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Almost neo cobalancing numbers

Ahmet Tekcan [1] and Ecem Akgüç [2]

[1] Department of Mathematics, Faculty of Science, Bursa Uludag University, Bursa, Türkiye
e-mail: tekcan@uludag.edu.tr

[2] Department of Mathematics, Faculty of Science, Bursa Uludag University, Bursa, Türkiye
e-mail: 502411005@ogr.uludag.edu.tr

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Abstract: In this work, we determined the general terms of almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers in terms of cobalancing and Lucas-cobalancing numbers. We also deduced some results on relationship with Pell, Pell–Lucas, triangular and square triangular numbers. Further we formulate the sum of first n terms of these numbers.
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1 Introduction

A positive integer B is called a balancing number [2] if the Diophantine equation

$$1 + 2 + \cdots + (B-1) = (B+1) + (B+2) + \cdots + (B+R) \tag{1.1}$$

holds for some $R \in \mathbb{Z}^+$ which is called balancer. A positive integer b is called a cobalancing number [16] if the Diophantine equation

$$1 + 2 + \cdots + b = (b+1) + (b+2) + \cdots + (b+r) \tag{1.2}$$

holds for some $r \in \mathbb{Z}^+$ which is called cobalancer.

Let B_n denote the n-th balancing number and let b_n denote the n-th cobalancing number. Then from (1.1), B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and from (1.2), b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. Thus $C_n = \sqrt{8B_n^2+1}$ and $c_n = \sqrt{8b_n^2 + 8b_n + 1}$ are integers which are called Lucas-balancing number and Lucas-cobalancing number, respectively. It is clear from (1.1) and (1.2) that every balancing number is a cobalancer and every cobalancing number is a balancer, that is, $B_n = r_{n+1}$ and $R_n = b_n$ for $n \geq 1$, where R_n is the n-th balancer and r_n is the n-th cobalancer.

Ray proved in his PhD thesis [18] that

$$B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}, b_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, C_n = \frac{\alpha^{2n} + \beta^{2n}}{2} \quad \text{and} \quad c_n = \frac{\alpha^{2n-1} + \beta^{2n-1}}{2}$$

for $n \geq 1$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ which are the roots of the characteristic equation of both Pell P_n and Pell–Lucas Q_n numbers (see also [12, 17]).

Balancing numbers and their generalisations are studied by many mathematicians nowadays (see [4, 8–11, 13–15, 19, 21–23, 25–28, 30]). Recently Chailangka and Pakapongpun [3] defined neo balancing numbers. They said that a positive integer n is called a neo balancing number if the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r)$$

holds for some $r \in \mathbb{Z}^+$, which is called neo balancer. In [29], Tekcan and Yıldız said that a positive integer n is called a neo cobalancing number if the Diophantine equation

$$1 + 2 + \cdots + n = (n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r)$$

holds for some $r \in \mathbb{Z}^+$, which is called neo cobalancer. In [24], Tekcan defined almost neo balancing numbers. He said that a positive integer n is called an almost neo balancing number if the Diophantine equation

$$|(n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r) - [1 + 2 + \cdots + (n-1)]| = 1$$

holds for some $r \in \mathbb{Z}^+$ which is called almost neo balancer.

2 Almost neo cobalancing numbers

In this section, we first define almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers and then try to determine the general terms of them in terms of cobalancing and Lucas-cobalancing numbers.

