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Power Fibonacci sequences in quadratic integer modulo m

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Abstract: The power Fibonacci sequence in $\mathbb{Z}_m[\sqrt{\delta}]$ is defined as a Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$ where $F_0 = 1$ and $F_1 = a$, such that $a \in \mathbb{Z}_m[\sqrt{\delta}]$ and $F_n \equiv a^n \pmod{m}$, for all $n \in \mathbb{N} \cup \{0\}$. In this paper, we investigated the existence of power Fibonacci sequences in $\mathbb{Z}_m[\sqrt{\delta}]$, and the number of power Fibonacci sequences in $\mathbb{Z}_m[\sqrt{\delta}]$ for a given m, where δ is a square-free integer. Furthermore, we determined explicitly all power Fibonacci sequences in $\mathbb{Z}_{p^k}[\sqrt{\delta}]$, where p is a prime number.

Keywords: Power Fibonacci sequences, Fibonacci sequence, Legendre symbol, Quadratic equation, Square-free integer.
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1 Introduction

The Fibonacci sequence is defined recursively as follows:

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

where $n \geq 2,\ F_0 = 0,\ F_1 = 1.$

In the study "Power Fibonacci Sequences" by Joshua Ide and Marc S. Renault, the authors introduced a sequence defined as a Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$ where $F_0 = 1$ and $F_1 = a$, such that for some $a \in \mathbb{Z}_m$,

$$F_n \equiv a^n \pmod m \tag{2}$$

for all $n \in \mathbb{N} \cup \{0\}$ [2]. If such a exists, they called $\{F_n\}_{n=1}^{\infty}$ a power Fibonacci sequence in $\mathbb{Z}_m$. For instance, in $\mathbb{Z}_5$, there is only one power Fibonacci sequence, that is, when a = 3, but in $\mathbb{Z}_{10}$ there are no such sequences. Also, there are four power Fibonacci sequences in $\mathbb{Z}_{209}$.

With these ideas in mind, we shall extend the past work of Ide and Renault and work on the power Fibonacci sequences in $\mathbb{Z}_m[\sqrt{\delta}]$ where δ is a square-free integer. We will determine all $m \in \mathbb{Z}^+$ for which power Fibonacci sequences exist in $\mathbb{Z}_m[\sqrt{\delta}]$ and the number of power Fibonacci sequences in $\mathbb{Z}_m[\sqrt{\delta}]$ there are for a given m, where δ is a square-free integer.

The following propositions are vital in the main result.

Proposition 1.1. *Let p be an odd prime and $a \in \mathbb{Z}$ with $\gcd(a, p) = 1$. The equation*

$$x^2 \equiv a \pmod{p^k} \tag{3}$$

either

- *has no solution if $\left(\frac{a}{p}\right) = -1$; or*
- *has 2 solutions x_1 and $-x_1$ if $\left(\frac{a}{p}\right) = 1$.*

Proposition 1.2. *The equation $x^2 \equiv 5 \pmod{5^e}$ has no solution $x \in \mathbb{Z}_{5^e}$ for $e > 1$.*

2 Working equations

In this section, we will derive working equations to characterize the power Fibonacci sequences in $\mathbb{Z}_m[\sqrt{\delta}]$, assuming that it exists.

Proposition 2.1. *Let $m \in \mathbb{N}$ where $m > 1$ and let δ be a square-free integer. Then, $\{a^n\}_{n=0}^{\infty}$ is a power Fibonacci sequence in $\mathbb{Z}_m[\sqrt{\delta}]$ if and only if $a = x + \sqrt{\delta}y$ is a root of $f(z) = z^2 - z - 1$ in $\mathbb{Z}_m[\sqrt{\delta}]$. Furthermore, if 2 is a unit in $\mathbb{Z}_m[\sqrt{\delta}]$, then $a = 2^{-1}(1 + r)$, where r is a root of $g(z) = z^2 - 5$ in $\mathbb{Z}_m[\sqrt{\delta}]$.*

Proof. Assume $\{a^n\}_{n=0}^{\infty}$ is a power Fibonacci sequence in $\mathbb{Z}_m[\sqrt{\delta}]$. Then,

$$a^2 = a^1 + a^0 = a + 1. \tag{4}$$

Therefore, $a \in \mathbb{Z}_m[\sqrt{\delta}]$ is a root of $f(z)$. Conversely, suppose that $a \in \mathbb{Z}_m[\sqrt{\delta}]$ is a root of $f(z)$. Then, $a^2 = a + 1$, and thus, $a^n = a^{n-1} + a^{n-2}$, for all $n \geq 2$, as desired. Furthermore, if 2 is a unit in $\mathbb{Z}_m[\sqrt{\delta}]$, then completing the square implies that $a = 2^{-1}(1 + r)$ where r is a root of $g(z) = z^2 - 5$ in $\mathbb{Z}_m[\sqrt{\delta}]$. □

Remark 2.1. *Counting the roots of $f(z) = z^2 - z - 1$ in $\mathbb{Z}_m[\sqrt{\delta}]$ determines the number of power Fibonacci sequences in $\mathbb{Z}_m[\sqrt{\delta}]$. Furthermore, if 2 is a unit in $\mathbb{Z}_m[\sqrt{\delta}]$, then counting the roots of $g(z) = z^2 - 5$ determines the number of power Fibonacci sequences in $\mathbb{Z}_m[\sqrt{\delta}]$.*

3 Preparatory lemmas

To prove the main result of this study, we need the following lemmas. In this study, if we have a system of equations

$$F(x, y) : \begin{cases} f(x, y) = 0 \\ g(x, y) = 0, \end{cases}$$

then we write $F(x, y) \equiv 0 \pmod{n}$ if and only if $f(x, y) \equiv 0 \pmod{n}$ and $g(x, y) \equiv 0 \pmod{n}$. In addition, we will use the notation $N(\mathbb{Z}_m[\sqrt{\delta}])$ to be the number of power Fibonacci sequences in $\mathbb{Z}_m[\sqrt{\delta}]$.

