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Josephus Nim

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Abstract: In this study, we propose a variant of Nim that uses two piles. In the first pile, we have stones with a weight of a, and in the second pile, we have stones with a weight of $-2a$, where a is a natural number. Two players take turns to remove stones from one of the piles. The total weight of the stones to be removed should be equal to or less than half of the total weight of the stones in the pile. Therefore, if there are x stones with weight a and y stones with weight $-2a$, then the total weight of the stones to be removed is less than or equal to $(ax - 2ay)/2$. The player who removes the last stone is the winner of the game. The authors proved that when (n, m) is the winning position of the previous player, $2m + 1$ is the last remaining number in the Josephus problem, where there are $n+1$ numbers, and every second number is to be removed. For any natural number s, there are similar relationships between the position at which the Grundy number is s and the $(n - s)$-th removed number in the Josephus problem with $n + 1$ numbers.
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1 Introduction

Let $\mathbb{Z}_{\geq 0}$ and $\mathbb{N}$ represent the sets of non-negative integers and natural numbers, respectively.

The classic game of Nim is played using stone piles. Players can remove any number of stones from any pile during their turn, and the player who removes the last stone is considered the winner [2].

Several variants of the classical Nim game exist. For the Maximum Nim, we place an upper bound $f(n)$ on the number of stones that can be removed in terms of the number n of stones in the pile (see [4]). For other articles on Maximum Nim, see [5] and [6].

In this study, we investigated a variant of the Maximum Nim, with stones of weights a and $-2a$. Simple relationships exist between the Grundy numbers in this game and the Josephus problem. This is remarkable because the games for Nim and Josephus' problems are entirely different. This game was proposed by Takahashi, one of the authors of this article.

2 A game of nim with two kinds of stones

Definition 2.1. *Suppose there are two piles of stones and two players take turns removing the stones from one pile. In the first pile, there are stones with a weight of a, and in the second pile, there are stones with a weight of $-2a$. When there are x stones with a weight of a and y stones with a weight of $-2a$, the total weight of the stones is $ax - 2ay \in \mathbb{Z}_{\geq 0}$. Then, a player is allowed to remove stones from one of the piles whose total weight is less than or equal to $\lfloor \frac{ax-2ay}{2} \rfloor$, where $\lfloor\ \rfloor$ denotes the floor function. The player who removes the last stone is the winner of the game.*

Definition 2.2. *We denote a position of the game of Definition 2.1 by (x, y), where x and y denote the numbers of stones with weights a and $2a$, respectively.*

Definition 2.3. (i) *For any position (x, y) in this game, the set of positions can be reached by precisely one move, denoted as move(x, y).*

(ii) *The minimum excluded value* (mex) *of a set S of non-negative integers is the smallest non-negative integer that is not in S.*

(iii) *Let (x, y) be a position in the game. The associated Grundy number is denoted by $\mathcal{G}(x, y)$ and is recursively defined as follows: $\mathcal{G}(x, y) = \text{mex}(\{\mathcal{G}(u, v) : (u, v) \in \text{move}(x, y)\})$.*

Definition 2.4. (a) *A position is referred to as a $\mathcal{P}$-position if it is a winning position for the previous player (the player who has just moved) as long as the player plays correctly at every stage.*

(b) *A position is referred to as an $\mathcal{N}$-position as long as the player plays correctly at every stage.*

The next result demonstrates the usefulness of the Sprague–Grundy theory in impartial games.

Theorem 2.1. *For any position (x, y), $\mathcal{G}(x, y) = 0$ if and only if (x, y) is the $\mathcal{P}$-position.*

See [1] for the proof of this theorem.

