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On the extensions of two arithmetical functions and some of their properties

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Abstract: In the paper, an extension of the well-known Jordan's totient function and generalized Dedekind psi-function are proposed and some properties of theirs are studied.
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1 Introduction

Let $s \geq 1$ be an integer. Jordan's totient function, denoted by φ_s, or J_s, is introduced, for integer $n \geq 1$, as the number of s-tuples of integers a_j, with $1 \leq a_j \leq n$, $j = 1, 2, \ldots, s$, such that $\gcd(a_1, \ldots, a_s, n) = 1$ (see [1, pp. 147–155]).

When $s = 1$, φ_s coincides with Euler's totient function φ.

The generalized Dedekind psi-function, denoted by ψ_s, could be introduced by:

$$\psi_s(n) = \frac{\varphi_{2s}(n)}{\varphi_s(n)}$$

for an integer $n \geq 1$.

When $s = 1$, ψ_s coincides with the well-known Dedekind psi-function.

Below, we introduce extensions of these functions on the complex plane setting:

$$\varphi_s(1) = \psi_s(1)$$

$$\varphi_s(n) = n^s \prod_p \left(1 - \frac{1}{p^s}\right) \quad p \text{ runs over the prime divisors of } n, \tag{1}$$

$$\psi_s(n) = n^s \prod_p \left(1 + \frac{1}{p^s}\right) \quad p \text{ runs over the prime divisors of } n, \tag{2}$$

for integer $n > 1$ and s being a complex number.

It is obvious that φ_s and ψ_s are entire functions with respect to s and are multiplicative functions with respect to n. φ_s is an extension of Jordan's totient function φ_s, since for integer $s \geq 1$ both functions coincide. The same is true for the extension ψ_s and the generalized Dedekind psi-function ψ_s.

2 Properties for real numbers s

It is clear that

$$\varphi_0(n) = \begin{cases} 1, & n = 1; \\ 0, & n > 1. \end{cases}$$

and thus

$$\varphi_0(n) = \sum_{d|n} \mu(d),$$

where μ is the Möbius function.

Also, we have

$$\psi_0(n) = 2^{\omega(n)},$$

where, for $n > 1$, $\omega(n)$ denotes the number of distinct prime divisors of n and $\omega(1) = 0$. It is clear that for prime n we have:

$$\varphi_s(n) = n^s - 1;$$
$$\psi_s(n) = n^s + 1.$$

For composite n the following assertion is true.

Lemma 1. *If $n \geq 4$ is a composite number and $s > 0$ is a real number, then the inequalities:*

$$\varphi_s(n) \leq n^s - \sqrt{n^s} \tag{3}$$

$$\psi_s(n) \geq n^s + \sqrt{n^s} \tag{4}$$

are valid, where equalities hold only for $n = p^2$, p is a prime number.

Proof. Let p be the greatest prime divisor of n. Then (1) yields:

$$\varphi_s(n) \ \leq \ n^s\left(1-\frac{1}{p^s}\right) \ = \ n^s - \frac{n^s}{p^s} \ \leq \ n^s - \sqrt{n^s},$$

since $p \leq \sqrt{n}$.

Thus (3) is proved.

In the same manner we have

$$\psi_s(n) \ \geq \ n^s\left(1+\frac{1}{p^s}\right) \ \geq \ n^s + \frac{n^s}{p^s} \ \geq \ n^s + \sqrt{n^s},$$

since $p \leq \sqrt{n}$. Thus (4) is proved. □

Since $\varphi_1(n) = \varphi(n)$, (3) is a generalization of the well-known inequality of Sierpiński for φ, see [2, p. 231, Theorem 5]:

$$\varphi(n) \leq n - \sqrt{n}$$

which is valid for any composite number $n \geq 4$, and the equality holds only for n^2, n is a prime number.

Let $s > 0$ be an arbitrary real number. Then for $n > 1$, we have:

$$\begin{aligned}
\varphi_s(n)\varphi_{-s}(n) &= n^s \prod_p \left(1-\frac{1}{p^s}\right)(-1)^{\omega(n)} n^{-s} \prod_p (p^s - 1) \\
&= (-1)^{\omega(n)} \left(\prod_p p\right)^s \left(\prod_p \left(1-\frac{1}{p^s}\right)\right)^2 \\
&= \frac{(-1)^{\omega(n)}}{n^{2s}} \left(\prod_p p\right)^s \left(n^s \prod_p \left(1-\frac{1}{p^s}\right)\right)^2 \\
&= \frac{(-1)^{\omega(n)}}{n^{2s}} \left(\varphi_s(n)\right)^2 \left(\prod_p p\right)^s.
\end{aligned}$$

Hence, the relation

$$\varphi_{-s}(n) = \frac{(-1)^{\omega(n)}}{n^{2s}} \varphi_s(n) \left(\prod_p p\right)^s \tag{5}$$

holds.