A positive integer n is called an almost neo cobalancing number if the Diophantine equation

$$|(n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r) - (1 + 2 + \cdots + n)| = 1 \tag{2.1}$$

holds for some $r \in \mathbb{Z}^+$, which is called almost neo cobalancer. From (2.1), we have two cases:

Case 1. If $(n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r) - (1 + 2 + \cdots + n) = 1$, then n is called an almost neo cobalancing number of first type, r is called an almost neo cobalancer of first type and hence

$$r = \frac{-2n - 1 + \sqrt{8n^2 - 8n + 17}}{2}. \tag{2.2}$$

Let b_n^{neo*} denote the almost neo cobalancing number of first type. Then from (2.2), b_n^{neo*} is an almost neo cobalancing number of first type if and only if $8(b_n^{neo*})^2 - 8b_n^{neo*} + 17$ is a perfect square. Thus

$$c_n^{neo*} = \sqrt{8(b_n^{neo*})^2 - 8b_n^{neo*} + 17} \tag{2.3}$$

is an integer which is called the almost Lucas-neo cobalancing number of first type. We denote the almost neo cobalancer of first type by r_n^{neo*}.

Case 2. If $(n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r) - (1 + 2 + \cdots + n) = -1$, then n is called an almost neo cobalancing number of second type, r is called an almost neo cobalancer of second type and hence

$$r = \frac{-2n - 1 + \sqrt{8n^2 - 8n + 1}}{2}. \tag{2.4}$$

Let b_n^{neo**} denote the almost neo cobalancing number of second type. Then from (2.4), b_n^{neo**} is an almost neo cobalancing number of second type if and only if $8(b_n^{neo**})^2 - 8b_n^{neo**} + 1$ is a perfect square. Thus

$$c_n^{neo**} = \sqrt{8(b_n^{neo**})^2 - 8b_n^{neo**} + 1} \tag{2.5}$$

is an integer which is called the almost Lucas-neo cobalancing number of second type. We denote the almost neo cobalancer of second type by r_n^{neo**}.

2.1 Almost neo cobalancing numbers of first type

From (2.3), we see that b_n^{neo*} is an almost neo cobalancing number of first type if and only if $8(b_n^{neo*})^2 - 8b_n^{neo*} + 17$ is a perfect square. So we set

$$8(b_n^{neo*})^2 - 8b_n^{neo*} + 17 = y^2$$

for some $y \in \mathbb{Z}^+$. Then $2[4(b_n^{neo*})^2 - 4b_n^{neo*}] + 17 = y^2$ and hence $2(2b_n^{neo*} - 1)^2 + 15 = y^2$. Taking $x = 2b_n^{neo*} - 1$, we get the Pell equation (see [1, 7])

$$2x^2 - y^2 = -15. \tag{2.6}$$

Let $\Omega^* = \{(x, y) : 2x^2 - y^2 = -15\}$. Then we can give the following theorem.

Theorem 2.1. $\Omega^* = \{\}$.

Proof. For the Pell equation in (2.6), the indefinite form is $F = (a, b, c) = (2, 0, -1)$ of discriminant $\Delta = 8$. So $\tau = 3 + 2\sqrt{2}$. Therefore for $m = -15$, we get

$$U = \left|\frac{am\tau}{\Delta}\right|^{\frac{1}{2}} (1 + \frac{1}{\tau}) = \left|\frac{(2)(-15)(3 + 2\sqrt{2})}{8}\right|^{\frac{1}{2}} (1 + \frac{1}{3 + 2\sqrt{2}}) \simeq 5.477.$$

But in the range $0 \leq y \leq 5$, $\Delta y^2 + 4am = 8y^2 - 120$ is not a perfect square (see [5, p. 121]). So there are no integer solutions, that is, $\Omega^* = \{\}$. □

From Theorem 2.1, we get the next theorem.

Theorem 2.2. *There are no almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of first type.*

Proof. Note that $\Omega^* = \{\}$. Since $x = 2b_n^{neo*} - 1$, there is no integer

$$b_n^{neo*} = \frac{x+1}{2}.$$

So from (2.3), there is no integer

$$c_n^{neo*} = \sqrt{8(b_n^{neo*})^2 - 8b_n^{neo*} + 17}$$

and from (2.2), there is no integer

$$r_n^{neo*} = \frac{-2b_n^{neo*} - 1 + c_n^{neo*}}{2}.$$

Therefore, there are no almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of first type. □