Lemma 3.1. *Let $f(x, y)$ and $g(x, y)$ be polynomials with integral coefficients and consider the system of equations*

$$F(x, y) : \begin{cases} f(x, y) = 0 \\ g(x, y) = 0. \end{cases} \tag{5}$$

Suppose that $\gcd(n, n') = 1$. *If*

$$F(x, y) \equiv 0 \pmod{n} \tag{6}$$

has N solutions and

$$F(x, y) \equiv 0 \pmod{n'} \tag{7}$$

has N' solutions, then

$$F(x, y) \equiv 0 \pmod{nn'} \tag{8}$$

has NN' solutions.

Proof. See lemma in [1]. □

By Lemma 3.1, it turns out that $h : \mathbb{N} \to \mathbb{N} \cup \{0\} : m \mapsto N(\mathbb{Z}_m[\sqrt{\delta}])$ is a multiplicative function.

Lemma 3.2. *Let* $h : \mathbb{N} \to \mathbb{N} \cup \{0\}$ *where* $h(1) = 1$ *and* $h(m) = N(\mathbb{Z}_m[\sqrt{\delta}])$*, for* $m > 1$*. Then,* h *is a multiplicative function. Consequently, if* $m = p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}$ *is the prime decomposition of* m*, for* $k \geq 1$*, then*

$$N(\mathbb{Z}_m[\sqrt{\delta}]) = \prod_{i=1}^{k} N(\mathbb{Z}_{p_i^{q_i}}[\sqrt{\delta}]). \tag{9}$$

Proof. It is enough to show that h is multiplicative. Indeed, let $m = ab$ where $\gcd(a, b) = 1$. Note that $N(\mathbb{Z}_m[\sqrt{\delta}])$ is the number of solutions to the equation $f(x + y\sqrt{\delta}) \equiv 0 \pmod{m}$. Meanwhile,

$$f(x + y\sqrt{\delta}) \equiv 0 \pmod{m}$$
$$(x^2 + \delta y^2 - x - 1) + (2xy - y)\sqrt{\delta} \equiv 0 \pmod{m}.$$

Thus, we have a system of equations

$$x^2 + \delta y^2 - x - 1 \equiv 0 \pmod{m}$$

and

$$2xy - y \equiv 0 \pmod{m}.$$

From Remark 2.1, $N(\mathbb{Z}_m[\sqrt{\delta}])$ is the number of solutions to $f(x + y\sqrt{\delta}) \equiv 0 \pmod{m}$ or

$$F(x, y) : \begin{cases} x^2 + \delta y^2 - x - 1 & \equiv 0 \pmod{m} \\ 2xy - y & \equiv 0 \pmod{m}. \end{cases} \tag{10}$$

Thus, the number of solutions to $F(x, y) \equiv 0 \pmod{m}$ is $N(\mathbb{Z}_m[\sqrt{\delta}])$. Also, the number of solutions to $F(x, y) \equiv 0 \pmod{a}$ and $F(x, y) \equiv 0 \pmod{b}$ is $N(\mathbb{Z}_a[\sqrt{\delta}])$ and $N(\mathbb{Z}_b[\sqrt{\delta}])$, respectively. Since $\gcd(a, b) = 1$, then by Lemma 3.1, the number of solutions to $F(x, y) \equiv 0 \pmod{m}$ is the product of the number of solutions to $F(x, y) \equiv 0 \pmod{a}$ and $F(x, y) \equiv 0 \pmod{b}$. Therefore,

$$h(ab) = h(m) = N(\mathbb{Z}_m[\sqrt{\delta}]) = N(\mathbb{Z}_a[\sqrt{\delta}])N(\mathbb{Z}_b[\sqrt{\delta}]) = h(a)h(b),$$

as desired. □

The system of equations in (10) implies that we will deal with equation $ab \equiv 0 \pmod{m}$ where $a, b \in \mathbb{Z}_m$. Lemma 3.3 gives a characterization to the elements a and b satisfying $ab \equiv 0 \pmod{p^k}$ where p is a prime number.

Lemma 3.3. *Let* p *be a prime and* $a, b \in \mathbb{Z}_{p^k}$*. Then,* $ab \equiv 0 \pmod{p^k}$ *if and only if there are positive integers* t_1 *and* t_2 *where* $a = p^r t_1$ *and* $b = p^s t_2$ *with* $r + s \geq k$*,* $r \leq k$*,* $s \leq k$*, and* $\gcd(p, t_1) = 1 = \gcd(p, t_2)$*.*

Proof. The converse is trivial. Assume $ab \equiv 0 \pmod{p^k}$. Then, $ab = \theta p^k$, for some $\theta \in \mathbb{N}$. In that case, there exists $a \in \mathbb{N}$ such that $a = \frac{\theta p^k}{b} = t_1 p^r$, for some $t_1 \in \mathbb{N}$ where $\gcd(t_1, p) = 1$. Similarly, $\mathbb{N} \ni b = \frac{\theta p^k}{a} = t_2 p^s$, for some $t_2 \in \mathbb{N}$ where $\gcd(t_2, p) = 1$. Since $a, b \in \mathbb{Z}_{p^k}$, then $r \leq k$, and $s \leq k$. If $r + s < k$, then $p^k \theta = ab = t_1 t_2 p^{r+s}$, and thus, $p | t_1 t_2$, which is a contradiction. Therefore, $r + s \geq k$. □

Lemma 3.4. *Let $p \neq 5$ be an odd prime. Then, $(x + y\sqrt{\delta})^2 \equiv 5 \pmod{p^k}$, for all $k \in \mathbb{N}$ if and only if exactly one of the following holds:*

1. $x^2 \equiv 5 \pmod{p^k}$ *and* $y \equiv 0 \pmod{p^k}$.
2. $\delta y^2 \equiv 5 \pmod{p^k}$ *and* $x \equiv 0 \pmod{p^k}$.

