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On Entry 8 of Chapter 19 of Ramanujan's Second Notebook

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Abstract: The aim of this paper is to give an alternative proof of Entry 8 of Chapter 19 of Ramanujan's Second Notebook. Further, we deduce certain modular equations of degree 5 as a consequence of Entry 8 of Chapter 19.

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1 Introduction

For any complex numbers a and q with $|q| < 1$, we define

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad \text{for } n \in \mathbb{Z},$$

and

$$(a_1, a_2, a_3, \ldots, a_n; q)_\infty := \prod_{k=1}^{n} (a_k; q)_\infty.$$

S. Ramanujan defined [5, p. 197] his general theta function $f(a,b)$ by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2},\ |ab| < 1.$$

By Jacobi triple product identity [2, p. 35],

$$f(a,b) = (-a; ab)_\infty (-b; ab)_\infty (ab, ab)_\infty, \quad |ab| < 1.$$

Further, he defined following three special cases of $f(a,b)$:

$$\varphi(q) := f(q,q) = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q,-q^2) = (q;q)_\infty.$$

On page 233 of his second notebook, Ramanujan recorded the following four interesting identities which relates Lambert series to his theta functions:

Theorem 1.1. [5, Entry 8] *We have*

$$q\psi^3(q)\psi(q^5) - 5q^2\psi(q)\psi^3(q^5) = \frac{q}{1-q^2} + \frac{2q^2}{1-q^4} - \frac{3q^3}{1-q^6} + \frac{4q^4}{1-q^8} + \frac{6q^6}{1-q^{12}} + \cdots, \tag{1.1}$$

$$5\varphi(q)\varphi^3(q^5) - \varphi^3(q)\varphi(q^5) = 4\left\{1 + \frac{q}{1+q} - \frac{2q^2}{1-q^2} - \frac{3q^3}{1+q^3} + \frac{4q^4}{1-q^4} + \frac{6q^6}{1-q^6} + \cdots\right\}, \tag{1.2}$$

$$25\varphi(q)\varphi^3(q^5) - \frac{\varphi^5(q)}{\varphi(q^5)} = 24 + 40\left\{\frac{q}{1+q} - \frac{3q^3}{1+q^3} - \frac{7q^7}{1+q^7} + \frac{9q^9}{1+q^9} + \cdots\right\}, \tag{1.3}$$

and

$$\frac{\psi^5(q)}{\psi(q^5)} - 25q^2\psi(q)\psi^3(q^5) = 1 + 5\left\{\frac{q}{1+q} - \frac{2q^2}{1+q^2} - \frac{3q^3}{1+q^3} + \frac{4q^4}{1+q^4} + \frac{6q^6}{1+q^6} \cdots\right\}. \tag{1.4}$$

B. C. Berndt [2, pp. 250–257] gave a proof of Theorem 1.1 using the modular equations found in Ramanujan's notebooks [5]. These identities can also be deduced from the series identities established by S. Cooper [3, pp. 533–534] by using the theory of modular forms.

In this paper, our aim is to give an alternative proof for the identities in Theorem 1.1. We establish three of them by employing W. N. Bailey's summation formula [1] and the remaining one by employing Ramanujan's ${}_1\psi_1$ summation formula [2, p. 34]. All the four identities will be proved in Section 3. At the end of Section 3, we deduce two modular equations of degree 5, originally due to Ramanujan as a consequence of our main results. The required preliminaries will be recalled in Section 2.

2 Preliminary results

For convenience, throughout this paper, we set for any positive integer n,

$$f_n = f(-q^n).$$

One can easily see that

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \quad \varphi(-q) = \frac{f_1^2}{f_2}, \quad \psi(q) = \frac{f_2^2}{f_1}, \quad \psi(-q) = \frac{f_1 f_4}{f_2}, \quad \text{and} \quad f(q) = \frac{f_2^3}{f_1 f_4}. \tag{2.1}$$

Modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)} = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)},$$

where

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!}, \quad |z| < 1$$

is the ordinary hypergeometric function with $(a)_0 = 1$ and $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$, for $n \geq 1$. If β has degree n over α, then Ramanujan defined the multiplier m by

$$m = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}.$$

The well-known Ramanujan's ${}_1\psi_1$ Summation formula as follows:

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(az, q/az, b/a, q; q)_\infty}{(z, b/az, b, q/a; q)_\infty}, \quad |b/a| < |z| < 1, |q| < 1. \tag{2.2}$$

We use the following Bailey's summation formula with suitable convergence conditions:

$$\sum_{n=-\infty}^{\infty} \frac{aq^n}{(1-aq^n)^2} - \sum_{n=-\infty}^{\infty} \frac{bq^n}{(1-bq^n)^2} = \frac{a(ab, q/ab, b/a, aq/b; q)_\infty f_1^4}{(a, q/a, b, q/b; q)_\infty^2}. \tag{2.3}$$

Along with these two summation formulas, we will be using the following identities while deducing above said identities:

$$\varphi^2(-q) - 5\varphi^2(-q^5) = -4\frac{f_2^3 f_5}{f_1 f_{10}}, \tag{2.4}$$

$$\psi^2(q) - 5q\psi^2(q^5) = \frac{f_1^3 f_{10}}{f_2 f_5}, \tag{2.5}$$

$$\varphi(q) - \sqrt{5}\varphi(q^5) = \frac{(1-\sqrt{5})f_2}{(-\omega q^2, \omega^2 q, \omega^3 q, -\omega^4 q^2; q^2)_\infty}, \tag{2.6}$$

$$\varphi(q) + \sqrt{5}\varphi(q^5) = \frac{(1+\sqrt{5})f_2}{(-\omega q, -\omega^2 q^2, -\omega^3 q^2, -\omega^4 q; q^2)_\infty}, \tag{2.7}$$

$$\psi(q^2) + q\sqrt{5}\psi(q^{10}) = \frac{f_2}{(\omega q, -\omega^2 q, -\omega^3 q, \omega^4 q; q^2)_\infty}, \tag{2.8}$$

and

$$\psi(q^2) - q\sqrt{5}\psi(q^{10}) = \frac{f_2}{(-\omega q, \omega^2 q, \omega^3 q, -\omega^4 q; q^2)_\infty}, \tag{2.9}$$

where $\omega = e^{2\pi i/5}$. In fact, S. Y. Kang [4] has given a proof of identities (2.4)–(2.9) by employing elementary methods.

3 Main results

We begin this section by recalling the following lemmas, which play very important role in our proofs.

Lemma 3.1. *Let ω be the fifth roots of unity. Then, we have*

(i) $\omega^n - \omega^{2n} - \omega^{3n} + \omega^{4n} = \left(\frac{n}{5}\right)\sqrt{5}$, *where n is any non-negative interger and $\left(\frac{n}{5}\right)$ denotes Legendre symbol of n modulo 5.*

(ii) $(\omega, \omega^2, \omega^3, \omega^4; q)_\infty = 5\frac{f_5}{f_1}$.

(iii) $(\omega q, \omega^2 q, \omega^3 q, \omega^4 q; q^2)_\infty = \frac{f_2 f_5}{f_1 f_{10}}$.

(iv) $(-\omega, -\omega^2, -\omega^3, -\omega^4; q)_\infty = \frac{f_1 f_{10}}{f_2 f_5}$.

Proof. The above lemma readily follows from the properties of the fifth roots of unity ω. One may refer to [1] and [4], where similar identities were used. □

Lemma 3.2. *We have*

(i) $$\sum_{n=-\infty}^{\infty} \frac{aq^n}{(1-aq^n)^2} - \sum_{n=-\infty}^{\infty} \frac{bq^n}{(1-bq^n)^2} = \frac{a}{(1-a)^2} - \frac{b}{(1-b)^2} + \sum_{n=1}^{\infty} \frac{n(a^n - b^n - b^{-n} + a^{-n})q^n}{1-q^n}.$$

(ii) $$\sum_{n=-\infty}^{\infty} \frac{aq^{2n-1}}{(1-aq^{2n-1})^2} - \sum_{n=-\infty}^{\infty} \frac{bq^{2n-1}}{(1-bq^{2n-1})^2} = \sum_{n=1}^{\infty} \frac{n(a^n - b^n - b^{-n} + a^{-n})q^n}{1-q^{2n}}.$$