Equality (5) shows us how does the extension of φ_s look for the negative real numbers. When n is a squarefree number, (5) yields

$$\varphi_{-s}(n) = (-1)^{\omega(n)} n^{-s} \varphi_s(n). \tag{6}$$

In the same way one may obtain

$$\psi_{-s}(n) = \frac{1}{n^{2s}} \psi_s(n) \left(\prod_p p\right)^s. \tag{7}$$

For the squarefree number $n > 1$, (7) yields

$$\psi_{-s}(n) = n^{-s} \psi_s(n). \tag{8}$$

Equalities (5)–(8) are valid for arbitrary comlex numbers, too.

3 Properties for complex numbers s

From the definitions of φ_s and ψ_s (see (1) and (2)), it is obvious that for any complex number s, the relation:

$$\varphi_{2s}(n) = \varphi_s(n)\psi_s(n) \tag{9}$$

or, which is the same,

$$\psi_s(n) = \frac{\varphi_{2s}(n)}{\varphi_s(n)}, \tag{10}$$

holds.

This means that the extension of the generalized Dedekind psi-function is expressed only by the extension of the Jordan's totient function. As an analogue of the famous Gauss's equality:

$$n = \sum_{d|n} \varphi(d) \tag{11}$$

(see [3, p. 141, Theorem 7.6.]), we have the equality

$$n^s = \sum_{d|n} \varphi_s(d), \tag{12}$$

which is valid for an arbitrary complex number s.

The proof of (12) follows from the multiplicativity of the function φ_s and from the multiplicativity of the function

$$F_s(n) = \sum_{d|n} \varphi_s(d)$$

(see [3, p. 109, Theorem 6.4.]).

Using the Möbius inversion formula, from (12), we obtain

$$\varphi_s(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d^s = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^s = n^s \sum_{d|n} \frac{\mu(d)}{d^s}. \tag{13}$$

Comparing (1) and (13), we obtain

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{d|n} \frac{\mu(d)}{d^s}. \tag{14}$$

4 Main results

Below, we will deduce an important connection between functions φ_s, ψ_s and the Riemann's function ζ.

Considering the sequence of all primes:

$$2, 3, 5, 7, 11, 13, \ldots,$$

we denote it by:

$$p_1, p_2, p_3, p_4, p_5, p_6, \ldots,$$

and observe that all prime divisors of $n!$ are

$$p_1, p_2, p_3, \ldots, p_{\pi(n)},$$

where $\pi(n)$ is the number of primes p, satisfying $p \leq n$. Hence, see (1):

$$\frac{(n!)^s}{\varphi_s(n!)} = \prod_{k=1}^{\pi(n)} \frac{1}{1 - \frac{1}{p_k^s}}. \tag{15}$$

Let n tend to $+\infty$. Then $\pi(n)$ tends to $+\infty$ and in the right-hand side of (15) all primes take part. If s is an arbitrary complex number with $\mathrm{Re}(s) > 1$, then from the Euler's identity

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}} \quad (\mathbb{P} \text{ is the set of all primes.}) \tag{16}$$

and from (15), we obtain the following theorem.

Theorem 1. *The relation*

$$\lim_{n\to\infty} \frac{(n!)^s}{\varphi_s(n!)} = \zeta(s) \tag{17}$$

holds for all complex numbers s, *such that* $\mathrm{Re}(s) > 1$.

Corollary 1. *For all complex numbers* s, *such that* $\mathrm{Re}(s) > \frac{1}{2}$,

$$\lim_{n\to\infty} \frac{(n!)^{2s}}{\varphi_{2s}(n!)} = \zeta(2s) \tag{18}$$

holds.

From Corollary 1, (18) and from (10) we obtain one more corollary.

Corollary 2. *For any complex numbers* s, *with* $\mathrm{Re}(s) > 1$,

$$\lim_{n\to\infty} \frac{(n!)^s}{\psi_s(n!)} = \frac{\zeta(2s)}{\zeta(s)} \tag{19}$$

holds.

In the particular case when s is a natural number, using Euler's relation:

$$\zeta(2s) = (-1)^{s-1} 2^{2s-1} \frac{\pi^{2s}}{(2s)!} B_{2s},$$

where B_{2s} are the Bernoulli numbers (see [4] and (18)), we obtain

$$\lim_{n\to\infty} \frac{(n!)^{2s}}{\varphi_{2s}(n!)} = (-1)^{s-1} 2^{2s-1} \frac{\pi^{2s}}{(2s)!} B_{2s}.$$

As a final illustration, we remark that if we substitute $s = 2$ in (19), we obtain:

$$\lim_{n\to\infty} \frac{(n!)^2}{\psi_2(n!)} = \frac{\pi^2}{15}.$$

5 Conclusion

The proposed extensions for the Jordan's totient function and the generalized Dedekind psi-function provide many new opportunities for further research, thus making them particularly meaningful.

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