \beta} = \frac{\alpha^n\beta^n(\alpha + 1 - \beta - 1)}{\alpha - \beta} = (-1)^n.$$

□

We examined this situation for three integer Diophantine equations $ax + by + cz = 1$. Let us choose integers among the Lucas numbers.

$$L_nx + L_{n+1}y + L_{n+2}z = 5$$

$$L_n = 4 \quad L_{n+1} = 7 \quad L_{n+2} = 11$$

$4x + 7y + 11z = 5$, from here we found this solution: $x = 7, y = -11, z = 4$. We can check this format is working in every Lucas number.

$$L_nL_{n+1} - L_{n+1}L_{n+2} + L_{n+2}L_n = 5(-1)^n$$

- for $n = 2$, $L_2L_3 - L_3L_4 + L_4L_2 = 3 \cdot 4 - 4 \cdot 7 + 7 \cdot 3 = 5$;
- for $n = 3$, $L_3L_4 - L_4L_5 + L_5L_3 = 4 \cdot 7 - 7 \cdot 11 + 11 \cdot 4 = -5$;
- for $n = 4$, $L_4L_5 - L_5L_6 + L_6L_4 = 18 \cdot 29 - 29 \cdot 47 + 47 \cdot 18 = 5$.

We reached this equation: $L_nL_{n+1} - L_{n+1}L_{n+2} + L_{n+2}L_n = 5(-1)^n$.

Lemma 2.4. *It holds that:*

$$L_nL_{n+1} - L_{n+1}L_{n+2} + L_{n+2}L_n = 5(-1)^n.$$

Proof.

$$\begin{aligned}
&(\alpha^n + \beta^n)(\alpha^{n+1} + \beta^{n+1}) - (\alpha^{n+1} + \beta^{n+1})(\alpha^{n+2} + \beta^{n+2}) + (\alpha^{n+2} + \beta^{n+2})(\alpha^n + \beta^n)\\
&= \alpha^{2n+1} + \alpha^n\beta^n + \alpha^{n+1}\beta^{n+1} + \beta^{2n+1} - \alpha^{2n+3} - \alpha^{n+1}\beta^{n+2} - \alpha^{n+2}\beta^{n+1} - \beta^{2n+3}\\
&\quad + \alpha^{2n+2} + \alpha^{n+2}\beta^n + \alpha^n\beta^{n+2} + \alpha^{n+2}\beta + \beta^{2n+2}\\
&= \alpha^{2n+1} + \alpha^{2n+2} - \alpha^{2n+3} + \beta^{2n+1}\beta^{2n+2} - \beta^{2n+3}\\
&\quad + (\alpha\beta)^n\beta + (\alpha\beta)^n\alpha - \alpha^n\beta^n(\alpha^2\beta) - \alpha^n\beta^n(\alpha\beta^2) + (\alpha\beta)^n(\alpha^2 + \beta^2)\\
&= \alpha^{2n}(\alpha + \alpha^2 - \alpha^3) + \beta^{2n}(\beta + \beta^2 - \beta^3) + (\alpha\beta)^n[\beta + \alpha + \alpha^2 + \beta^2 - \alpha^2\beta - \alpha\beta^2]\\
&= \alpha^{2n}\cdot 0 + \beta^{2n}\cdot 0 + (\alpha\beta)^n[\beta + \alpha + \alpha + \beta + 1 + \alpha + \beta]\\
&= (-1)^n(3(\beta + \alpha) + 2) = 5(-1)^n.
\end{aligned}$$

□

Lemma 2.5. *It holds that:*

$$L_nL_{n+3} - L_{n+1}L_{n+2} + L_{n+2}L_n - L_{n+3}L_{n+2} = 5(-1)^n.$$

Proof.