2.2 Almost neo cobalancing numbers of second type

From (2.5), we note that b_n^{neo**} is an almost neo cobalancing number of second type if and only if $8(b_n^{neo**})^2 - 8b_n^{neo**} + 1$ is a perfect square. So we set

$$8(b_n^{neo**})^2 - 8b_n^{neo**} + 1 = y^2$$

for some $y \in \mathbb{Z}^+$. Then $2[4(b_n^{neo**})^2 - 4b_n^{neo**}] + 1 = y^2$ and hence $2(2b_n^{neo**} - 1)^2 - 1 = y^2$. Taking $x = 2b_n^{neo**} - 1$, we get the Pell equation

$$2x^2 - y^2 = 1. \tag{2.7}$$

Let $\Omega^{**} = \{(x, y) : 2x^2 - y^2 = 1\}$. Then we can give the following theorem.

Theorem 2.3. $\Omega^{**} = \{(2b_n + 1, c_n) : n \geq 1\}$.

Proof. For the Pell equation in (2.7), the indefinite form is $F = (2, 0, -1)$. The set of representatives is $\text{Rep} = \{[\pm 1 \quad 1]\}$ and $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Here we notice that $[x_n \quad y_n] = [1 \quad 1]M^{n-1}$ for $n \geq 1$. Since the n-th power of M is $M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}$, we get

$$[x_n \quad y_n] = [1 \quad 1]M^{n-1} = [2B_{n-1} + C_{n-1} \quad 4B_{n-1} + C_{n-1}].$$

Thus $\Omega^{**} = \{(2B_{n-1} + C_{n-1}, 4B_{n-1} + C_{n-1}) : n \geq 1\}$. It can be easily seen that $2B_{n-1} + C_{n-1} = 2b_n + 1$ and $4B_{n-1} + C_{n-1} = c_n$. Consequently, $\Omega^{**} = \{(2b_n + 1, c_n) : n \geq 1\}$. □

From Theorem 2.3, we deduce that:

Theorem 2.4. *The general terms of almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of second type are*

$$b_n^{neo**} = b_n + 1,\ c_n^{neo**} = c_n \ \text{ and } \ r_n^{neo**} = \frac{-2b_n + c_n - 3}{2}$$

for $n \geq 1$.

Proof. Note that $\Omega^{**} = \{(2b_n + 1, c_n) : n \geq 1\}$ by Theorem 2.3. Since $x = 2b_n^{neo**} - 1$, we get

$$b_n^{neo**} = \frac{x_n + 1}{2} = \frac{2b_n + 1 + 1}{2} = b_n + 1$$

for $n \geq 1$. Thus from (2.5), we obtain

$$\begin{aligned} c_n^{neo**} &= \sqrt{8(b_n^{neo**})^2 - 8b_n^{neo**} + 1} \\ &= \sqrt{8(b_n + 1)^2 - 8(b_n + 1) + 1} \\ &= \sqrt{8b_n^2 + 8b_n + 1} \\ &= c_n \end{aligned}$$

for $n \geq 1$, and from (2.4), we get

$$r_n^{neo**} = \frac{-2b_n^{neo**} - 1 + c_n^{neo**}}{2} = \frac{-2b_n + c_n - 3}{2}$$

for $n \geq 1$. □

Here we note that almost neo cobalancing numbers of second type must be positive form definition. But for $n = 0$, $8(0)^2 - 8(0) + 1 = 1^2$ is a perfect square. So we can accept 0 to be an almost neo cobalancing numbers of second type, that is, $b_0^{neo**} = 0$ just like 0 and 1 accepted to be balancing numbers by [2]. Thus from (2.4), $r_0^{neo**} = 0$ and from (2.5), $c_0^{neo**} = 1$.

