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On the set of $\underline{\text{Set}}(n)$'s. Part 2

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Abstract: In the previous author's research, the set of $\underline{\text{Set}}(n)$'s for natural numbers n was constructed. For this set it was proved that it is a commutative semi-group. The condition for which it is a monoid was given. The present leg of research continues by demonstrating that for any n, $\underline{\text{Set}}(n)$ is a lattice, and by the introduction of four new operations over the elements of $\underline{\text{Set}}(n)$.

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1 Introduction

In a series of papers [1–6], we discussed the idea of generating a special set for an arbitrary natural number $n \geq 2$ that has the canonical form

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where $k, \alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ are natural numbers and $p_1 < p_2 < \cdots < p_k$ are different prime numbers. In [1], this set was defined by:

$$\underline{\text{Set}}(n) = \{m \mid m = \prod_{i=1}^{k} p_i^{\beta_i} \ \& \ \delta(n) \leq \beta_i \leq \Delta(n)\},$$

where *

$$\delta(n) = \min(\alpha_1, \ldots, \alpha_k),$$
$$\Delta(n) = \max(\alpha_1, \ldots, \alpha_k),$$
$$\underline{\text{set}}(n) = \{p_1, \ldots, p_k\}.$$

In the present paper, we will continue to study the properties of $\underline{\text{Set}}(n)$.

Let for every $l = \prod_{i=1}^{k} p_i^{\beta_i}, m = \prod_{i=1}^{k} p_i^{\gamma_i} \in \underline{\text{Set}}(n)$,

$$[l, m] = \prod_{i=1}^{k} p_i^{\max(\beta_i, \gamma_i)},$$
$$(l, m) = \prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i)},$$
$$l \times m = \prod_{i=1}^{k} p_i^{\min(\beta_i + \gamma_i, \Delta(n))}.$$

Let also:

$$\boxdot\, n = (\underline{\text{mult}}(n))^{\delta(n)},$$
$$\boxtimes\, n = (\underline{\text{mult}}(n))^{\Delta(n)}.$$

2 Lattice generated by $\underline{\text{Set}}(n)$

As we saw in [6], $\langle \underline{\text{Set}}(n), (.), \boxtimes\, n\rangle$ and $\langle \underline{\text{Set}}(n), [.], \boxdot\, n\rangle$ are commutative monoids, or at least commutative semi-groups, i.e., for every two elements $l, m \in \underline{\text{Set}}(n) : (l, m) \in \underline{\text{Set}}(n)$ and $[l, m] \in \underline{\text{Set}}(n)$. Obviously, the elements of $\underline{\text{Set}}(n)$ are partially ordered (because as natural numbers, they are ordered). Let numbers l and m be connected with a directed arch ($\rightarrow$) of a graph if and only if $l, m \in \underline{\text{Set}}(n), p \in \underline{\text{set}}(n)$ and $m = lp$.

For example, the graph from Figure 1 is related to $\underline{\text{Set}}(60)$ and this from Figure 2 is related to $\underline{\text{Set}}(24)$.

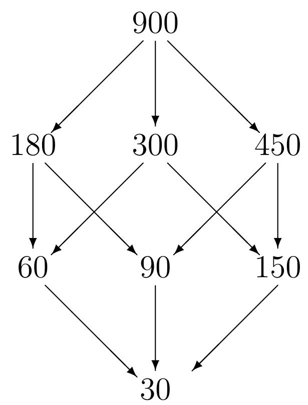

Figure 1. Lattice for $\underline{\text{Set}}(60)$.

* Other authors (see, e.g. [9]) denote the functions δ and Δ by h and H, respectively.