Lemma 2.1. *For the game in Definition 2.1,*

$$\begin{aligned}\text{move}(x, y) = \{(x - t, y) : t \in \mathbb{N} \textit{ and } at \leq \lfloor \frac{ax - 2ay}{2} \rfloor\} \\ \cup \{(x, y - u) : u \in \mathbb{N} \textit{ and } -2au \leq \lfloor \frac{ax - 2ay}{2} \rfloor\}.\end{aligned} \tag{1}$$

Proof. When we have position (x, y), we can remove the stones whose total weight is $\lfloor \frac{ax-2ay}{2} \rfloor$. Then, we can remove t stones from the first pile when $at \leq \lfloor \frac{ax-2ay}{2} \rfloor$, and remove u stones from the second pile when $-2au \leq \lfloor \frac{ax-2ay}{2} \rfloor$. Therefore, we have (1). □

Lemma 2.2. *Let $a \in \mathbb{N}$. Then, we have (i) and (ii).*

(i) *For any integers t, w,*

$$t \leq \lfloor \frac{w}{2} \rfloor \textit{ if and only if } at \leq \lfloor \frac{aw}{2} \rfloor.$$

(ii) *For any integers t, w,*

$$-2u \leq \lfloor \frac{w}{2} \rfloor \textit{ if and only if } -2au \leq \lfloor \frac{aw}{2} \rfloor.$$

Proof. When a is a natural number,

$$at \leq \lfloor \frac{aw}{2} \rfloor \text{ if and only if } at \leq \frac{aw}{2}.$$

Because

$$t \leq \lfloor \frac{w}{2} \rfloor \text{ if and only if } t \leq \frac{w}{2},$$

we obtain (i). We prove (ii) using a method similar to that used in (i). □

Lemma 2.3. *For the game in Definition 2.1,*

$$\begin{aligned}\text{move}(x, y) = \{(x - t, y) : t \in \mathbb{N} \textit{ and } t \leq \lfloor \frac{x - 2y}{2} \rfloor\} \\ \cup \{(x, y - u) : u \in \mathbb{N} \textit{ and } -2u \leq \lfloor \frac{x - 2y}{2} \rfloor\}.\end{aligned}$$

Proof. This is directly from Lemmas 2.1 and 2.2. □

From Lemma 2.3, move of the game in Definition 2.1 for any natural number a is the same as that in Definition 2.1 for a natural number $a = 1$.

The aim of the present section is to determine the formulas for the set of positions whose Grundy number is s for the game of Definition 2.1, which is sufficient to study this game under condition $a = 1$.

Remark 2.1. *When a is not a natural number, the result of Lemma 2.2 is invalid. For instance,*

$$t \leq \lfloor \frac{2}{2} \rfloor$$

is not the same as

$$1.5t \leq \lfloor \frac{1.5 \times 2}{2} \rfloor.$$

Definition 2.5. *For $s, n \in \mathbb{Z}_{\geq 0}$, let*

$$\mathcal{G}_{s,n} = \{((2s+1) \times 2^n - 1 + m, m) : m \in \mathbb{Z}_{\geq 0} \text{ and } 0 \leq m \leq (2s+1) \times 2^n - 1\},$$
$$\mathcal{G}_{0,a} = \mathcal{G}_{0,b} = \varnothing.$$

For $s \in \mathbb{N}$, let

$$\mathcal{G}_{s,a} = \{(2k, j) : k, j \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq s-1 \text{ and } 2^{s-k-1} + k \leq j \leq 2^{s-k} + k - 1\},$$
$$\mathcal{G}_{s,b} = \{(2k+1, j) : k, j \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq s-1 \text{ and } 2^{s-k-1} + k \leq j \leq 2^{s-k} + k - 1\},$$

and

$$\mathcal{G}_s = (\bigcup_{n=0}^{\infty} \mathcal{G}_{s,n}) \cup \mathcal{G}_{s,a} \cup \mathcal{G}_{s,b}.$$

Lemma 2.4. (i) *For $s, n, h \in \mathbb{Z}_{\geq 0}$ such that $h \leq s-2$, we have the following* (2) *and* (3).