Proof. The converse is trivial. Assume $(x + y\sqrt{\delta})^2 \equiv 5 \pmod{p^k}$. Then,

$$x^2 + 2xy\sqrt{\delta} + \delta y^2 \equiv 5 \pmod{p^k}.$$

Thus, we have the system of equation

$$x^2 + \delta y^2 \equiv 5 \pmod{p^k} \tag{11}$$

and

$$2xy \equiv 0 \pmod{p^k}. \tag{12}$$

Since $2xy \equiv 0 \pmod{p^k}$, then $x = p^r x_0$ and $y = p^s y_0$ where $r + s \geq k$, $r \leq k$, $s \leq k$, and $\gcd(x_0, p^k) = 1 = \gcd(y_0, p^k)$. For the first case, assume $r \leq s$. Suppose $r \neq 0$. Then,

$$(p^r x_0)^2 + \delta(p^s y_0)^2 \equiv 5 \pmod{p^k}.$$

Since $r \leq k$, it follows that

$$p^{2r} x_0^2 + \delta p^{2s} y_0^2 - 5 \equiv 0 \pmod{p^r}. \tag{13}$$

Observe that $p^{2r} \equiv 0 \pmod{p^r}$ and $p^{2s} \equiv 0 \pmod{p^r}$. Hence, Equation (13) reduces to

$$-5 \equiv 0 \pmod{p^r}.$$

However, since $p \neq 5$, this is a contradiction for any $r \in \mathbb{N}$. Thus, $r = 0$ and $x = p^0 x_0 = x_0$, implying $\gcd(x, p^k) = 1$. Consequently, since p is an odd prime then $\gcd(2x, p^k) = 1$, which implies $(2x)^{-1}$ exists in $\mathbb{Z}_{p^k}$. That being so, Equation (12) reduces to

$$y \equiv 0 \pmod{p^k}.$$

Therefore, Equation (11) now reduces to $x^2 \equiv 5 \pmod{p^k}$, as desired. For the second case, assume $s < r$. Suppose $s \neq 0$. Then

$$(p^r x_0)^2 + \delta(p^s y_0)^2 \equiv 5 \pmod{p^k}.$$

Since $s \leq k$, it follows that

$$p^{2r} x_0^2 + \delta p^{2s} y_0^2 - 5 \equiv 0 \pmod{p^s}. \tag{14}$$

Observe that $p^{2r} \equiv 0 \pmod{p^s}$ and $p^{2s} \equiv 0 \pmod{p^s}$. Hence, Equation (14) reduces to

$$-5 \equiv 0 \pmod{p^s}.$$

However, since $p \neq 5$, this is a contradiction for any $s \in \mathbb{N}$. Thus, $s = 0$ and $y = p^0 y_0 = y_0$, implying $\gcd(y, p^k) = 1$. Consequently, since p is an odd prime, then $\gcd(2y, p^k) = 1$, which implies $(2y)^{-1}$ exists in $\mathbb{Z}_{p^k}$. That being so, Equation (12) reduces to

$$x \equiv 0 \pmod{p^k}.$$

Therefore, Equation (11) now reduces to $\delta y^2 \equiv 5 \pmod{p^k}$, as desired.

Lastly, suppose $x^2 \equiv 5 \pmod{p^k}$ and $\delta y^2 \equiv 5 \pmod{p^k}$. Since $x^2 \equiv 5 \pmod{p^k}$ and $\delta y^2 \equiv 5 \pmod{p^k}$, then Equation (11) becomes

$$5 \equiv 0 \pmod{5^k}.$$

This is a contradiction for $p \neq 5$ and for any $k \in \mathbb{N}$. Therefore, $x^2 \equiv 5 \pmod{p^k}$ and $\delta y^2 \equiv 5 \pmod{p^k}$ cannot happen at the same time. □

Lemma 3.5. *Let* $e \in \mathbb{N}, e > 1$. *Then,* $(x + y\sqrt{\delta})^2 \equiv 5 \pmod{5^e}$ *if and only if* $\delta y^2 \equiv 5 \pmod{5^e}$ *and* $x \equiv 0 \pmod{5^e}$.

Proof. The converse is trivial. Assume $(x + y\sqrt{\delta})^2 \equiv 5 \pmod{5^e}$. Then, we have

$$(x^2 + \delta y^2) \equiv 5 \pmod{5^e} \tag{15}$$

and

$$2xy \equiv 0 \pmod{5^e}. \tag{16}$$

Since $2xy \equiv 0 \pmod{5^e}$, then $x = 5^r x_0$ and $y = 5^s y_0$ where $r + s \geq e$, $r \leq e$, $s \leq e$ and $\gcd(x_0, 5^e) = 1 = \gcd(y_0, 5^e)$. For the first case, assume $r \leq s$. Suppose $r \neq 0$, then

$$(5^r x_0)^2 + \delta(5^s y_0)^2 \equiv 5 \pmod{5^e}. \tag{17}$$

Since $r \leq e$, then Equation (17) implies that

$$5^{2r} x_0^2 + \delta 5^{2s} y_0^2 \equiv 5 \pmod{5^r}. \tag{18}$$

Since $r \leq s$, then $5^s \equiv 0 \pmod{5^r}$. Thus, Equation (18) will be reduced to