Proof. In the series
$$\sum_{n=1}^{\infty} \frac{aq^n}{(1-aq^n)^2},$$
expanding each of the summands into geometric series, interchanging the order of summation and then summing into geometric series, we obtain
$$\sum_{n=1}^{\infty} \frac{aq^n}{(1-aq^n)^2} = \sum_{n=1}^{\infty} \frac{na^n q^n}{1-q^n}. \tag{3.1}$$
From the above, the lemma follows. □

Lemma 3.3. *We have*
$$\frac{a}{(1-a)^2} - \sum_{n=1}^{\infty} \frac{n(a^n + a^{-n})q^n}{1+q^n} = \frac{a(-a;q)_\infty(-q/a;q)_\infty(a^2q;q^2)_\infty(q/a^2;q^2)_\infty f_1^2 f_2^2}{(a;q)_\infty(q/a;q)_\infty(a^2;q^2)_\infty(q^2/a^2;q^2)_\infty}.$$

Proof. By replacing b with aq in (2.2), we obtain
$$\sum_{n=-\infty}^{\infty} \frac{z^n}{1-aq^n} = \frac{(az;q)_\infty(q/az;q)_\infty(q;q)_\infty^2}{(a;q)_\infty(q/a;q)_\infty(z;q)_\infty(q/z;q)_\infty}. \tag{3.2}$$

Differentiating (3.2) with respect to a, and then using the fact that $f'(a) = f(a)(ln(f(a)))'$ and (3.1), we obtain

$$\frac{a}{(1-a)^2}+\sum_{n=1}^{\infty}\frac{nza^nq^n}{1-zq^n}+\sum_{n=1}^{\infty}\frac{nz^{-1}a^{-n}q^n}{1-z^{-1}q^n}$$
$$=\frac{(az;q)_\infty(q/az;q)_\infty(q;q)_\infty^2}{(a;q)_\infty(q/a;q)_\infty(z;q)_\infty(q/z;q)_\infty}\sum_{n=0}^{\infty}\left\{\frac{aq^n}{1-aq^n}-\frac{azq^n}{1-azq^n}+\frac{\frac{q^{n+1}}{az}}{1-\frac{q^{n+1}}{az}}-\frac{\frac{q^{n+1}}{a}}{1-\frac{q^{n+1}}{a}}\right\}. \tag{3.3}$$

Letting z to -1 in the above, we find that

$$\frac{a}{(1-a)^2}+\sum_{n=1}^{\infty}\frac{n(a^n+a^{-n})q^n}{1+q^n}=\frac{(-a;q)_\infty(-q/a;q)_\infty(q;q)_\infty^2}{(a;q)_\infty(q/a;q)_\infty(-q;q)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{aq^n}{1-a^2q^{2n}}.$$

Using (3.2) in the right-hand side of the above, we establish the required result. □

We return to the proof of Theorem 1.1.

Proof of (1.1). Replacing q by q^2 in (2.3) and then in the resultant identity, replacing a by aq^{-1} and b by bq^{-1}, we obtain

$$\sum_{n=-\infty}^{\infty}\frac{aq^{2n-1}}{(1-aq^{2n-1})^2}-\sum_{n=-\infty}^{\infty}\frac{bq^{2n-1}}{(1-bq^{2n-1})^2}=\frac{a}{q}\frac{(ab/q^2,q^4/ab,b/a,aq^2/b;q^2)_\infty}{(a/q,q^3/a,b/q,q^3/b;q^2)_\infty^2}f_2^4. \tag{3.4}$$

Using Lemma 3.2(ii) in the above, we find that

$$\sum_{n=1}^{\infty}\frac{n(a^n-b^n-b^{-n}+a^{-n})q^n}{1-q^{2n}}=\frac{a}{q}\cdot\frac{(ab/q^2,q^4/ab,b/a,aq^2/b;q^2)_\infty}{(a/q,q^3/a,b/q,q^3/b;q^2)_\infty^2}f_2^4. \tag{3.5}$$

Setting $a=\omega$ and $b=\omega^2$ in the above, then employing Lemma 3.1(i) to the left-hand side of the resulting identity and employing Lemma 3.1(i), (ii) and (iii) in the right-hand side of the resulting identity, and after simplification, we obtain