$$\{(2h, j) : h+1 \leq j \leq 2^{s-h-1} + h - 1\} \subset \cup_{i=h+1}^{s-1} \mathcal{G}_{i,a} \tag{2}$$

and

$$\{(2h+1, j) : h+1 \leq j \leq 2^{s-h-1} + h - 1\} \subset \cup_{i=h+1}^{s-1} \mathcal{G}_{i,b}. \tag{3}$$

(ii) *We have that (4) and (5).*

$$\{(2h, x) : 0 \leq x \leq h\} \subset \cup_{i=0}^{h} \cup_{n=0}^{\infty} \mathcal{G}_{i,n} \tag{4}$$

and

$$\{(2h+1, x) : 0 \leq x \leq h\} \subset \cup_{i=0}^{h} \cup_{n=0}^{\infty} \mathcal{G}_{i,n}. \tag{5}$$

(iii) *For any $(x, y) \in \mathcal{G}_{s,a} \cup \mathcal{G}_{s,b}$, $x \leq 2s-1$ and $y > \frac{x}{2}$.*

Proof. (i) First, we prove (2). Let $s, n, h \in \mathbb{Z}_{\geq 0}$ such that $h \leq s-2$. Then,

$$\begin{aligned}
&\{(2h, j) : h+1 \leq j \leq 2^{s-h-1} + h - 1\} \\
&= \cup_{i=h+1}^{s-1} \{(2h, j) : 2^{i-h-1} + h \leq j \leq 2^{i-h} + h - 1\} \\
&= \cup_{i=h+1}^{s-1} \{(2h, j) : h \leq i-1, 2^{i-h-1} + h \leq j \leq 2^{i-h} + h - 1\} \\
&\subset \cup_{i=h+1}^{s-1} \mathcal{G}_{i,a}.
\end{aligned}$$

Similarly, we obtain (3).

(ii) Let $x, h \in \mathbb{Z}_{\geq 0}$ such that $0 \leq x \leq h$. For $2h+1-x \in \mathbb{N}$, there exist $n, t \in \mathbb{Z}_{\geq 0}$ such that $t \leq h$ and $2h+1-x = (2t+1)2^n$. Since $x \leq h$,

$$(2t+1)2^n = 2h+1-x \geq 2x+1-x = x+1,$$

and hence, $(2h, x) = ((2t+1)2^n - 1 + x, x) \in \mathcal{G}_{t,n}$. Therefore, we obtain (4). Similarly, we prove (5).

(iii) For $(x, y) = (2k, j)$ or $(2k+1, j)$ such that $2^{s-k-1} + k \leq j$, we have $y = j \geq k+1 > \frac{x}{2}$. Since $k \leq s-1$, $x \leq 2s-1$. □

Lemma 2.5. *Suppose that we start with a position, $(x, y) \in \mathcal{G}_s$. Then,*

$$\text{move}(x, y) \cap \mathcal{G}_s = \varnothing. \tag{6}$$

Proof. Suppose we start with a position $(x, y) \in \mathcal{G}_s$.

(i) Suppose $(x, y) = ((2s+1) \times 2^n - 1 + m, m) \in \mathcal{G}_{s,n}$, where

$$m \leq (2s+1) \times 2^n - 1. \tag{7}$$

(i-1) We assume that $n = 0$. From (7), we have

$$m \leq 2s \tag{8}$$

and $(x, y) = (2s+m, m)$, and from (7), the total weight of the stones is $2s+m-2m$ $= 2s - m \geq 0$. From Definition 2.1, the total weight of stones that can be removed is

$$\lfloor \frac{2s-m}{2} \rfloor = s - \lceil \frac{m}{2} \rceil. \tag{9}$$

We prove (6) by contradiction.

(i-1-1) We assume that we remove the stones from the first pile and move to the position

$$(2s+m-i, m) = ((2s+1) \times 2^k - 1 + m, m) \in \mathcal{G}_{s,k},$$

where $k \in \mathbb{Z}_{\geq 0}$ and

$$0 < i \leq s - \lceil \frac{m}{2} \rceil. \tag{10}$$

Then, $i = 2s - (2s+1) \times 2^k + 1 \leq 0$, which contradicts (10).