$$-5 \equiv 0 \pmod{5^r}. \tag{19}$$

Observe that Equation (19) holds if and only if $r = 1$. However, if $r = 1$, then $x = 5x_0$, and thus, by invoking Equation (17), we have

$$25x_0^2 + \delta(5^s y_0)^2 \equiv 5 \pmod{5^e}. \tag{20}$$

Since $e > 1$, then from Equation (20) we have

$$5x_0^2 + \delta 5^{2s-1} y_0^2 - 1 \equiv 0 \pmod{5^{e-1}}. \tag{21}$$

Consequently, from Equation (21), it follows that

$$5x_0^2 + \delta 5^{2s-1} y_0^2 - 1 \equiv 0 \pmod{5}. \tag{22}$$

Thus, Equation (22) reduces to

$$-1 \equiv 0 \pmod{5}.$$

However, this is a contradiction. Thus, $r = 0$ and $x = 5^0 x_0 = x_0$, implying $\gcd(x, 5^e) = 1$. Consequently, $\gcd(2x, 5^e) = 1$, which implies $(2x)^{-1}$ exists in $\mathbb{Z}_{5^e}$. That being so, Equation (16) reduces to

$$y \equiv 0 \pmod{5^e}.$$

Hence, Equation (15) now reduces to $x^2 \equiv 5 \pmod{5^e}$. However, since $e > 1$, this is a contradiction by Proposition 1.2. Thus, $s < r$. Suppose $s \neq 0$. Since $s \leq e$, from Equation (17) it follows that

$$5^{2r} x_0^2 + \delta 5^{2s} y_0^2 \equiv 5 \pmod{5^s}. \tag{23}$$

Since $s < r$, then $5^r \equiv 0 \pmod{5^s}$. So, Equation (23) implies that

$$-5 \equiv 0 \pmod{5^s}. \tag{24}$$

Observe that Equation (24) holds if and only if $s = 1$. But, if $s = 1$, then $y = 5y_0$, and thus, by invoking Equation (17), we have

$$(5^r x_0)^2 + \delta(5y_0)^2 \equiv 5 \pmod{5^e}. \tag{25}$$

Since $e > 1$, then by Equation (25) we have

$$5^{2r-1} x_0^2 + \delta 5 y_0^2 - 1 \equiv 0 \pmod{5}. \tag{26}$$

Hence, Equation (26) reduces to

$$-1 \equiv 0 \pmod{5}.$$

Again, this is a contradiction. Thus, $s = 0$ and $y = 5^0 y_0 = y_0$, implying $\gcd(y, 5^e) = 1$. Consequently, $\gcd(2y, 5^e) = 1$, which implies $(2y)^{-1}$ exists in $\mathbb{Z}_{5^e}$. That being so, Equation (16) reduces to

$$x \equiv 0 \pmod{5^e}.$$

Therefore, Equation (15) now reduces to $\delta y^2 \equiv 5 \pmod{5^e}$, as desired. □

4 Power Fibonacci sequences

Since Lemma 3.2 depends on the prime decomposition of m, we consider four specific cases of $N(\mathbb{Z}_m[\sqrt{\delta}])$:

1. $N(\mathbb{Z}_{2^k}[\sqrt{\delta}])$ where $k \geq 1$; to be proven in Proposition 4.1,
2. $N(\mathbb{Z}_{p^k}[\sqrt{\delta}])$ where $p \neq 5$ is an odd prime; to be proven in Proposition 4.2 through Proposition 4.5,
3. $N(\mathbb{Z}_5[\sqrt{\delta}])$; to be proven in Proposition 4.6,
4. $N(\mathbb{Z}_{5^e}[\sqrt{\delta}])$ where $e > 1$; to be proven in Proposition 4.7.

For the first case, we have Proposition 4.1.

Proposition 4.1. $N(\mathbb{Z}_{2^k}[\sqrt{\delta}]) = 0$ *for all* $k \in \mathbb{N}$.

Proof. By Remark 2.1, $N(\mathbb{Z}_{2^k}[\sqrt{\delta}])$ is the number of solutions to $f(x + y\sqrt{\delta}) \equiv 0 \pmod{2^k}$. Suppose $N(\mathbb{Z}_{2^k}[\sqrt{\delta}]) > 0$. Then, there exists $a := x + y\sqrt{\delta}$, where $x, y \in \mathbb{Z}_2$ and $f(a) \equiv 0 \pmod 2$. Thus, we have a system of equations

$$x^2 + \delta y^2 - x - 1 \equiv 0 \pmod 2 \tag{27}$$

and

$$(2x - 1)y \equiv 0 \pmod 2. \tag{28}$$

Since $(2x - 1)y \equiv 0 \pmod 2$, then $y \equiv 0 \pmod 2$. Hence, Equation (27) reduces to

$$x^2 - x - 1 \equiv 0 \pmod 2,$$

which is a contradiction since $x^2 - x - 1 = 0$ does not have a solution in $\mathbb{Z}_2$. Therefore, $N(\mathbb{Z}_{2^k}[\sqrt{\delta}]) = 0$. □

Now, for the second case, that is, $N(\mathbb{Z}_{p^k}[\sqrt{\delta}])$ where $p \neq 5$ is an odd prime, we consider four sub-cases:

1. $\left(\frac{5}{p}\right) = 1$; to be proven in Proposition 4.2,
2. $\left(\frac{5}{p}\right) = -1$ and $\left(\frac{\delta}{p}\right) = 1$; to be proven in Proposition 4.3,
3. $\left(\frac{5}{p}\right) = -1$ and $\left(\frac{\delta}{p}\right) = 0$; to be proven in Proposition 4.4,
4. $\left(\frac{5}{p}\right) = -1$ and $\left(\frac{\delta}{p}\right) = -1$; to be proven in Proposition 4.5.