Theorem 2.5. *Binet formulas for almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of second type are*

$$\begin{aligned} b_n^{neo**} &= \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} + \frac{1}{2} \\ c_n^{neo**} &= \frac{\alpha^{2n-1} + \beta^{2n-1}}{2} \\ r_n^{neo**} &= \frac{\alpha^{2n-2} - \beta^{2n-2}}{4\sqrt{2}} - 1 \end{aligned}$$

for $n \geq 1$. *The recurrence relations are*

$$\begin{aligned} b_{n+1}^{neo**} &= 6b_n^{neo**} - b_{n-1}^{neo**} - 2 \\ c_{n+1}^{neo**} &= 6c_n^{neo**} - c_{n-1}^{neo**} \\ r_{n+1}^{neo**} &= 6r_n^{neo**} - r_{n-1}^{neo**} + 4 \end{aligned}$$

for $n \geq 2$.

Proof. It can be easily derived from Theorem 2.4. □

In Theorem 2.4, we deduced the general terms of almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of second type in terms of cobalancing and Lucas-cobalancing numbers. Conversely, we can give the general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers in terms of almost neo cobalancing numbers and almost Lucas-neo cobalancing numbers of second type as follows.

Theorem 2.6. *The general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers are*

$$B_n = \frac{2b_n^{neo**} + c_n^{neo**} - 1}{2},\ b_n = b_n^{neo**} - 1,\ C_n = 4b_n^{neo**} + c_n^{neo**} - 2 \ \textit{and}\ \ c_n = c_n^{neo**}$$

for $n \geq 1$.

Proof. Applying Theorem 2.5, we get

$$\begin{aligned} B_n &= \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} \\ &= \frac{\alpha^{2n-1}(1+\sqrt{2}) - \beta^{2n-1}(1-\sqrt{2})}{4\sqrt{2}} \\ &= \frac{\alpha^{2n-1}(\frac{1}{2\sqrt{2}} + \frac{1}{2}) - \beta^{2n-1}(\frac{1}{2\sqrt{2}} - \frac{1}{2})}{2} \\ &= \frac{2\left(\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} + \frac{1}{2}\right) + \left(\frac{\alpha^{2n-1}+\beta^{2n-1}}{2}\right) - 1}{2} \\ &= \frac{2b_n^{neo**} + c_n^{neo**} - 1}{2}. \end{aligned}$$

The others can be proved similarly. □

3 Sums of almost neo cobalancing numbers of second type

Theorem 3.1. *The sum of first the* n *terms of almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of second type are*

$$\begin{aligned} \sum_{i=1}^{n} b_i^{neo**} &= \frac{b_{n+1}^{neo**} - b_n^{neo**} + 2n}{4} \\ \sum_{i=1}^{n} c_i^{neo**} &= \frac{c_{n+1}^{neo**} - c_n^{neo**} - 2}{4} \\ \sum_{i=1}^{n} r_i^{neo**} &= \frac{r_{n+1}^{neo**} - r_n^{neo**} - 4n - 1}{4} \end{aligned}$$

for $n \geq 1$.

Proof. Recall that $b_{n+1}^{neo**} = 6b_n^{neo**} - b_{n-1}^{neo**} - 2$ by Theorem 2.5. So

$$b_{n+1}^{neo**} - b_n^{neo**} = 5b_n^{neo**} - b_{n-1}^{neo**} - 2$$

and hence

$$\begin{aligned}
b_2^{neo**} - b_1^{neo**} &= 5b_1^{neo**} - b_0^{neo**} - 2\\
b_3^{neo**} - b_2^{neo**} &= 5b_2^{neo**} - b_1^{neo**} - 2\\
b_4^{neo**} - b_3^{neo**} &= 5b_3^{neo**} - b_2^{neo**} - 2 \qquad (3.1)\\
&\vdots\\
b_{n+1}^{neo**} - b_n^{neo**} &= 5b_n^{neo**} - b_{n-1}^{neo**} - 2.
\end{aligned}$$

If we sum both sides of (3.1), then we obtain

$$b_{n+1}^{neo**} - b_1^{neo**} = 5(b_1^{neo**} + b_2^{neo**} + \cdots + b_n^{neo**}) - (b_0^{neo**} + b_1^{neo**} + \cdots + b_{n-1}^{neo**}) - 2n.$$