(i-1-2) We assume that we remove the stones from the second pile and move to the position

$$(2s+m, m-i) = ((2s+1) \times 2^k - 1 + m - i, m - i) \in \mathcal{G}_{s,k}, \tag{11}$$

where $k \in \mathbb{Z}_{\geq 0}$ and

$$i \leq m. \tag{12}$$

From (11) and $k \geq 1$, we have that $i \geq 2(2s+1) - 2s - 1 = 2s+1$, contradicting (12) and (8).

(i-1-3) We assume that the stones are removed from the first pile and we moved to

$$(u, m) = (2s+m-i, m) \in \mathcal{G}_{s,a} \cup \mathcal{G}_{s,b}.$$

From (8) and (9), we obtain:

$$u = 2s+m-i \geq 2s+m-(s - \lceil \frac{m}{2} \rceil) = s+m+\lceil \frac{m}{2} \rceil \geq 2m.$$

This result contradicts (iii) in Lemma 2.4.

(i-1-4) We assumed that we removed the stones from the second pile and moved to the position

$$(u, v) = (2s+m, m-i) \in \mathcal{G}_{s,a} \cup \mathcal{G}_{s,a}.$$

Then, from (8), we have

$$u = 2s+m \geq 2m > 2(m-i) = 2v.$$

This result contradicts (iii) in Lemma 2.4.

(i-2) Suppose that $n \geq 1$. Then, the total weight of the stones is

$$(2s+1) \times 2^n - 1 + m - 2m = (2s+1) \times 2^n - 1 - m.$$

Hence, by Definition 2.1, the total weight of the stones that can be removed is

$$\lfloor \frac{(2s+1) \times 2^n - 1 - m}{2} \rfloor \leq (2s+1) \times 2^{n-1} - 1. \quad (13)$$

(i-2-1) Suppose that we remove stones from the first pile, that is, reduce the first coordinate x. Then, by (13), we move to the position

$$(u, m) = ((2s+1) \times 2^n - 1 + m - i, m) \quad (14)$$

for $i \in \mathbb{N}$ such that

$$i \leq (2s+1) \times 2^{n-1} - 1. \quad (15)$$

We prove (6) through contradiction.

(i-2-1-1) We assume that

$$((2s+1) \times 2^n - 1 + m - i, m) \in \mathcal{G}_{s,n'}$$

for $n' \in \mathbb{Z}_{\geq 0}$ such that $n' < n$ and

$$((2s+1) \times 2^n - 1 + m - i, m) = ((2s+1) \times 2^{n'} - 1 + m, m).$$

Then,

$$i = (2s+1) \times (2^n - 2^{n'}) \geq (2s+1) \times 2^{n-1},$$

which contradicts (15). Therefore, we have (6).

(i-2-1-2) We assume that

$$((2s+1) \times 2^n - 1 + m - i, m) \in \mathcal{G}_{s,a} \cup \mathcal{G}_{s,b}, \quad (16)$$

where

$$i \leq (2s+1) \times 2^{n-1} - 1. \quad (17)$$

Then, from (14), and (17),we obtain:

$$u = (2s+1) \times 2^n - 1 + m - i \geq (2s+1) \times 2^{n-1} + m.$$

Because $n \geq 1$, $(2s+1) \times 2^n - 1 + m - i \geq 2s$. Therefore, this contradicts (iii) of Lemma 2.4.

(i-2-2) Suppose we remove stones from the second pile, that is, reduce the second coordinate y. Because each stone in the second pile weighs -2, from (13), we can remove i stones using

$$i \leq m \leq (2s+1) \times 2^n - 1. \quad (18)$$

(i-2-2-1) Suppose that we move to the position

$$\begin{aligned}&((2s+1)\times 2^n - 1 + m, m - i)\\&= ((2s+1)\times 2^{n'} - 1 + (m-i), m - i) \in \mathcal{G}_{s,n'}\end{aligned} \tag{19}$$

such that $n' \geq n+1$ and

$$i = (2s+1)(2^{n'} - 2^n) \geq (2s+1)2^n,$$

and this contradicts (18). Therefore, we have (6).