Proposition 4.2. *Let* $p \neq 5$ *be an odd prime such that* $\left(\frac{5}{p}\right) = 1$. *Then, the following holds:*

1. $N(\mathbb{Z}_{p^k}[\sqrt{\delta}]) = 2$, *for all* $k \in \mathbb{N}$.
2. *The equation* $r^2 \equiv 5 \pmod{p^k}$ *has two incongruent solution* $r_1, r_2 \in \mathbb{Z}_{p^k}$.
3. *The two power Fibonacci sequences in* $\mathbb{Z}_{p^k}[\sqrt{\delta}]$ *are* $\{(2^{-1} + 2^{-1}r_1)^n \bmod p^k\}_{n=0}^{\infty}$ *and* $\{(2^{-1} + 2^{-1}r_2)^n \bmod p^k\}_{n=0}^{\infty}$.

Proof. First, we will prove Statement 1. By Remark 2.1, $N(\mathbb{Z}_{p^k}[\sqrt{\delta}])$ is the number of solutions to $(x + y\sqrt{\delta})^2 \equiv 5 \pmod{p^k}$, given that $p \neq 5$ is an odd prime and $\left(\frac{5}{p}\right) = 1$. Since $\left(\frac{5}{p}\right) = 1$ and $\gcd(5, p) = 1$, then $r^2 \equiv 5 \pmod{p^k}$ for some $r \in \mathbb{Z}_{p^k}$. By Lemma 3.4, $x^2 \equiv 5 \pmod{p^k}$ and $y \equiv 0 \pmod{p^k}$. Thus, it suffices to count the solutions to $x^2 \equiv 5 \pmod{p^k}$ in $\mathbb{Z}_{p^k}$ to determine the number of power Fibonacci sequences in $\mathbb{Z}_{p^k}[\sqrt{\delta}]$. From Proposition 1.1, it follows that $x^2 \equiv 5 \pmod{p^k}$ has two solutions. Therefore, $N(\mathbb{Z}_{p^k}[\sqrt{\delta}]) = 2$.

Statements 2. and 3. follow directly from Proposition 1.1, with the assumption that $\left(\frac{5}{p}\right) = 1$, Lemma 3.4, and Proposition 2.1. □

Proposition 4.3. *Let $p \neq 5$ be an odd prime. If $\left(\frac{5}{p}\right) = -1$ and $\left(\frac{\delta}{p}\right) = 1$, then $N(\mathbb{Z}_{p^k}[\sqrt{\delta}]) = 0$, for all $k \in \mathbb{N}$.*

Proof. Since $\left(\frac{\delta}{p}\right) = 1$, then $\mathbb{Z}_{p^k}[\sqrt{\delta}] = \mathbb{Z}_{p^k}$.

Since $\left(\frac{5}{p}\right) = -1$, then by [2], $N(\mathbb{Z}_{p^k}[\sqrt{\delta}]) = N(\mathbb{Z}_{p^k}) = 0$, as desired. □

Proposition 4.4. *Let $p \neq 5$ be an odd prime. If $\left(\frac{5}{p}\right) = -1$ and $\left(\frac{\delta}{p}\right) = 0$, then $N(\mathbb{Z}_{p^k}[\sqrt{\delta}]) = 0$, for all $k \in \mathbb{N}$.*

Proof. By Remark 2.1, $N(\mathbb{Z}_{p^k}[\sqrt{\delta}])$ is the number of solutions to $(x + y\sqrt{\delta})^2 \equiv 5 \pmod{p^k}$, given that $p \neq 5$ is an odd prime and $\left(\frac{5}{p^k}\right) = -1$ and $\left(\frac{\delta}{p^k}\right) = 0$. From Lemma 3.4, either

$$x^2 \equiv 5 \pmod{p^k} \tag{29}$$

or

$$\delta y^2 \equiv 5 \pmod{p^k}, \tag{30}$$

but not both. Since $\left(\frac{5}{p}\right) = -1$, then by Proposition 1.1, $x^2 \not\equiv 5 \pmod{p^k}$ for any $x \in \mathbb{Z}_{p^k}$. Thus, Equation (30) should have a solution. Suppose, $\delta y^2 \equiv 5 \pmod{p^k}$ for some $y \in \mathbb{Z}_{p^k}$. Since $\left(\frac{\delta}{p}\right) = 0$, then $\gcd(\delta, p^k) \neq 1$. Now, given that δ is a square-free integer, we can express $\delta := pd$ where $\gcd(d, p^k) = 1$. Hence, Equation (30) will be

$$pdy^2 - 5 \equiv 0 \pmod{p^k}. \tag{31}$$

It follows that

$$pdy^2 - 5 \equiv 0 \pmod{p}. \tag{32}$$

Equation (32) reduces to

$$-5 \equiv 0 \pmod{p}.$$

However, $p \neq 5$, which is a contradiction. This implies $\delta y^2 \not\equiv 5 \pmod{p^k}$ for any $y \in \mathbb{Z}_{p^k}$. Therefore, $N(\mathbb{Z}_{p^k}[\sqrt{\delta}]) = 0$. □

Proposition 4.5. *Let $p \neq 5$ be an odd prime such that $\left(\frac{5}{p}\right) = -1$ and $\left(\frac{\delta}{p}\right) = -1$. Then, the following holds:*

1. $N(\mathbb{Z}_{p^k}[\sqrt{\delta}]) = 2$, *for all $k \in \mathbb{N}$.*
2. *The equation $\delta r^2 \equiv 5 \pmod{p^k}$ has two incongruent solutions $r_1, r_2 \in \mathbb{Z}_{p^k}$.*
3. *The two power Fibonacci sequences in $\mathbb{Z}_{p^k}[\sqrt{\delta}]$ are $\{(2^{-1} + 2^{-1}r_1)^n \bmod p^k\}_{n=0}^{\infty}$ and $\{(2^{-1} + 2^{-1}r_2)^n \bmod p^k\}_{n=0}^{\infty}$.*