$$\begin{aligned}
&(\alpha^n + \beta^n)(\alpha^{n+3} + \beta^{n+3}) + (\alpha^{n+1} + \beta^{n+1})(\alpha^{n+2} + \beta^{n+2}) + (\alpha^{n+2} + \beta^{n+2})(\alpha^n + \beta^n)\\
&\quad - (\alpha^{n+3} + \beta^{n+3})(\alpha^{n+2} + \beta^{n+2})\\
&= \alpha^{2n+3} + \alpha^n\beta^n + 3\alpha^{n+3} + \alpha^2\beta^{n+1} + \alpha^{n+1}\beta^2 + \beta^{2n+3} + \alpha^{2n+3} + \alpha^{n+1}\beta^n + \alpha^n\beta^{n+2} + \beta^{2n+3}\\
&\quad + \alpha^{2n+2} + \alpha^{n+2}\beta^n + \alpha^n\beta^{n+2} + \alpha^{n+2}\beta + \beta^{2n+2}\\
&\quad - (\alpha^{2n+5} + \alpha^{n+3}\beta^2 + \alpha^{n+2}\beta^3 + \alpha^3\beta^{n+2} + \alpha^3 + \beta^3 + \alpha^2\beta^2 - \alpha^3\beta^2 - \alpha^2\beta^3)\\
&= \alpha^{2n+3} + \alpha^{2n+3} + \alpha^{2n+2} - \alpha^{2n+5} + \beta^{2n+3} + \beta^{2n+3} + \beta^{2n+2} - \beta^{2n+5}\\
&\quad + \alpha^n\beta^n(\beta^3 + \alpha^3 + \alpha\beta^2 + \beta\alpha^2 + \alpha^2 + \beta^2 - \alpha^3\beta^2 - \alpha^2\beta^3)\\
&= \alpha^{2n}(2\alpha + 1 + 2\alpha + 1 + \alpha + 1 - (5\alpha + 3)) + \beta^{2n}(2\beta + 1 + 2\beta + 1 + \beta + 1 - (5\beta + 3))\\
&\quad + \alpha^n\beta^n(2\beta + 1 + 2\alpha + 1 + \beta + \alpha)\\
&= \alpha^{2n}\cdot 0 + \beta^{2n}\cdot 0 + \alpha^n\beta^n(3(\alpha + \beta) + 2)\\
&= 5(-1)^n.
\end{aligned}$$

□

Lemma 2.6. *It holds that:*

$$L_nL_{n+2} + L_{n+1}L_{n+2} + L_{n+2}L_{n+3} + L_{n+3}L_{n+4} + L_{n+4}L_{n+5} - {L_{n+5}}^2 = 5(-1)^n.$$

Proof.

$$\begin{aligned}
&(\alpha^n + \beta^n)(\alpha^{n+2} + \beta^{n+2}) + (\alpha^{n+1} + \beta^{n+1})(\alpha^{n+2} + \beta^{n+2})\\
&\quad + (\alpha^{n+2} + \beta^{n+2})(\alpha^{n+3} + \beta^{n+3}) + (\alpha^{n+3} + \beta^{n+3})(\alpha^{n+4} + \beta^{n+4})\\
&\quad + (\alpha^{n+4} + \beta^{n+4})(\alpha^{n+5} + \beta^{n+5}) - (\alpha^{n+5} + \beta^{n+5})(\alpha^{n+5} + \beta^{n+5})\\
&= \alpha^{2n}[\alpha^2 + \alpha^3 + \alpha^5 + \alpha^7 + \alpha^9 - \alpha^{10}] + \beta^{2n}[\beta^2 + \beta^3 + \beta^5 + \beta^9 - \beta^{10}]\\
&\quad + \alpha^n\beta^n[\alpha^2 + \beta^2 + \alpha\beta^2 + \alpha^2\beta + \alpha^2\beta^3 + \alpha^3\beta^2 + \alpha^3\beta^4 + \alpha^4\beta^3 + \alpha^4\beta^5 + \alpha^5\beta^4 - 2\alpha^5\beta^5]
\end{aligned}$$

$$
\begin{aligned}
&= \alpha^{2n}[\alpha + 1 + 2\alpha + 1 + 5\alpha + 3 + 13\alpha + 8 + 34\alpha + 21 - 55\alpha + 34] \\
&\quad + \beta^{2n}[\beta + 1 + 2\beta + 1 + 5\beta + 3 + 13\beta + 8 + 34\beta + 21 - 55\beta + 34] \\
&\quad + \alpha^n\beta^n[\alpha + 1 + \beta + 1 + \alpha\beta[\beta + \alpha + \alpha\beta^2 + \alpha^2\beta + \alpha^2\beta^3 + \alpha^3\beta^2 + \alpha^3\beta^4 + \alpha^4\beta^3 - 2\alpha^4\beta^4]] \\
&= (-1)^n[\alpha + 1 + \beta + 1 + 2] \\
&= 5(-1)^n.
\end{aligned}
$$
□

There is a connection between the equations we have found.