(i-2-2-2) Suppose that we move to the position

$$(u,v) = ((2s+1)\times 2^n - 1 + m, m - i) \in \mathcal{G}_{s,a} \cup \mathcal{G}_{s,b}.$$

Since $m \geq 1$,

$$u \geq (2s+1) - 1 + m \geq 2s+1$$

and this contradicts (18).

(ii) Suppose that $(x,y) = (2k,j)$ or $(2k+1,j) \in \mathcal{G}_{s,a} \cup \mathcal{G}_{s,a}$ such that $0 \leq k \leq s-1$ and

$$2^{s-k-1} + k \leq j \leq 2^{s-k} + k - 1. \tag{20}$$

(ii-1) We prove that

$$\text{move}(x,y) \cap \mathcal{G}_{s,n} = \varnothing \tag{21}$$

for any $n \in \mathbb{Z}_{\geq 0}$. From Lemma 2.4, $x \leq 2s-1$ and $u \geq 2s$ for any $(u,v) \in \mathcal{G}_{s,n}$, we have (21).

(ii-2) The total weight of the stones is $2k-2j$ or $2k+1-2j$, and hence the total weight of the stones that can be removed is

$$\lfloor \frac{2k-2j}{2} \rfloor = \lfloor \frac{2k+1-2j}{2} \rfloor = k - j. \tag{22}$$

By (20),

$$2k - 2j \leq 0,$$

and hence, we cannot remove stones from the first pile.

Next, we remove the stones from the second pile and move to the position $(2k,i)$ or $(2k+1,i)$. We prove that $(2k,i),(2k+1,i) \notin \mathcal{G}_{s,a} \cup \mathcal{G}_{s,b}$ by contradiction. We suppose that

$$2^{s-k-1} + k \leq i < j \leq 2^{s-k} + k - 1. \tag{23}$$

Then, we remove $j-i$ stones from the second pile and the total weight of stones that were removed is

$$-2(j-i). \tag{24}$$

From (22) and (24), we have that

$$k - j \geq -2j + 2i,$$

and hence by (23),

$$k + j \geq 2i \geq 2^{s-k} + 2k.$$

Then, we have

$$j \geq 2^{s-k} + k,$$

which contradicts (23). Therefore, we have (6). □

Lemma 2.6. *We suppose that we start with the position $(x, y) \notin \mathcal{G}_s$, such that $x \geq 2s$. Then, there exists $n \in \mathbb{Z}_{\geq 0}$, for which*

$$\text{move}(x, y) \cap \mathcal{G}_{s,n} \neq \varnothing.$$

Proof. Suppose that we start with a position $(x, y) \notin \mathcal{G}_s$ such that $x \geq 2s$. Then, there exists $n \in \mathbb{Z}_{\geq 0}$ such that

$$(2s + 1) \times 2^n \leq x + 1 < (2s + 1) \times 2^{n+1}. \tag{25}$$

Let

$$m = x - ((2s + 1) \times 2^n - 1). \tag{26}$$

Then, by (25), it follows that

$$\begin{aligned} 0 &\leq m \\ &= x - ((2s + 1) \times 2^{n-1} - 1) \\ &\leq (2s + 1) \times 2^{n+1} - 2 - ((2s + 1) \times 2^n - 1) \\ &= (2s + 1) \times 2^n - 1. \end{aligned} \tag{27}$$

Therefore, from (26) and (27), we have that

$$x \geq 2m. \tag{28}$$

The total weight of the stones is $x - 2y$, and we can remove stones whose total weight is

$$\frac{x - 2y}{2}. \tag{29}$$

If $y = m$, $(x, y) = ((2s + 1) \times 2^n - 1 + m, m) \in \mathcal{G}_{s,n}$. Therefore, two cases exist: (i) and (ii).