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^{2n}}=q\frac{f_1^2f_2f_{10}^2}{f_5^2}. \tag{3.6}$$

We have

$$q\psi^3(q)\psi(q^5)-5q^2\psi(q)\psi^3(q^5)=q\psi(q)\psi(q^5)\{\psi^2(q)-5q\psi^2(q^5)\}. \tag{3.7}$$

Employing (2.5) and (2.1) in the above, we find that

$$q\psi^3(q)\psi(q^5)-5q^2\psi(q)\psi^3(q^5)=q\frac{f_1^2f_2f_{10}^2}{f_5^2}. \tag{3.8}$$

Finally, by comparing (3.6) and (3.8), we complete the proof of (1.1). □

Proof of (1.2). From (2.3) and Lemma 3.2(i), we obtain

$$\frac{a}{(1-a)^2}-\frac{b}{(1-b)^2}+\sum_{n=1}^{\infty}\frac{n(a^n-b^n-b^{-n}+a^{-n})q^n}{1-q^n}=\frac{a(ab,q/ab,b/a,aq/b;q)_\infty f_1^4}{(a,q/a,b,q/b;q)_\infty^2}. \tag{3.9}$$

By setting $a = -\omega$ and $b = -\omega^2$ in the above, on employing Lemma 3.1(i) to the left-hand side and Lemma 3.1(ii) and (iv) to the right-hand side and then on simplification, we arrive at

$$1+\sum_{n=1}^{\infty}\binom{n}{5}\frac{n(-1)^nq^n}{1-q^n}=\frac{f_1f_2^2f_5^3}{f_{10}^2}. \tag{3.10}$$

Consider

$$5\varphi(-q)\varphi^3(-q^5)-\varphi^3(-q)\varphi(-q^5)=\varphi(-q)\varphi(-q^5)\{5\varphi^2(-q^5)-\varphi^2(-q)\}.$$

Employing (2.4) and (2.1) in the above, we obtain

$$5\varphi(-q)\varphi^3(-q^5)-\varphi^3(-q)\varphi(-q^5)=4\,\frac{f_1f_2^2f_5^3}{f_{10}^2}. \tag{3.11}$$

By (3.10) and (3.11), it follows that

$$5\varphi(-q)\varphi^3(-q^5)-\varphi^3(-q)\varphi(-q^5)=4\left\{1+\sum_{n=1}^{\infty}\binom{n}{5}\frac{(-1)^nnq^n}{1-q^n}\right\}.$$

Finally, by replacing q with $-q$, we establish (1.2). $\square$

Proof of (1.3). By (3.10), it follows that

$$\sqrt{5}+\sqrt{5}\sum_{n=1}^{\infty}\binom{n}{5}\frac{n(-1)^nq^n}{1-q^n}=\sqrt{5}\frac{f_1f_2^2f_5^3}{f_{10}^2}. \tag{3.12}$$

By setting $a = \omega$ and $b = \omega^2$ in (3.9), and employing the same method used to deduce (3.10) from (3.9), we find that

$$\frac{-1}{\sqrt{5}}+\sqrt{5}\sum_{n=1}^{\infty}\binom{n}{5}\frac{nq^n}{1-q^n}=-\frac{1}{\sqrt{5}}\frac{f_1^5}{f_5}. \tag{3.13}$$

Subtracting (3.12) from (3.13) and multiplying the resultant equation throughout by $-4\sqrt{5}$, we get

$$24-40\sum_{n=1}^{\infty}\binom{2n-1}{5}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}=4\left\{\frac{f_1^5}{f_5}+5\frac{f_1f_2^2f_5^3}{f_{10}^2}\right\}. \tag{3.14}$$

On the other hand, we have

$$25\varphi(-q)\varphi^3(-q^5)-\frac{\varphi^5(-q)}{\varphi(-q^5)}=\frac{\varphi(-q)}{\varphi(-q^5)}\{5\varphi^2(-q^5)-\varphi^2(-q)\}\{5\varphi^2(-q^5)+\varphi^2(-q)\}. \tag{3.15}$$

Employing (2.4) and (2.1) in the above, we obtain

$$25\varphi(-q)\varphi^3(-q^5)-\frac{\varphi^5(-q)}{\varphi(-q^5)}=\left\{\frac{f_1^5}{f_5}+5\frac{f_1f_2^2f_5^3}{f_{10}^2}\right\}. \tag{3.16}$$