Since $b_0^{neo**} = b_1^{neo**} = 1$, we get

$$b_{n+1}^{neo**} = 5(b_1^{neo**} + b_2^{neo**} + \cdots + b_n^{neo**}) - (b_1^{neo**} + \cdots + b_{n-1}^{neo**} + b_n^{neo**}) + b_n^{neo**} - 2n$$

and hence

$$b_{n+1}^{neo**} = 4(b_1^{neo**} + b_2^{neo**} + \cdots + b_n^{neo**}) + b_n^{neo**} - 2n.$$

Thus

$$b_1^{neo**} + b_2^{neo**} + \cdots + b_n^{neo**} = \frac{b_{n+1}^{neo**} - b_n^{neo**} + 2n}{4}.$$

The others can be proved similarly. □

For the sums of Pell numbers, it is proved in [20, Lemma 1] that

$$\sum_{i=1}^{4n+1} P_i = \left[\sum_{i=0}^{n} \binom{2n+1}{2i} 2^i\right]^2.$$

Similarly we can give the following result.

Theorem 3.2. *The sum of the first* $4n+1$ *nonzero terms of Pell numbers is*

$$\sum_{i=1}^{4n+1} P_i = (-8b_{n+2}^{neo**} + 3c_{n+2}^{neo**} + 4)^2.$$

Proof. Note that $\sum_{i=1}^{n} P_i = \frac{P_{n+1}+P_n-1}{2}$ and $P_n = \frac{\alpha^n-\beta^n}{2\sqrt{2}}$. So we get

$$\begin{aligned}
\sum_{i=1}^{4n+1} P_i &= \frac{P_{4n+2} + P_{4n+1} - 1}{2}\\
&= \frac{\frac{\alpha^{4n+2}-\beta^{4n+2}}{2\sqrt{2}} + \frac{\alpha^{4n+1}-\beta^{4n+1}}{2\sqrt{2}} - 1}{2}\\
&= \frac{\frac{\alpha^{4n+2}(1+\alpha^{-1})+\beta^{4n+2}(-1-\beta^{-1})}{2\sqrt{2}}}{2} - \frac{1}{2}\\
&= \frac{\alpha^{4n+2} + \beta^{4n+2} - 2}{4}\\
&= \left(\frac{\alpha^{2n+1} + \beta^{2n+1}}{2}\right)^2\\
&= \left[\alpha^{2n+3}\left(\frac{-2}{\sqrt{2}} + \frac{3}{2}\right) + \beta^{2n+3}\left(\frac{2}{\sqrt{2}} + \frac{3}{2}\right)\right]^2
\end{aligned}$$

$$= \left[-8\left(\frac{\alpha^{2n+3}-\beta^{2n+3}}{4\sqrt{2}}+\frac{1}{2}\right)+3\left(\frac{\alpha^{2n+3}+\beta^{2n+3}}{2}\right)+4\right]^2$$
$$= (-8b_{n+2}^{neo**}+3c_{n+2}^{neo**}+4)^2$$

by Theorem 2.5. □

Apart from Theorem 3.2, we can also give the following theorem which can be proved similarly.

Theorem 3.3. *For the sums of Pell, Pell–Lucas, balancing and Lucas-cobalancing numbers, we have*

$$1+\sum_{i=1}^{4n-1} P_i = (4b_{n+1}^{neo**}-c_{n+1}^{neo**}-2)^2,$$
$$\sum_{i=1}^{2n} Q_{2i-1} = (-4b_{n+1}^{neo**}+2c_{n+1}^{neo**}+2)^2,$$
$$\frac{\sum_{i=0}^{2n} Q_{2i+1}}{2} = (-8b_{n+2}^{neo**}+3c_{n+2}^{neo**}+4)^2,$$
$$\sum_{i=1}^{2n} B_{2i-1} = (\frac{-2b_{2n+1}^{neo**}+c_{2n+1}^{neo**}+1}{2})^2,$$
$$1+\sum_{i=1}^{4n+2} c_i = (4b_{2n+2}^{neo**}-c_{2n+2}^{neo**}-2)^2.$$