Theorem 2.1. *It holds that:*

$$
L_nL_{n+2} + \left(\sum_{k=1}^{x} L_{n+k}L_{n+k+1}\right) - L_{n+x-1}L_{n+x+3} = \begin{cases} 5(-1)^n, & \text{if } x \text{ is odd.} \\ 10(-1)^n, & \text{if } x \text{ is even.} \end{cases} \tag{1}
$$

Proof. We first proof the equation when x is an even number. First consider the base case for $x = 2$.

$$
\begin{aligned}
&L_nL_{n+2} + L_{n+1}L_{n+2} + L_{n+2}L_{n+3} - L_{n+1}L_{n+5} \\
&= L_{n+2}(L_n + L_{n+1} + L_{n+3}) - L_{n+1}L_{n+5} \\
&= L_{n+2}(L_{n+2} + L_{n+3}) - L_{n+1}L_{n+5} \\
&= L_{n+2}L_{n+4} - L_{n+1}L_{n+5} \\
&= 10(-1)^{n+1}
\end{aligned}
$$

The last step can be easily proven by Binet's formulas, thus the theorem is correct for $x = 2$. Now, assume that the theorem holds for x,

$$
L_nL_{n+2} + \left(\sum_{k=1}^{x} L_{n+k}L_{n+k+1}\right) - L_{n+x-1}L_{n+x+3} = 10(-1)^n.
$$

We will prove that it holds for $x + 2$.

$$
\begin{aligned}
&L_nL_{n+2} + \left(\sum_{k=1}^{x+2} L_{n+k}L_{n+k+1}\right) - L_{n+x+1}L_{n+x+5} \\
&L_nL_{n+2} + \left(\sum_{k=1}^{x} L_{n+k}L_{n+k+1}\right) + L_{n+x+1}L_{n+x+2} + L_{n+x+2}L_{n+x+3} - L_{n+x+1}L_{n+x+5}
\end{aligned}
$$

Proving $L_{n+x+1}L_{n+x+2} + L_{n+x+2}L_{n+x+3} - L_{n+x+1}L_{n+x+5} = -L_{n+x-1}L_{n+x+3}$ is enough to proof the main theorem for even x.

$$
\begin{aligned}
&L_{n+x-1}L_{n+x+3} + L_{n+x+1}L_{n+x+2} + L_{n+x+2}L_{n+x+3} - L_{n+x+1}L_{n+x+5} \\
&= L_{n+x+3}(L_{n+x-1} + L_{n+x+2}) + L_{n+x+1}(L_{n+x+2} - L_{n+x+5}) \\
&= 2L_{n+x+3}L_{n+x+1} - 2L_{n+x+1}L_{n+x+3} \\
&= 0.
\end{aligned}
$$

Now consider x is an odd number. We first inspect the case $x = 1$.

$$\begin{aligned}&L_nL_{n+2} + L_{n+1}L_{n+2} - L_nL_{n+4}\\&= {L_{n+2}}^2 - L_nL_{n+4}\\&= 5(-1)^n.\end{aligned}$$

Note that we used Catalan's identity in the last line. The rest of the induction steps are the same as in the case for an even x. □

Theorem 2.2. *It holds that:*

$$F_nF_{n+2} + \left(\sum_{k=1}^{x} F_{n+k}F_{n+k+1}\right) - F_{n+x-1}F_{n+x+3} = \begin{cases}(-1)^n, & \textit{if } x \textit{ is odd.}\\ 2(-1)^n, & \textit{if } x \textit{ is even.}\end{cases} \tag{2}$$

Proof. First, assume x is even. We show that theorem the holds for the base case, $x = 2$.

$$F_nF_{n+2} + F_{n+1}F_{n+2} + F_{n+2}F_{n+3} - F_{n+1}F_{n+5} = F_{n+2}F_{n+4} - F_{n+1}F_{n+5}.$$

By Vajda's Identity [10] $F_{n+2}F_{n+4} - F_{n+1}F_{n+5} = 2(-1)^n$, assuming the theorem holds for x, we show that it holds for $x + 2$.

$$\begin{aligned}&F_nF_{n+2} + \left(\sum_{k=1}^{x+2} F_{n+k}F_{n+k+1}\right) - F_{n+x+1}F_{n+x+5}\\&\quad = F_nF_{n+2} + \left(\sum_{k=1}^{x} F_{n+k}F_{n+k+1}\right) + F_{n+x+1}F_{n+x+2} + F_{n+x+2}F_{n+x+3} - F_{n+x+1}F_{n+x+5}\end{aligned}$$

Consider the expression $F_{n+x+1}F_{n+x+2} + F_{n+x+2}F_{n+x+3} - F_{n+x+1}F_{n+x+5} + F_{n+x-1}F_{n+x+3}$.

$$\begin{aligned}&F_{n+x-1}F_{n+x+3} + F_{n+x+1}F_{n+x+2} + F_{n+x+2}F_{n+x+3} - F_{n+x+1}F_{n+x+5}\\&= F_{n+x+3}(F_{n+x-1} + F_{n+x+2}) + F_{n+x+1}(F_{n+x+2} - F_{n+x+5})\\&= 2F_{n+x+3}F_{n+x+1} - 2F_{n+x+1}F_{n+x+3}\\&= 0\end{aligned}$$

The proof is complete by induction.