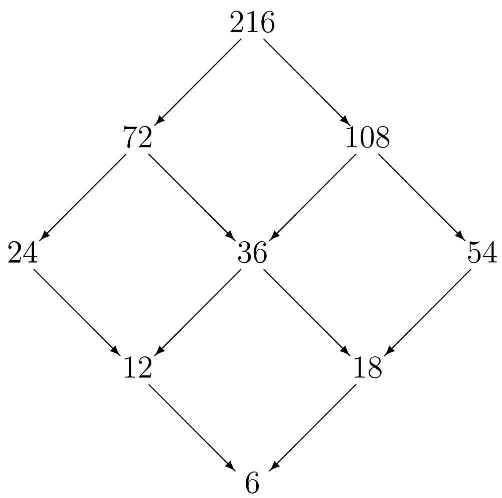

Figure 2. Lattice for $\underline{\mathrm{Set}}(24)$.

As we discussed for every $l, m \in \underline{\mathrm{Set}}(n)$: $(l, m), [l, m] \in \underline{\mathrm{Set}}(n)$ and these operations are interpretations of operations $\wedge$ and $\vee$ from the definition of a lattice in [8].

Therefore, the following theorem is valid:

Theorem 1. *Let* $\langle \underline{\mathrm{Set}}(n), (.), [.], \boxast n, \boxcircle n \rangle$ *be a lattice with minimal element* $\boxcircle n$*, maximal element* $\boxast n$ *and (standard) order* $\leq$*.*

We can immediately check that the elements of l, m, s of $\langle \underline{\mathrm{Set}}(n), (.), [.], \boxast n, \boxcircle n \rangle$ satisfy the conditions (cf. [7, 8]):

(a) $(l, l) = l$ and $[l, l] = l$,

(b) $(l, m) = (m, l)$ and $[l, m] = [m, l]$,

(c) $(l, (m, s)) = ((l, m), s)$ and $[l, [m, s]] = [[l, m], s]$,

(d) $(l, [m, l]) = l = [l, (m, l)]$.

Then, we prove:

Theorem 2. *For every two* $l, m \in \underline{\mathrm{Set}}(n)$*:*

$$l \leq m \quad \Longleftrightarrow \quad (l, m) = l \quad \Longleftrightarrow \quad [l, m] = m.$$

Proof. Let $l, m \in \underline{\mathrm{Set}}(n)$ and let $l \leq m$. Then, for the first equivalence we see that

$$\begin{aligned} (l, m) = l &\Longleftrightarrow \prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i)} = \prod_{i=1}^{k} p_i^{\beta_i} \\ &\Longleftrightarrow (\forall i)(\min(\beta_i, \gamma_i) = \beta_i) \\ &\Longleftrightarrow (\forall i)(\beta_i \leq \gamma_i) \\ &\Longleftrightarrow l \leq m. \end{aligned}$$

The second equivalence is checked in the same manner. $\square$

3 Other results related to $\underline{\mathrm{Set}}(n)$

Having in mind the well-known equalities

$$[(l,m),u] = ([l,u],[m,u]),$$
$$([l,m],u) = [(l,u),(m,u)],$$

we will prove the following assertion.

Theorem 3. *For every three natural numbers $l, m, u \in \underline{\mathrm{Set}}(n)$:*

(a) $(l \times m, u) \leq (l, u) \times (m, u)$,

(b) $[l \times m, u] \leq [l, u] \times [m, u]$,

(c) $(l, m) \times u = (l \times u, m \times u)$,

(d) $[l, m] \times u = [l \times u, m \times u]$.

To prove Theorem 3, we need to prove first two lemmas.

Lemma 1. *For every three positive real numbers x, y, z:*

$$\min(x,z) + \min(y,z) \geq \min(x+y,z).$$

Proof. Let

$$X \equiv \min(x,z) + \min(y,z) - \min(x+y,z).$$

We will discuss three cases regarding the order in which x, y, z appear:

- Case 1: $x, y \leq z$. Then
$$X = x + y - \min(x+y,z) \geq 0.$$

- Case 2: $x, y \geq z$. Then
$$X = 2z - z = z > 0.$$

- Case 3: $x \leq z \leq y$. Then
$$X = x + z - z = x > 0.$$

The remaining cases are checked analogously. □

Lemma 2. *For every four positive real numbers x, y, z, t, so that $t \geq x, y, z$:*

$$\min(\max(x,z) + \max(y,z), t) \geq \max(\min(x+y,t),z).$$

Proof. Let

$$X \equiv \min(\max(x,z) + \max(y,z), t) - \max(\min(x+y,t),z).$$

Again, we will discuss three cases as above.