Panda and Ray proved in [17, Theorem 3.4] that

$$\sum_{i=1}^{2n-1} P_i = B_n + b_n. \tag{3.2}$$

Later Gözeri, Özkoç and Tekcan proved in [6, Theorem 2.5] that

$$\sum_{i=0}^{2n-1} Q_i = C_n + c_n.$$

Since $R_n = b_n$, (3.2) becomes

$$\sum_{i=1}^{2n-1} P_i = B_n + R_n. \tag{3.3}$$

As in (3.3), we can give the following result.

Theorem 3.4. *The sum of the first* $2n-2$ *Pell numbers is*

$$\sum_{i=1}^{2n-2} P_i = b_n^{neo**} + r_n^{neo**}.$$

Proof. Since $\sum_{i=1}^{n} P_i = \frac{P_{n+1}+P_n-1}{2}$ and $P_n = \frac{\alpha^n-\beta^n}{2\sqrt{2}}$, we observe that

$$\begin{aligned}\sum_{i=1}^{2n-2} P_i &= \frac{P_{2n-1}+P_{2n-2}-1}{2}\\ &= \frac{\frac{\alpha^{2n-1}-\beta^{2n-1}}{2\sqrt{2}}+\frac{\alpha^{2n-2}-\beta^{2n-2}}{2\sqrt{2}}-1}{2}\\ &= \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}+\frac{\alpha^{2n-2}-\beta^{2n-2}}{4\sqrt{2}}-\frac{1}{2}\\ &= \left(\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}+\frac{1}{2}\right)+\left(\frac{\alpha^{2n-2}-\beta^{2n-2}}{4\sqrt{2}}-1\right)\\ &= b_n^{neo**}+r_n^{neo**}\end{aligned}$$

by Theorem 2.5. □

4 Relationship with Pell and Pell–Lucas numbers

We can give the general terms of almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of second type in terms of Pell numbers as follows.

Theorem 4.1. *The general terms of almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of second type are*

$$b_n^{neo**} = \frac{P_{2n-1}+1}{2}, \quad c_n^{neo**} = P_{2n-1}+P_{2n-2} \quad \textit{and} \quad r_n^{neo**} = \frac{P_{2n-2}-2}{2}$$

for $n \geq 1$.

Proof. Recall that $b_n^{neo**} = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} + \frac{1}{2}$ by Theorem 2.5. So we get

$$b_n^{neo**} = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}+\frac{1}{2} = \frac{\frac{\alpha^{2n-1}-\beta^{2n-1}}{2\sqrt{2}}+1}{2} = \frac{P_{2n-1}+1}{2}.$$

The others are similar. □

Conversely, we can give the general terms of the even and odd ordered Pell numbers in terms of almost neo cobalancing numbers and almost Lucas-neo cobalancing numbers of second type as follows.

Theorem 4.2. *The general terms of the even and odd ordered Pell numbers are*

$$P_{2n} = -2b_{n+1}^{neo**}+c_{n+1}^{neo**}+1 \quad \textit{and} \quad P_{2n-1} = 6b_{n+1}^{neo**}-2c_{n+1}^{neo**}-3$$

for $n \geq 1$.