(i) Suppose that

$$y > m. \tag{30}$$

From (28) and (30), we have that

$$x + 2y > 4m,$$

and hence

$$\frac{x - 2y}{2} > -2(y - m)$$

Therefore, from (29), we can remove $y - m$ stones from the second pile and move to the position $((2s + 1) \times 2^n - 1 + m, m) \in \mathcal{G}_{s,n}$.

(ii) Suppose that $y < m$. From (28), we have that

$$\begin{aligned}\frac{x-2y}{2} - (m-y) &= \frac{x-2y-2(m-y)}{2} \\ &= \frac{x-2m}{2} \\ &\geq 0. \end{aligned} \tag{31}$$

From (31), we obtain that

$$\frac{x-2y}{2} \geq m-y,$$

and hence we can remove $m-y$ stones from the first pile and move to the position $(x,m) = ((2s+1)\times 2^n - 1 + y, y) \in \mathcal{G}_{s,n}$. □

Lemma 2.7. *If*

$$(x,y) \notin \cup_{i=0}^{s} \mathcal{G}_i$$

and $x \leq 2s-1$*, then*

$$\text{move}(x,y) \cap \mathcal{G}_s \neq \varnothing.$$

Proof. Let

$$(x,y) \notin \cup_{i=0}^{s} \mathcal{G}_i \tag{32}$$

and $x \leq 2s-1$. Then, there exists k such that: $0 \leq k \leq s-1$ and $x = 2k$ or $x = 2k+1$. We have two cases.

(i) First, we suppose that $x = 2s-1$ or $2s-2$. From Definition 2.5

$$(2s-1, s), (2s-2, s) \in \mathcal{G}_{s,a} \cup \mathcal{G}_{s,b}. \tag{33}$$

From (ii) of Lemma 2.4,

$$\{(2s-1, j) : 0 \leq j \leq s-1\} \subset \cup_{i=0}^{s-1} \cup_{n=0}^{\infty} \mathcal{G}_{i,n} \tag{34}$$

and

$$\{(2s-2, j) : 0 \leq j \leq s-1\} \subset \cup_{i=0}^{s-1} \cup_{n=0}^{\infty} \mathcal{G}_{i,n}. \tag{35}$$

From (32), (33), (34), and (35),

$$y \geq s+1. \tag{36}$$

At positions $(2s-1, y)$ or $(2s-2, y)$, the total weight of the stones is $2s-1-2y$ or $2s-2-2y$, and we can remove the total weight of $s-y-1$ stones. From (36), we obtain:

$$s-y-1 \geq -2y+2s = -2(y-s). \tag{37}$$

Therefore, we remove $y-s$ stones from the second pile to the positions $(2s-1, s)$ or $(2s-2, s) \in \mathcal{G}_{s,a} \cup \mathcal{G}_{s,b}$.

(ii) Next, we suppose that $x = 2k$ or $x = 2k + 1$ with $k \leq s - 2$. From (i) and (ii) of Lemma 2.4,

$$\{(2k, j) : 0 \leq j \leq 2^{s-k-1} + k - 1\} \subset \cup_{i=0}^{s-1} \mathcal{G}_i \tag{38}$$

and

$$\{(2k + 1, j) : 0 \leq j \leq 2^{s-k-1} + k - 1\} \subset \cup_{i=0}^{s-1} \mathcal{G}_i. \tag{39}$$

From Definition 2.5,

$$\{(2k, j) : 2^{s-k-1} + k \leq j \leq 2^{s-k} + k - 1\} \subset \mathcal{G}_s \tag{40}$$

and

$$\{(2k + 1, j) : 2^{s-k-1} + k \leq j \leq 2^{s-k} + k - 1\} \subset \mathcal{G}_s. \tag{41}$$

From (32), (38), (39), (40), and (41), we obtain:

$$y \geq 2^{s-k} + k. \tag{42}$$

At position (x, y), the total weight of the stones is $x - 2y = 2k - 2y$ or $x - 2y = 2k + 1 - 2y$, and and that of the stones that can be removed is

$$\lfloor \frac{2k + 1 - 2y}{2} \rfloor = \lfloor \frac{2k - 2y}{2} \rfloor = k - y. \tag{43}$$