- Case 1: $x, y \le z$. Then

$$X = \min(2z, t) - \max(\min(x + y, t), z).$$

Now, there are two sub-cases.

  - Sub-case 1.1: $x + y \le t$. Then

$$X = \min(2z, t) - \max(x + y, z) \ge 0.$$

  - Sub-case 1.2: $x + y > t$. Then $2z > x + y > t$ and

$$X = \min(2z, t) - \max(t, z) = t - t = 0.$$

- Case 2: $x, y \ge z$. Then $x + y > z$ and

$$X = \min(x + y, t) - \max(\min(x + y, t), z) = \min(x + y, t) - \min(x + y, t) = 0.$$

- Case 3: $x \le z \le y$. Then $x + y > z$ and

$$X = \min(z + y, t) - \max(\min(x + y, t), z) = \min(z + y, t) - \min(x + y, t) \ge 0.$$

This completes the proof. □

Proof of Theorem 3. Now, for the validity of (a) – (d), using Lemmas 1 and 2, we obtain sequentially:

(a)
$$\begin{aligned}
(l \times m, u) &= \left(\prod_{i=1}^{k} p_i^{\min(\beta_i + \gamma_i, \Delta(n))}, u \right) \\
&= \prod_{i=1}^{k} p_i^{\min(\min(\beta_i + \gamma_i, \Delta(n)), \varepsilon_i)} \\
&= \prod_{i=1}^{k} p_i^{\min(\beta_i + \gamma_i, \Delta(n), \varepsilon_i)} \\
&= \prod_{i=1}^{k} p_i^{\min(\beta_i + \gamma_i, \varepsilon_i)} \\
&\le \prod_{i=1}^{k} p_i^{\min(\beta_i, \varepsilon_i) + \min(\gamma_i, \varepsilon_i)} \\
&\le \prod_{i=1}^{k} p_i^{\min(\min(\beta_i, \varepsilon_i) + \min(\gamma_i, \varepsilon_i)), \Delta(n)} \\
&= \prod_{i=1}^{k} p_i^{\min(\beta_i, \varepsilon_i, \Delta(n))} \times \prod_{i=1}^{k} p_i^{\min(\gamma_i, \varepsilon_i, \Delta(n))} \\
&= (l, u) \times (m, u);
\end{aligned}$$

$$\text{(b)} \qquad [l \times m, u] = \left[\prod_{i=1}^{k} p_i^{\min(\beta_i+\gamma_i, \Delta(n))}, u \right]$$

$$= \prod_{i=1}^{k} p_i^{\max(\min(\beta_i+\gamma_i,\Delta(n)),\varepsilon_i)}$$

$$\leq \prod_{i=1}^{k} p_i^{\min(\max(\beta_i,\varepsilon_i)+\min(\gamma_i,\varepsilon_i),\Delta(n))}$$

$$= \prod_{i=1}^{k} p_i^{\max(\beta_i,\varepsilon_i)} \times \prod_{i=1}^{k} p_i^{\max(\gamma_i,\varepsilon_i)}$$

$$= [l, u] \times [m, u];$$

$$\text{(c)} \qquad (l, m) \times u = \prod_{i=1}^{k} p_i^{\min(\beta_i,\gamma_i)} \times u$$

$$= \prod_{i=1}^{k} p_i^{\min(\min(\beta_i,\gamma_i)+\varepsilon_i,\Delta(n))}$$

$$= \prod_{i=1}^{k} p_i^{\min(\beta_i+\varepsilon_i,\gamma_i+\varepsilon_i,\Delta(n))}$$