Proof. First, we will prove Statement 1. By Remark 2.1, $N(\mathbb{Z}_{p^k}[\sqrt{\delta}])$ is the number of solutions to $(x + y\sqrt{\delta})^2 \equiv 5 \pmod{p^k}$, given that $p \neq 5$ is an odd prime and $\left(\frac{5}{p}\right) = -1$ and $\left(\frac{\delta}{p}\right) = -1$. Since $\left(\frac{5}{p}\right) = -1$, then by Proposition 1.1, $x^2 \not\equiv 5 (\text{mod} p^k)$ for any $x \in \mathbb{Z}_{p^k}$. Thus, by Lemma 3.4,

$\delta y^2 \equiv 5 \pmod{p^k}$ and $x \equiv 0 \pmod{p^k}$. So, it is enough to count the solutions of $\delta y^2 \equiv 5 \pmod{p^k}$ in $\mathbb{Z}_{p^k}$. Since $\left(\frac{\delta}{p}\right) = 1$, then $\gcd(\delta, p^k) = 1$. As a consequence, we have

$$y^2 \equiv 5\delta^{-1} \pmod{p^k}.$$

Now,

$$\left(\frac{5\delta^{-1}}{p}\right) = \left(\frac{5}{p}\right)\left(\frac{\delta^{-1}}{p}\right) = \left(\frac{5}{p}\right)\left(\frac{\delta}{p}\right) = (-1)(-1) = 1. \tag{33}$$

Since $\left(\frac{5\delta^{-1}}{p}\right) = 1$, then by Proposition 1.1, it follows that $y^2 \equiv 5\delta^{-1} (\text{mod} p^k)$ or $\delta y^2 \equiv 5(\text{mod} p^k)$ has two solutions. Therefore, $N(\mathbb{Z}_{p^k}[\sqrt{\delta}]) = 2$.

Statements 2. and 3. follow directly from Proposition 1.1, with the assumption that $\left(\frac{5\delta^{-1}}{p}\right) = 1$, Lemma 3.4, and Remark 2.1. □

For the third case, we have Proposition 4.6.

Proposition 4.6. *If δ is a square-free integer, then $N(\mathbb{Z}_5[\sqrt{\delta}]) = 1$. Furthermore, the only power Fibonacci sequence in $\mathbb{Z}_5[\sqrt{\delta}]$ is $\{1, 3, 4, 2, 1, \ldots\}$.*

Proof. By Remark 2.1, the roots of $f(x) = x^2 - x - 1$ in $\mathbb{Z}_5[\sqrt{\delta}]$ are those residues of the form $2^{-1}(1 \pm m)$, where $m^2 = 5$ in $\mathbb{Z}_5[\sqrt{\delta}]$. However, $m^2 = 5$ in $\mathbb{Z}_5[\sqrt{\delta}]$ implies that $m^2 = 0$ in $\mathbb{Z}_5[\sqrt{\delta}]$. Consequently, $m = 0$, and thus, $2^{-1}(1 \pm m) = 2^{-1} = 3$. Hence, the only root of f in $\mathbb{Z}_5[\sqrt{\delta}]$ is 3. Thus, there exists one distinct root of f in $\mathbb{Z}_5[\sqrt{\delta}]$. Therefore, $N([\mathbb{Z}_5[\sqrt{\delta}]) = 1$. Furthermore, by Remark 2.1, the only power Fibonacci sequence in $\mathbb{Z}_5[\sqrt{\delta}]$ is

$$\{3^n \bmod 5\}_{n=0}^{\infty} = \{1, 3, 4, 2, 1, \ldots\},$$

as desired. □

Finally, for the last case, we have Proposition 4.7.

Proposition 4.7. *Let $e > 1$ and $\lambda = \frac{\delta}{5}$. Then, $5 \mid \delta$ and $\lambda \equiv \pm 1 \pmod 5$ if and only if $N(\mathbb{Z}_{5^e}[\sqrt{\delta}]) > 0$. Consequently, $N(\mathbb{Z}_{5^e}[\sqrt{\delta}]) = 10$. Furthermore, the ten power Fibonacci sequences in $\mathbb{Z}_{5^e}[\sqrt{\delta}]$ are*

$$\{(2^{-1} + (2^{-1}y_1 + 5^{e-1}(i-1))\sqrt{\delta})^n : n \in \mathbb{N} \cup \{0\}\}$$

and

$$\{(2^{-1} + (2^{-1}y_2 + 5^{e-1}(i-1))\sqrt{\delta})^n : n \in \mathbb{N} \cup \{0\}\},$$

where y_1 and y_2 are two incongruent roots of $\lambda y^2 \equiv 1 \pmod{5^{e-1}}$ and $i = 1, 2, 3, 4, 5$.