Comparing (3.16) and (3.14), we deduce that

$$25\varphi(-q)\varphi^3(-q^5)-\frac{\varphi^5(-q)}{\varphi(-q^5)}=24-40\sum_{n=1}^{\infty}\binom{2n-1}{5}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}. \tag{3.17}$$

By replacing q with $-q$, we establish (1.3). $\square$

Proof of (1.4). Substituting $a=\omega^2$ in Lemma 3.3 and then subtracting the resulting identity by the identity obtained by substituting $a=\omega$ in Lemma 3.3, we deduce that

$$\frac{\omega}{(1-\omega)^2}-\frac{\omega^2}{(1-\omega^2)^2}-\sum_{n=1}^{\infty}\frac{n(\omega^n-\omega^{2n}-\omega^{3n}+\omega^{4n})q^n}{1+q^n}=Z\left\{\frac{1}{X}-\frac{\omega}{Y}\right\}, \tag{3.18}$$

where

$$\begin{aligned}Z&=\omega(-\omega,-\omega^2,-\omega^3q,-\omega^4q;q)_\infty(\omega q,\omega^2q,\omega^3q,\omega^4q;q^2)_\infty,\\ X&=(\omega,-\omega^2,-\omega^3q,\omega^4q;q)_\infty(\omega q,\omega^2,\omega^3q^2,\omega^4q;q^2)_\infty,\end{aligned}$$

and

$$Y=(-\omega,\omega^2,\omega^3q,-\omega^4q;q)_\infty(\omega q^2,\omega^2q,\omega^3q,\omega^4;q^2)_\infty.$$

From Lemma 3.1(i), it follows that

$$\frac{\omega}{(1-\omega)^2}-\frac{\omega^2}{(1-\omega^2)^2}-\sum_{n=1}^{\infty}\frac{n(\omega^n-\omega^{2n}-\omega^{3n}+\omega^{4n})q^n}{1+q^n}=-\frac{1}{\sqrt{5}}\left\{1+5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1+q^n}\right\}. \tag{3.19}$$

From Lemma 3.1(iii) and (iv), one can easily see that

$$Z=-f_1^2f_2^2. \tag{3.20}$$

Next by employing Lemma 3.1(ii), (2.7), (2.8) and then simplifying using (2.1), we establish

$$\frac{1}{X}=\frac{1}{2\sqrt{5}}\frac{1}{f_2f_{10}}\{\psi^2(q)+\sqrt{5}\varphi(q^5)\psi(q^2)-q\sqrt{5}\varphi(q)\psi(q^{10})+5q\psi^2(q^5)\}. \tag{3.21}$$

Similarly, using Lemma 3.1(ii), (2.6), (2.9) and then using (2.1), we obtain

$$\frac{\omega}{Y}=\frac{-1}{2\sqrt{5}}\cdot\frac{1}{f_2f_{10}}\{\psi^2(q)-\sqrt{5}\varphi(q^5)\psi(q^2)-q\sqrt{5}\varphi(q)\psi(q^{10})+5q\psi^2(q^5)\}. \tag{3.22}$$

Using (3.19), (3.20), (3.21) and (3.22) in (3.18), we obtain

$$1+5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1+q^n}=\frac{f_1^2f_2}{f_{10}}\{\psi^2(q)+5q\psi^2(q^5)\}. \tag{3.23}$$

We have

$$\frac{\psi^5(q)}{\psi(q^5)}-25q^2\psi(q)\psi^3(q^5)=\frac{\psi(q)}{\psi(q^5)}\{\psi^2(q)-5q\psi^2(q^5)\}\{\psi^2(q)+5q\psi^2(q^5)\}. \tag{3.24}$$

From (2.5) and (2.1), we find that

$$\frac{\psi^5(q)}{\psi(q^5)}-25q^2\psi(q)\psi^3(q^5)=\frac{f_1^2f_2}{f_{10}}\{\psi^2(q)+5q\psi^2(q^5)\}. \tag{3.25}$$