$$= \prod_{i=1}^{k} p_i^{\min(\min(\beta_i,\varepsilon_i,\Delta(n)),\min(\gamma_i,\varepsilon_i,\Delta(n)))}$$

$$= \left(\prod_{i=1}^{k} p_i^{\min(\beta_i+\varepsilon_i,\Delta(n))}, \prod_{i=1}^{k} p_i^{\min(\gamma_i+\varepsilon_i,\Delta(n))} \right)$$

$$= (l \times u, m \times u);$$

Statement (d) is proved in the same manner. □

Now, we will introduce a new operation over $\underline{\mathrm{Set}}(n)$ in the form:

$$l : m = \prod_{1=1}^{k} p_i^{\max(\beta_i-\gamma_i,\delta(n))}.$$

Obviously, for every $l, m \in \underline{\mathrm{Set}}(n)$, $l : m \in \underline{\mathrm{Set}}(n)$, because $\delta(n) \leq \max(\beta_i - \gamma_i, \delta(n)) \leq \Delta(n)$. For these numbers, we also see that:

(a) $l : m = \boxcircle\, n$ if and only if for each i $(1 \leq i \leq k) : \beta_i - \gamma_i \leq \delta(n)$

(b) $l \times m = \boxast\, n$ if and only if for each i $(1 \leq i \leq k) : \beta_i + \gamma_i \geq \Delta(n)$.

Theorem 4. *For every two natural numbers $l, m \in \underline{\mathrm{Set}}(n)$:*

(a) $(l \times m) : m = l$ *if and only if for each* i $(1 \leq i \leq k) : \beta_i + \gamma_i \leq \Delta(n)$,

(b) $(l : m) \times m = l$ *if and only if for each* i $(1 \leq i \leq k) : \beta_i - \gamma_i \geq \delta(n)$.

Proof. (a) Let $l, m \in \underline{\mathrm{Set}}(n)$. Then

$$\begin{aligned}(l \times m) : m &= \prod_{i=1}^{k} p_i^{\min(\beta_i+\gamma_i,\Delta(n))} : m \\ &= \prod_{i=1}^{k} p_i^{\max(\min(\beta_i+\gamma_i,\Delta(n))-\gamma_i,\delta(n))} \\ &= \prod_{i=1}^{k} p_i^{\max(\min(\beta_i,\Delta(n)-\gamma_i),\delta(n))}.\end{aligned}$$

Obviously, the equality $(l \times m) : m = l$ will be valid if and only if

$$\max(\min(\beta_i, \Delta(n) - \gamma_i), \delta(n)) = \beta_i$$

for each i $(1 \leq i \leq k)$. Because $\beta_i \geq \delta(n)$, the equality is valid when

$$\min(\beta_i, \Delta(n) - \gamma_i) = \beta_i,$$

i.e., $\Delta(n) - \gamma_i \geq \beta_i$ or $\beta_i + \gamma_i \leq \Delta(n)$.

Statement (b) is proved analogously. □

Theorem 5. *For every two natural numbers $l, m \in \underline{\mathrm{Set}}(n)$:*

(a) $(l, m) : l = \boxcircle n$,

(b) $[l, m] : l = m : l$,

(c) $l : (l, m) = l : m$,

(d) $l : [l, m] = \boxcircle n$.

Proof. Let $l, m \in \underline{\mathrm{Set}}(n)$. Then, we check sequentially:

(a)

$$\begin{aligned}(l, m) : l &= \prod_{i=1}^{k} p_i^{\min(\beta_i,\gamma_i)} : l \\ &= \prod_{i=1}^{k} p_i^{\max(\min(\beta_i,\gamma_i)-\beta_i,\delta(n))} \\ &= \prod_{i=1}^{k} p_i^{\max(\min(0,\gamma_i-\beta_i),\delta(n))} \\ &= \prod_{i=1}^{k} p_i^{\min(\max(\gamma_i-\beta_i,\delta(n)),\max(0,\delta(n)))} \\ &= \prod_{i=1}^{k} p_i^{\min(\max(\gamma_i-\beta_i,\delta(n)),\delta(n))} \\ &= \boxcircle n;\end{aligned}$$