Proof. Suppose $5 \mid \delta$ and $\frac{\delta}{5} = \lambda \equiv \pm 1 \pmod 5$. Then, $\delta = 5\lambda$ for some $\lambda \in \mathbb{Z}$. Since $\lambda \equiv \pm 1 \pmod 5$, then $\left(\frac{\lambda}{5}\right) = 1$, and thus, $\left(\frac{\lambda^{-1}}{5}\right) = 1$. Consequently, $\gcd(\lambda^{-1}, 5) = 1$. Now, by Proposition 1.1,

$$y^2 \equiv \lambda^{-1} \pmod{5^{e-1}} \tag{34}$$

has two solutions for $e > 1$, implying

$$\lambda y^2 - 1 = 5^{e-1}t \tag{35}$$

for some $t \in \mathbb{Z}$. From (35), we have

$$\begin{aligned} 5\lambda y^2 &\equiv 5 \pmod{5^e} \\ \delta y^2 &\equiv 5 \pmod{5^e}. \end{aligned}$$

This implies $\delta y^2 \equiv 5 \pmod{5^e}$ has a solution. Therefore, by Lemma 3.5 and Remark 2.1, $N(\mathbb{Z}_{5^e}[\sqrt{\delta}]) > 0$. Now, we claim that $N(\mathbb{Z}_{5^e}[\sqrt{\delta}]) = 10$. Let $f(x) = x^2 - x - 1$. By Remark 2.1 and Lemma 3.5, note that if y_0 is a solution of $\delta y^2 \equiv 5 \pmod{5^e}$, then

$$2^{-1}(1 + y_0\sqrt{\delta}) = 2^{-1} + 2^{-1}y_0\sqrt{\delta}$$

is a solution to

$$f(x + y\sqrt{\delta}) \equiv 0 \pmod{5^e}. \tag{36}$$

Thus, solutions to Equation (36) are of the form $2^{-1} + 2^{-1}y\sqrt{\delta}$ where $\delta y^2 \equiv 5 \pmod{5^e}$ or $y^2 \equiv \lambda^{-1} \pmod{5^{e-1}}$. Now, let y_1 and y_2 be the two incongruent solutions of Equation (34) where $y_1 \neq y_2$. Set

$$r_i = 2^{-1} + (2^{-1}y_1 + 5^{e-1}(i-1))\sqrt{\delta}$$

and

$$s_i = 2^{-1} + (2^{-1}y_2 + 5^{e-1}(i-1))\sqrt{\delta}$$

where $i = 1, 2, 3, 4, 5$. Observe that $f(r_i) = 0$ and $f(s_i) = 0$, and thus, r_1, r_2, r_3, r_4, r_5, s_1, s_2, s_3, s_4, and s_5 are ten incongruent solutions to Equation (36). Therefore, by Remark 2.1, $N(\mathbb{Z}_{5^e}[\sqrt{\delta}]) \geq 10$.

Assume t is another incongruent solution of Equation (36) that is, $t \neq r_i$ and $t \neq s_i$ for $i = 1, 2, 3, 4, 5$. Then, there exists $c \in \mathbb{Z}_{5^e}$ such that $t = 2^{-1} + 2^{-1}c\sqrt{\delta}$ where $\delta c^2 \equiv 5 \pmod{5^e}$. Thus, we have

$$\begin{aligned} 5\lambda c^2 &\equiv 5 \pmod{5^e} \\ \lambda c^2 &\equiv 1 \pmod{5^{e-1}} \end{aligned} \tag{37}$$

However, it was initially stated that Equation (34) or (37) has exactly two incongruent solutions y_1 and y_2 in $\mathbb{Z}_{5^{e-1}}$, and thus, either $y_1 = c$ or $y_2 = c$. By Hensel's lemma, either $t = r_i$ or $t = s_i$, a contradiction of the assumption that t is another incongruent solution. Therefore, $N(\mathbb{Z}_{5^e}[\sqrt{\delta}]) = 10$. Furthermore, by Proposition 2.1, the ten power Fibonacci sequences in $\mathbb{Z}_{5^e}[\sqrt{\delta}]$ are

$$\{(2^{-1} + (2^{-1}y_1 + 5^{e-1}(i-1))\sqrt{\delta})^n : n \in \mathbb{N} \cup \{0\}\}$$

and

$$\{(2^{-1} + (2^{-1}y_2 + 5^{e-1}(i-1))\sqrt{\delta})^n : n \in \mathbb{N} \cup \{0\}\}$$

where y_1 and y_2 are two incongruent roots of $\lambda y^2 \equiv 1 \pmod{5^{e-1}}$ and $i = 1, 2, 3, 4, 5$.

Conversely, by Remark 2.1, $N(\mathbb{Z}_{5^e}[\sqrt{\delta}])$ is the number of solutions to $(x+y\sqrt{\delta})^2 \equiv 5 \pmod{5^e}$. Assume $N(\mathbb{Z}_{5^e}[\sqrt{\delta}]) > 0$. By Lemma 3.5, the equation

$$\delta y^2 \equiv 5 \pmod{5^e} \tag{38}$$

must have a solution. Suppose $5 \nmid \delta$. Then, $5 \mid y$, that is, $y = 5y_0$ for some $y_0 \in \mathbb{Z}$. From Equation (38), we have

$$5\delta y_0^2 - 1 \equiv 0 \pmod{5^{e-1}}. \tag{39}$$

Since $e - 1 \geq 1$, it follows that

$$-1 \equiv 0 \pmod 5,$$

which is a contradiction. Therefore, $5 \mid \delta$. Now, assume $5 \mid \delta$ where $\delta = 5\lambda$ for some $\lambda \in \mathbb{Z}$. From Equation (38), we have

$$\begin{aligned} 5\lambda y^2 - 5 &\equiv 0 \pmod{5^e} \\ \lambda y^2 - 1 &\equiv 0 \pmod{5^{e-1}}. \end{aligned}$$

Since $e - 1 \geq 1$, it follows that,

$$\lambda y^2 \equiv 1 \pmod 5. \tag{40}$$

By inspection, only $\lambda \equiv \pm 1 \pmod 5$ satisfies Equation (40). Therefore, $5 \mid \delta$ and $\frac{\delta}{5} = \lambda \equiv \pm 1 \pmod 5$. □

Now, we accomplished the objective of this study and proved the main theorem by applying Lemma 3.2.