Next, assume x is an odd integer. Again, we prove by induction. It is easy to see that the theorem holds for $x = 1$. The rest of the induction steps are exactly same as the case when x is even. □

3 Polygons and their areas

Define the point in the Cartesian plane $P_n = (F_n, L_n)$. For a polygon with z edges and vertices $P_n, P_{n+k}, P_{n+2k}, \ldots, P_{n+(z-1)k}$, we can calculate its area:

- For triangles: If k is an odd number, $\text{Area} = F_{2k}$, otherwise $F_{2k} - 2F_k$.
- For quadrilaterals: If k is an odd number, $\text{Area} = F_{3k} - F_k$, otherwise $F_{3k} - 3F_k$.

We found these equations by using a program that calculates areas of polygons. Firstly, we tried to write equations and checked them by hand. Then, we thought about what general form could there exist and then proved it.

We use the shoelace method to calculate the areas. For example, the area of the triangle with vertices $(a, b), (c, d)$ and (e, f) is

$$\frac{1}{2}\begin{vmatrix} a & b \\ c & d \\ e & f \\ a & b \end{vmatrix} = \frac{|ad - bc + cf - de + eb - fa|}{2}$$

Table 1. Area of the triangle which has the corners
$(F_n, L_n), (F_{n+k}, L_{n+k})$ and (F_{n+2k}, L_{n+2k})

Triangle	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=3$	1	1	8	15	55	128	377	945
$n=4$	1	1	8	15	55	128	377	945
$n=5$	1	1	8	15	55	128	377	945
$n=6$	1	1	8	15	55	128	377	945
$n=7$	1	1	8	15	55	128	377	945

Table 2. Area of the quadrilateral which has the corners
$(F_n, L_n), (F_{n+k}, L_{n+k}), (F_{n+2k}, L_{n+2k})$ and (F_{n+3k}, L_{n+3k})

Quadrilateral	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=3$	1	5	32	135	605	2560	10933	46305
$n=4$	1	5	32	135	605	2560	10933	46305
$n=5$	1	5	32	135	605	2560	10933	46305
$n=6$	1	5	32	135	605	2560	10933	46305

Lemma 3.1. *The area of a triangle with vertices* $\begin{bmatrix} F_a & L_a \\ F_b & L_b \\ F_c & L_c \end{bmatrix}$ *is equal to the area of the triangle with vertices* $\begin{bmatrix} F_{a+1} & L_{a+1} \\ F_{b+1} & L_{b+1} \\ F_{c+1} & L_{c+1} \end{bmatrix}$.

Proof. The area of the first triangle is

$$\frac{1}{2}\begin{vmatrix} F_a & L_a \\ F_b & L_b \\ F_c & L_c \\ F_a & L_a \end{vmatrix} = \frac{|F_aL_b + F_bL_c + F_cL_a - F_bL_a - F_cL_b - F_aL_c|}{2}.$$

The area of the second triangle is

$$\frac{1}{2}\begin{vmatrix} F_{a+1} & L_{a+1} \\ F_{b+1} & L_{b+1} \\ F_{c+1} & L_{c+1} \\ F_{a+1} & L_{a+1} \end{vmatrix} = \frac{|F_{a+1}L_{b+1} + F_{b+1}L_{c+1} + F_{c+1}L_{a+1} - F_{b+1}L_{a+1} - F_{c+1}L_{b+1} - F_{a+1}L_{c+1}|}{2}.$$

These two are synchronized.

$$F_aL_b + F_bL_c + F_cL_a - F_bL_a - F_cL_b - F_aL_c = -F_{a+1}L_{b+1} - F_{b+1}L_{c+1} - F_{c+1}L_{a+1} + F_{b+1}L_{a+1} + F_{c+1}L_{b+1} + F_{a+1}L_{c+1}.$$

This is a symmetric equation. $L_n = F_{n+1} + F_{n-1}$ implies that

$$F_aL_b - F_bL_a + F_{a+1}L_{b+1} - F_{b+1}L_{a+1} = 0.$$

The proof is complete. □

Theorem 3.1. *Let $P_n = (F_n, L_n)$. Then, the area of the polygon with vertices $P_n, P_{n+k}, P_{n+2k}, \dots, P_{n+(z-1)k}$ is*

$$A(P_n, P_{n+k}, P_{n+2k}, \dots, P_{n+(z-1)k}) = \begin{cases} (1-z)F_k + F_{(z-1)k}, & \text{if } k \text{ is even.} \\ F_{(z-1)k} - F_k, & \text{if } k \text{ is odd and } z \text{ is even.} \\ F_{(z-1)k} & \text{if } k \text{ is odd and } z \text{ is odd.} \end{cases} \quad (3)$$