(b)
$$\begin{aligned}
[l,m]:l &= \prod_{i=1}^{k} p_i^{\max(\beta_i,\gamma_i)} : l \\
&= \prod_{i=1}^{k} p_i^{\max(\max(\beta_i,\gamma_i)-\beta_i,\delta(n))} \\
&= \prod_{i=1}^{k} p_i^{\max(0,\gamma_i-\beta_i,\delta(n))} \\
&= \prod_{i=1}^{k} p_i^{\max(\gamma_i-\beta_i,\delta(n))} \\
&= m:l;
\end{aligned}$$

(c)
$$\begin{aligned}
l:(l,m) &= l : \prod_{i=1}^{k} p_i^{\min(\beta_i,\gamma_i)} \\
&= \prod_{i=1}^{k} p_i^{\max(\beta_i-\min(\beta_i,\gamma_i),\delta(n))} \\
&= \prod_{i=1}^{k} p_i^{\max(\max(0,\beta_i-\gamma_i),\delta(n))} \\
&= \prod_{i=1}^{k} p_i^{\max(0,\beta_i-\gamma_i,\delta(n))} \\
&= \prod_{i=1}^{k} p_i^{\max(\beta_i-\gamma_i,\delta(n))} \\
&= l:m;
\end{aligned}$$

(d)
$$\begin{aligned}
l:[l,m] &= l : \prod_{i=1}^{k} p_i^{\max(\beta_i,\gamma_i)} \\
&= \prod_{i=1}^{k} p_i^{\max(\beta_i-\max(\beta_i,\gamma_i),\delta(n))} \\
&= \prod_{i=1}^{k} p_i^{\max(\min(0,\beta_i-\gamma_i),\delta(n))} \\
&= \prod_{i=1}^{k} p_i^{\min(\max(0,\delta(n)),\max(\beta_i-\gamma_i),\delta(n))} \\
&= \prod_{i=1}^{k} p_i^{\min(\max(\beta_i-\gamma_i,\delta(n)),\delta(n))} \\
&= \boxcircle\, n.
\end{aligned}$$

This completes the proof. □

Finally, for any two natural numbers $l \in \underline{\mathrm{Set}}(n)$ and $s \geq 1$, let

$$l^s = \prod_{i=1}^{k} p_i^{\min(s\beta_i,\Delta(n))}, \qquad \sqrt[s]{l}^{-} = \prod_{i=1}^{k} p_i^{\max(\lfloor \frac{\beta_i}{s} \rfloor,\delta(n))}, \qquad \sqrt[s]{l}^{+} = \prod_{i=1}^{k} p_i^{\max(\lceil \frac{\beta_i}{s} \rceil,\delta(n))}.$$

Obviously,

$$\sqrt[s]{l}^{-} \leq \sqrt[s]{l}^{+}.$$

Theorem 6. *For every two natural numbers $l \in \underline{\mathrm{Set}}(n)$ and $s \geq 1$:*

(a) $\sqrt[s]{l^s}^+ \leq l$,

(b) $\sqrt[s]{l^s}^- \leq l$.

First, we must prove the following lemma:

Lemma 3. *For every four natural numbers a, b, c, d, so that $c \geq b \geq d$:*

$$\max\left(\left\lceil \frac{\min(ab, c)}{a} \right\rceil, d\right) \leq b.$$

Proof. We must check the following two cases.