Theorem 4.8. *Let δ be a square-free integer. Then,*

$$N(\mathbb{Z}_m[\sqrt{\delta}]) = \begin{cases} 2^k, & \text{if } m = p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k} \text{ or } m = 5p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}, \\ & \quad \text{where } p_i \neq 5 \text{ is an odd prime,} \\ & \quad \text{and there is no } p_i \text{ such that } \left(\frac{5}{p_i}\right) = -1 \text{ and } \left(\frac{\delta}{p_i}\right) = 1 \\ & \quad \text{or there is no } p_i \text{ such that } \left(\frac{5}{p_i}\right) = -1 \text{ and } \left(\frac{\delta}{p_i}\right) = 0, \\ & \quad \text{for } i = 1, 2, \ldots, k; k \geq 0. \\ 10 \cdot 2^k, & \text{if } m = 5^n p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}, \\ & \quad \text{where } n > 1 \text{ and } p_i \neq 5 \text{ is an odd prime,} \\ & \quad \text{whenever } 5 \mid \delta \text{ and } \frac{\delta}{5} = \lambda \equiv \pm 1 \pmod 5, \\ & \quad \text{and there is no } p_i \text{ such that } \left(\frac{5}{p_i}\right) = -1 \text{ and } \left(\frac{\delta}{p_i}\right) = 1 \\ & \quad \text{or there is no } p_i \text{ such that } \left(\frac{5}{p_i}\right) = -1 \text{ and } \left(\frac{\delta}{p_i}\right) = 0, \\ & \quad \text{for } i = 1, 2, \ldots, k; k \geq 0. \\ 0, & \text{otherwise.} \end{cases}$$

5 Examples

Example 5.1. Count and determine all power Fibonacci sequences in $\mathbb{Z}_{605}[i]$ where $i = \sqrt{-1}$.

Solution. Note that $605 = 5 \cdot 11^2$ and $\left(\frac{5}{11}\right) = 1$. Applying Theorem 4.8, then $N(\mathbb{Z}_{605}[i]) = 2$. The two Power Fibonacci sequences in $\mathbb{Z}_{605}[i]$ are

$$\{1, 158, 159, 317, 476, \ldots\} \text{ and } \{1, 448, 449, 292, 136, \ldots\}.$$

Example 5.2. Count and determine all power Fibonacci sequences in $\mathbb{Z}_{21}[i]$ where $i = \sqrt{-1}$.

Solution. Note that $21 = 3 \cdot 7$. Since $\left(\frac{5}{3}\right) = -1$, $\left(\frac{-1}{3}\right) = -1$, $\left(\frac{5}{7}\right) = -1$ and $\left(\frac{-1}{7}\right) = -1$, then $N(\mathbb{Z}_{21}[\sqrt{-1}]) = 2^2 = 4$. The power Fibonacci sequences in $\mathbb{Z}_{21}[i]$ are:

$$\begin{gathered}
\{1, 11+2i, 12+2i, 2+4i, 14+6i, 16+10i, \ldots\}, \\
\{1, 11+5i, 12+5i, 2+10i, 14+15i, 16+4i, \ldots\}, \\
\{1, 11+16i, 12+16i, 2+11i, 14+6i, 16+17i, \ldots\}, \\
\{1, 11+19i, 12+19i, 2+17i, 14+15i, 16+11i, \ldots\}.
\end{gathered}$$

Example 5.3. Count and determine all power Fibonacci sequences in $\mathbb{Z}_{25}[\sqrt{5}]$.

Solution. Applying Theorem 4.8, then $N(\mathbb{Z}_{25}\sqrt{5}]) = 10 \cdot 2^0 = 10$. The ten power Fibonacci sequences in $\mathbb{Z}_{25}[\sqrt{5}]$ are the following:

$$\begin{gathered}
\left\{1, 13+2\sqrt{5}, 14+2\sqrt{5}, 2+4\sqrt{5}, 16+6\sqrt{5}, 18+10\sqrt{5}, \ldots\right\}, \\
\left\{1, 13+3\sqrt{5}, 14+3\sqrt{5}, 2+6\sqrt{5}, 16+9\sqrt{5}, 18+15\sqrt{5}, \ldots\right\}, \\
\left\{1, 13+7\sqrt{5}, 14+7\sqrt{5}, 2+14\sqrt{5}, 16+21\sqrt{5}, 18+10\sqrt{5}, \ldots\right\}, \\
\left\{1, 13+8\sqrt{5}, 14+8\sqrt{5}, 2+16\sqrt{5}, 16+24\sqrt{5}, 18+15\sqrt{5}, \ldots\right\}, \\
\left\{1, 13+12\sqrt{5}, 14+12\sqrt{5}, 2+24\sqrt{5}, 16+11\sqrt{5}, 18+10\sqrt{5}, \ldots\right\}, \\
\left\{1, 13+13\sqrt{5}, 14+13\sqrt{5}, 2+\sqrt{5}, 16+14\sqrt{5}, 18+15\sqrt{5}, \ldots\right\}, \\
\left\{1, 13+17\sqrt{5}, 14+17\sqrt{5}, 2+9\sqrt{5}, 16+\sqrt{5}, 18+10\sqrt{5}, \ldots\right\}, \\
\left\{1, 13+18\sqrt{5}, 14+18\sqrt{5}, 2+11\sqrt{5}, 16+4\sqrt{5}, 18+15\sqrt{5}, \ldots\right\}, \\
\left\{1, 13+22\sqrt{5}, 14+22\sqrt{5}, 2+19\sqrt{5}, 16+16\sqrt{5}, 18+10\sqrt{5}, \ldots\right\}, \\
\left\{1, 13+23\sqrt{5}, 14+23\sqrt{5}, 2+21\sqrt{5}, 16+19\sqrt{5}, 18+15\sqrt{5}, \ldots\right\}.
\end{gathered}$$

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