Proof. Since $P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$, we deduce that

$$\begin{aligned}
P_{2n} &= \frac{\alpha^{2n} - \beta^{2n}}{2\sqrt{2}} \\
&= \frac{\alpha^{2n+1}(-1+\sqrt{2}) - \beta^{2n+1}(-1-\sqrt{2})}{2\sqrt{2}} \\
&= \alpha^{2n+1}(\frac{-2}{4\sqrt{2}} + \frac{1}{2}) - \beta^{2n+1}(\frac{-2}{4\sqrt{2}} - \frac{1}{2}) \\
&= -2\left(\frac{\alpha^{2n+1} - \beta^{2n+1}}{4\sqrt{2}} + \frac{1}{2}\right) + \left(\frac{\alpha^{2n+1} + \beta^{2n+1}}{2}\right) + 1 \\
&= -2b_{n+1}^{neo**} + c_{n+1}^{neo**} + 1
\end{aligned}$$

by Theorem 2.5. Similarly we can prove that $P_{2n-1} = 6b_{n+1}^{neo**} - 2c_{n+1}^{neo**} - 3$. □

We can also give the general terms of almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of second type in terms of Pell–Lucas numbers as follows.

Theorem 4.3. *The general terms of almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of second type are*

$$b_n^{neo**} = \frac{Q_{2n} - Q_{2n-1} + 4}{8}, \quad c_n^{neo**} = \frac{Q_{2n-1}}{2} \quad \text{and} \quad r_n^{neo**} = \frac{-Q_{2n} + 3Q_{2n-1} - 8}{8}$$

for $n \geq 1$.

Proof. It can be proved similarly to Theorem 4.1. □

Conversely, we can give the general terms of the even and odd ordered Pell–Lucas numbers in terms of almost neo cobalancing numbers and almost Lucas-neo cobalancing numbers of second type as follows.

Theorem 4.4. *The general terms of the even and odd ordered Pell–Lucas numbers are*

$$Q_{2n} = 8b_{n+1}^{neo**} - 2c_{n+1}^{neo**} - 4 \quad \text{and} \quad Q_{2n-1} = -16b_{n+1}^{neo**} + 6c_{n+1}^{neo**} + 8$$

for $n \geq 1$.

Proof. It can be proved similarly to Theorem 4.2 was proved. □

5 Relationship with triangular and square triangular numbers

Triangular numbers are the numbers of the form $\frac{n(n+1)}{2}$ and denoted by T_n. There is a correspondence between balancing (and also cobalancing) numbers and triangular numbers. Indeed from (1.1), we note that

$$\frac{(n+r)(n+r+1)}{2} = n^2.$$

So

$$T_{B_n+R_n} = B_n^2.$$

Similarly from (1.2), we note that

$$\frac{(n+r)(n+r+1)}{2} = n^2 + n.$$

So

$$T_{b_n+r_n} = b_n^2 + b_n. \tag{5.1}$$

As in (5.1), we can give the following theorem.

Theorem 5.1. *For b_n^{neo**} and r_n^{neo**}, we have*

$$T_{b_n^{neo**}+r_n^{neo**}} = (b_n^{neo**})^2 - b_n^{neo**}.$$

Proof. Let n be an almost neo cobalancing number of second type. Then from (2.1), n satisfies the equation

$$2n - 1 + nr + \frac{r(r+1)}{2} - \frac{n(n+1)}{2} = -1.$$

If we rearrange the last equation, then we get

$$\frac{(n+r)(n+r+1)}{2} = n^2 - n.$$

Thus,

$$T_{b_n^{neo**}+r_n^{neo**}} = (b_n^{neo**})^2 - b_n^{neo**}$$

is obvious. □

Square triangular numbers are the numbers that are both squares and triangular numbers and are denoted by S_n, and can be written

$$S_n = s_n^2 = \frac{t_n(t_n+1)}{2}$$

for some s_n and t_n which are the sides of the corresponding square and triangle. Binet formulas are

$$S_n = \frac{\alpha^{4n} + \beta^{4n} - 2}{32}, \quad s_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} \quad \text{and} \quad t_n = \frac{\alpha^{2n} + \beta^{2n} - 2}{4} \tag{5.2}$$

for $n \geq 1$. We can deduce the general terms of almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of second type in terms of s_n and t_n as follows.