By comparing (3.23) and (3.25), we complete the proof of (1.4). □

Corollary 3.1. [5, Entry 13(iv), p. 236)]. *Let β has degree 5 over α and m is a multiplier of degree 5. Then, we have*

$$\frac{5}{m}=1+2^{4/3}\left\{\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right\}^{1/24}.$$

Proof. Ramanujan recorded the following identity in his lost notebook [6, p. 139]:

$$1 + \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{nq^n}{1-q^n} = \frac{f_1^5}{f_5}. \tag{3.26}$$

From (1.2) and (1.3), we find that

$$25\varphi(q)\varphi^3(q^5) - 10\varphi^3(q)\varphi(q^5) + \frac{\varphi^5(q)}{\varphi(q^5)} = 16\left\{1 + \sum_{n=1}^{\infty} \left(\frac{2n}{5}\right) \frac{2nq^{2n}}{1-q^{2n}}\right\}. \tag{3.27}$$

Employing (3.26) in (3.27), we obtain

$$\begin{aligned} 25\varphi(q)\varphi^3(q^5) - 10\varphi^3(q)\varphi(q^5) &= 16\frac{f_2^5}{f_{10}} - \frac{\varphi^5(q)}{\varphi(q^5)} \\ \text{or} \quad \left\{5\frac{\varphi^2(q^5)}{\varphi^2(q)} - 1\right\}^2 &= 16\frac{f_2^5\varphi(q^5)}{f_{10}\varphi^5(q)}, \end{aligned}$$

which implies

$$5\frac{\varphi^2(q^5)}{\varphi^2(q)} - 1 = 4\left\{\frac{f_2^5\varphi(q^5)}{f_{10}\varphi^5(q)}\right\}^{1/2}. \tag{3.28}$$

Transforming (3.28) in terms of α, β, and m, we obtain the corollary. □

Corollary 3.2. [5, Entry 13(vi), p. 236)]. *If β has degree 5 over α and m is a multiplier of degree 5, then we have*

$$\frac{5}{m} = \frac{1 + \left(\frac{\alpha^5}{\beta}\right)^{1/8}}{1 + (\alpha\beta^3)^{1/8}}. \tag{3.29}$$

Proof. From (1.3) and (1.4), we obtain

$$25\varphi(q)\varphi^3(q^5) - \frac{\varphi^5(q)}{\varphi(q^5)} - 8\frac{\psi^5(q)}{\psi(q^5)} + 200\psi(q)\psi^3(q^5) = 16\{\frac{\psi^5(q)}{\psi(q^5)} + 25\psi(q)\psi^3(q^5)\}.$$

Rearranging the terms, we simplify that

$$5\frac{\varphi^2(q^5)}{\varphi^2(q)}\left\{1 + 2q^2\frac{\psi(q)\psi^3(q^5)}{\varphi(q)\varphi^3(q^5)}\right\} = 1 + 4\frac{\psi^5(q)\varphi(q^5)}{\psi(q^5)\varphi(q)}. \tag{3.30}$$

It is easy to see that the Corollary 3.2 is equivalent to the above theta function identity. □

4 Conclusion

The results recorded by Ramanujan in his notebooks surprised many mathematicians around the world. The goal is not only to verify his results, but also to understand the methods he might have employed to derive them. While the initial goal of this article was to prove Entry 8 of Chapter 19 of Ramanujan's second notebook in the spirit of Ramanujan, our approach evolved to incorporate Bailey's summation—a simpler and classical alternative, while still helping us understand his methods.

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References

[1] Bailey, W. N. (1952). A further note on two of Ramanujan's formulae. *The Quarterly Journal of Mathematics*, 3(1), 158–160.

[2] Berndt, B. C. (1991). *Ramanujan's Notebooks: Part III*. Springer Science & Business Media, New York.

[3] Cooper, S. (2017). *Ramanujan's Theta Functions*. Springer, Cham.

[4] Kang, S.-Y. (1999). Some theorems on the Rogers–Ramanujan continued fraction and associated theta function identities in Ramanujan's lost notebook. *Ramanujan Journal*, 3, 91–111.

[5] Ramanujan, S. (1957). *Notebooks, Volume I & II*. Tata Institute of Fundamental Research, Bombay.

[6] Ramanujan, S. (1988). *The Lost Notebook and Other Unpublished Papers*. Narosa, New Delhi.