- Case 1: $ab \leq c$. Then

$$\begin{aligned} \max\left(\left\lceil \tfrac{\min(ab,c)}{a} \right\rceil, d\right) &= \max\left(\left\lceil \tfrac{ab}{a} \right\rceil, d\right) \\ &= \max(b, d) \\ &= b. \end{aligned}$$

- Case 2: $ab > c$. Then, if $c = ra$ for some natural number r, then $ab > ra$, i.e., $b > r$ and

$$\begin{aligned} \max\left(\left\lceil \tfrac{\min(ab,c)}{a} \right\rceil, d\right) &= \max(r, d) \\ &\leq b. \end{aligned}$$

If $c = ra + t$ for some natural number r and $0 < t < 1$, then $ab > ra + t$, i.e., $b > r + \frac{t}{a}$ and hence $b \geq r + 1$. Then then

$$\begin{aligned} \max\left(\left\lceil \tfrac{\min(ab,c)}{a} \right\rceil, d\right) &= \max(r + 1, d) \\ &\leq b \end{aligned}$$

and the Lemma 3 is proved. □

Proof of Theorem 6. (a) Let $l \in \underline{\mathrm{Set}}(n)$ and $s \geq 1$. Then, we check sequentially:

$$\begin{aligned} \sqrt[s]{l^s}^+ &= \sqrt[s]{\left(\prod_{i=1}^{k} p_i^{\beta_i}\right)^s} \\ &= \sqrt[s]{\prod_{i=1}^{k} p_i^{\min(s\beta_i, \Delta(n))}} \\ &= \prod_{i=1}^{k} p_i^{\max\left(\left\lceil \frac{\min(s\beta_i, \Delta(n))}{s} \right\rceil, \delta(n)\right)} \\ &\leq \prod_{i=1}^{k} p_i^{\beta_i} \\ &= l, \end{aligned}$$

because from Lemma 3 we have that

$$\max\left(\left\lceil\frac{\min(s\beta_i,\Delta(n))}{s}\right\rceil,\delta(n)\right)\le\beta_i$$

for each i ($1\le i\le k$).

The validity of (b) follows from (a). □

Theorem 7. *For every two natural numbers* $l\in\underline{\mathrm{Set}}(n)$ *and* $1\le s\le\left\lfloor\frac{\delta(l)}{\delta(n)}\right\rfloor$*:*

(a) $\left(\sqrt[s]{l}^-\right)^s\le l$,

(b) $\left(\sqrt[s]{l}^+\right)^s\ge l$.

Proof. (a) is valid, because for each $l\in\underline{\mathrm{Set}}(n)$, obviously, $\delta(l)\ge\delta(n)$, i.e., $s\ge 1$, for $s=1$ both inequalities are valid (as equalities) and for each i ($1\le i\le k$):

$$\begin{aligned}\left(\sqrt[s]{l}^-\right)^s &= \left(\prod_{i=1}^{k}p_i^{\max\left(\left\lfloor\frac{\beta_i}{s}\right\rfloor,\delta(n)\right)}\right)^s\\ &=\prod_{i=1}^{k}p_i^{\min\left(s\max\left(\left\lfloor\frac{\beta_i}{s}\right\rfloor,\delta(n)\right),\Delta(n)\right)}\\ &=\prod_{i=1}^{k}p_i^{\min\left(\max\left(s\left\lfloor\frac{\beta_i}{s}\right\rfloor,s\delta(n)\right),\Delta(n)\right)}\\ &\le\prod_{i=1}^{k}p_i^{\min\left(\max\left(\beta_i,\left\lfloor\frac{\delta(l)}{\delta(n)}\right\rfloor\delta(n)\right),\Delta(n)\right)}\\ &\le\prod_{i=1}^{k}p_i^{\min(\max(\beta_i,\delta(l)),\Delta(n))}\\ &=l.\end{aligned}$$

The validity of (b) is proved in the same manner as (a). □

4 Conclusion

In the paper, four new operations over the elements of the set $Set(n)$ were introduced and some of their properties wer studied. An **Open problem** is what other interesting operations can be defined over $\underline{\mathrm{Set}}(n)$.

We showed also that $\underline{\mathrm{Set}}(n)$ can be represented as a lattice. Another **Open problem** is what other properties does the object $\underline{\mathrm{Set}}(n)$ have.

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