Proof. By Lemma 3.1, we can see that integer n does not affect the equation. Therefore, it is enough to prove the theorem for $n = 0$.

$$\begin{vmatrix} F_0 & L_0 \\ F_k & L_k \\ F_{2k} & L_{2k} \\ \vdots & \vdots \\ F_{(z-1)k} & L_{(z-1)k} \\ F_0 & L_0 \end{vmatrix} = F_0L_k - F_kL_0 + F_kL_{2k} - F_{2k}L_k + \cdots + F_{(z-1)k}L_0 - F_0L_{(z-1)k}$$

First, assume k is even. For even integers i and j, $F_j = -F_{-j}$ and $L_j = L_{-j}$. Modifying the equation $2F_{i+j} = F_iL_j + F_jL_i$ we get $2F_{i-j} = F_iL_j - F_jL_i$ which is similar to pairs of terms in the expression above for area of the polygon.

$$F_0L_k - F_kL_0 + F_kL_{2k} - F_{2k}L_k + \cdots + F_{(z-1)k}L_0 - F_0L_{(z-1)k} = (1-z)2F_k + 2F_{(z-1)k}$$

The area is half of this result, as desired.

Next, assume k is odd. We split this into two cases: z is odd and z is even. Assume z is odd. Note that for odd j, $F_j = F_{-j}$ and $L_j = -L_{-j}$.

$$\begin{aligned} F_0L_k - F_kL_0 + F_kL_{2k} - F_{2k}L_k + \cdots + F_{(z-1)k}L_0 - F_0L_{(z-1)k} &= -2F_k + 2F_k + \cdots + 2F_{(z-1)k} \\ &= 2F_{(z-1)k} \end{aligned}$$

The area is half of this result, as desired.

Assume z is even.

$$\begin{aligned} F_0L_k - F_kL_0 + \cdots + F_{(z-1)k}L_0 - F_0L_{(z-1)k} &= -2F_k + 2F_k + \cdots - 2F_k + 2F_{(z-1)k} \\ &= -2F_k + 2F_{(z-1)k} \end{aligned}$$

The area is half of this result, as desired. □

4 Conclusion

Diophantine equations which include Fibonacci and Lucas numbers were solved and their solutions were generalized using Fibonacci and Lucas numbers. From here, equations were generalized into formulas which contain products of Lucas numbers with Lucas numbers or Fibonacci numbers with Fibonnaci numbers. These series were analyzed in the Cartesian plane, and points with coordinates Fibonacci and Lucas numbers were consecutively chosen. Polygons were constructed with these points and a general formula was found, giving the areas of these polygons.

References

[1] Belbachir, H., & Bencherif, F. (2008). On some properties of bivariate Fibonacci and Lucas polynomials. *Journal of Integer Sequences*, 11, Article 08.2.6.

[2] Edson, M., & Yayenie, O. (2009). A new generalization of Fibonacci sequence & extended Binet's formula. *Integers*, 9(6), 639–654.

[3] Falcon, S. (2011). On the k-Lucas numbers. *International Journal of Contemporary Mathematical Sciences*, 6(21), 1039–1050.

[4] Grechuk, B. (2011). *Diophantine equations: A systematic approach*. Preprint. arXiv.org, Available online at: `https://arxiv.org/abs/2108.08705`

[5] Hoggatt, V. E. Jr., & Bicknell, M. (1973). Generalized Fibonacci polynomials. *The Fibonacci Quarterly*, 11(5), 457–465.

[6] Koshy, T. (2001). *Fibonacci and Lucas Numbers with Applications*. A Wiley-Interscience Publication, USA.

[7] Koshy, T. (2014). *Pell and Pell–Lucas Numbers with Applications*. Springer New York Heidelberg Dordrecht, London.

[8] Prodinger, H. (2009). On the expansion of Fibonacci and Lucas polynomials. *Journal of Integer Sequences*, 12, Article 09.1.6.

[9] Thapa, G. B., & Thapa, R. (2018). The relation of golden ratio, mathematics and aesthetics. *Journal of the Institute of Engineering*, 14(1), 188–199.

[10] Vajda, S. (1989). *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*. Dover Publications, USA.

[11] Xiong, M. (2022). Solving linear Diophantine equation and simultaneous linear Diophantine equations with minimum principles. *International Mathematical Forum*, 17(4), 143–161.