Theorem 5.2. *The general terms of almost neo cobalancing numbers, almost Lucas-neo cobalancing numbers and almost neo cobalancers of second type are*

$$\begin{aligned} b_n^{neo**} &= s_{n-1} + t_{n-1} + 1 \\ c_n^{neo**} &= 4s_{n-1} + 2t_{n-1} + 1 \\ r_n^{neo**} &= s_{n-1} - 1 \end{aligned}$$

for $n \geq 1$.

Proof. Recall that $b_n^{neo**} = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} + \frac{1}{2}$ by Theorem 2.5. Thus we get

$$\begin{aligned}
b_n^{neo**} &= \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} + \frac{1}{2} \\
&= \alpha^{2n-1}\Big(\frac{\alpha^{-1}}{4\sqrt{2}} + \frac{\alpha^{-1}}{4}\Big) - \beta^{2n-1}\Big(\frac{\beta^{-1}}{4\sqrt{2}} - \frac{\beta^{-1}}{4}\Big) - \frac{1}{2} + 1 \\
&= \frac{\alpha^{2n-2}-\beta^{2n-2}}{4\sqrt{2}} + \frac{\alpha^{2n-2}+\beta^{2n-2}-2}{4} + 1 \\
&= s_{n-1} + t_{n-1} + 1
\end{aligned}$$

by (5.2). The others are proven in a similar way. □

Conversely, we can give the following theorem.

Theorem 5.3. *General terms of* S_n, s_n *and* t_n *are*

$$\begin{aligned}
S_n &= \frac{4(b_{n+1}^{neo**})^2 + (c_{n+1}^{neo**})^2 - 4b_{n+1}^{neo**}c_{n+1}^{neo**} - 4b_{n+1}^{neo**} + 2c_{n+1}^{neo**} + 1}{4} \\
s_n &= \frac{-2b_{n+1}^{neo**} + c_{n+1}^{neo**} + 1}{2} \\
t_n &= \frac{4b_{n+1}^{neo**} - c_{n+1}^{neo**} - 3}{2}
\end{aligned}$$

for $n \geq 1$.

Proof. Since $S_n = \frac{\alpha^{4n}+\beta^{4n}-2}{32}$ by (5.2), we deduce that

$$\begin{aligned}
S_n &= \frac{\alpha^{4n}+\beta^{4n}-2}{32} \\
&= \frac{\alpha^{4n}\big(\frac{3\alpha^2-2\sqrt{2}\alpha^2}{8}\big) + \beta^{4n}\big(\frac{3\beta^2+2\sqrt{2}\beta^2}{8}\big) - \frac{1}{4}}{4} \\
&= \frac{\left\{\begin{array}{c} 4\big(\frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2}\big)^2 + 4\big(\frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2}\big) + \big(\frac{\alpha^{2n+1}+\beta^{2n+1}}{2}\big)^2 \\ -4\big(\frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2}\big)\big(\frac{\alpha^{2n+1}+\beta^{2n+1}}{2}\big) - 2\big(\frac{\alpha^{2n+1}+\beta^{2n+1}}{2}\big) + 1 \end{array}\right\}}{4} \\
&= \frac{\left\{\begin{array}{c} 4\big(\frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}} + \frac{1}{2}\big)^2 + \big(\frac{\alpha^{2n+1}+\beta^{2n+1}}{2}\big)^2 - 4\big(\frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}} + \frac{1}{2}\big) \\ \times\big(\frac{\alpha^{2n+1}+\beta^{2n+1}}{2}\big) - 4\big(\frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}} + \frac{1}{2}\big) + 2\big(\frac{\alpha^{2n+1}+\beta^{2n+1}}{2}\big) + 1 \end{array}\right\}}{4} \\
&= \frac{4(b_{n+1}^{neo**})^2 + (c_{n+1}^{neo**})^2 - 4b_{n+1}^{neo**}c_{n+1}^{neo**} - 4b_{n+1}^{neo**} + 2c_{n+1}^{neo**} + 1}{4}
\end{aligned}$$

by Theorem 2.5. The others can be proved similarly. □

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