We prove that we can move to position $(2k, 2^{s-k-1} + k)$ or $(2k + 1, 2^{s-k-1} + k) \in \mathcal{G}_{s,a} \cup \mathcal{G}_{s,b}$. From (42), we obtain that:

$$\begin{aligned} k - y - (-2)(y - (2^{s-k-1} + k)) &= k - y - 2^{s-k} - 2k + 2y \\ &= y - k - 2^{s-k} \geq 0. \end{aligned}$$

Hence, we can move to $(2k, 2^{s-k-1} + k)$ or $(2k + 1, 2^{s-k-1} + k) \in \mathcal{G}_{s,a} \cup \mathcal{G}_{s,b}$ by removing $y - (2^{s-k-1} + k)$ stones from the second pile. □

Theorem 2.2. *$\mathcal{G}_s$ is the set of positions with Grundy number s.*

Proof. This is directly derived from Definition 2.3, and Lemmas 2.5, 2.6, and 2.7. □

3 Non-negative total weight of stones

Let $X = \{(x, y) : x - 2y \geq 0\}$. We define a game on X. In this game, the total weight of stones should not be negative, and the set of positions a player can go from the position $(x, y) \in X$ is given by move_X defined in the following definition.

Definition 3.1. *For $(x, y) \in X$*

$$\begin{aligned} \text{move}_X(x, y) = \{&(x - t, y) : t \in \mathbb{N}, (x - t) - 2y \geq 0 \text{ and } t \leq \lfloor \frac{x - 2y}{2} \rfloor\} \\ &\cup \{(x, y - u) : u \in \mathbb{N} \text{ and } -2u \leq \lfloor \frac{x - 2y}{2} \rfloor\}. \end{aligned} \tag{44}$$

Lemma 3.1. *For* $(x, y) \in X$, $\text{move}_X(x, y) = \text{move}(x, y) \cap X$.

Proof. Let $(x, y) \in X$. Then, for any $u \in \mathbb{N}$,

$$x - 2(y - u) \geq x - 2y \geq 0. \tag{45}$$

By (45),

$$\begin{aligned}
\text{move}(x, y) \cap X = (&\{(x - t, y) : t \in \mathbb{N} \text{ and } t \leq \lfloor \frac{x - 2y}{2} \rfloor\} \cap X) \\
&\cup (\{(x, y - u) : u \in \mathbb{N} \text{ and } -2u \leq \lfloor \frac{x - 2y}{2} \rfloor\} \cap X) \\
= &\{(x - t, y) : t \in \mathbb{N}, (x - t) - 2y \geq 0 \text{ and } t \leq \lfloor \frac{x - 2y}{2} \rfloor\} \\
&\cup \{(x, y - u) : u \in \mathbb{N}, (x - t) - 2y \geq 0 \text{ and } -2u \leq \lfloor \frac{x - 2y}{2} \rfloor\} \\
= &\{(x - t, y) : t \in \mathbb{N}, (x - t) - 2y \geq 0 \text{ and } t \leq \lfloor \frac{x - 2y}{2} \rfloor\} \\
&\cup \{(x, y - u) : u \in \mathbb{N} \text{ and } -2u \leq \lfloor \frac{x - 2y}{2} \rfloor\} \\
= &\text{move}_X(x, y).
\end{aligned}$$

□

Lemma 3.2. *Let* $s, n \in \mathbb{Z}_{\geq 0}$. *Then, we have* (i) *and* (ii).

(i) $\mathcal{G}_{s,n} \subset X$.

(ii) $(\mathcal{G}_{s,a} \cup \mathcal{G}_{s,b}) \cap X = \varnothing$.

Proof. (i) and (ii) are direct from Definition 2.5. □

Definition 3.2. *For* $s \in \mathbb{Z}_{\geq 0}$, *let*

$$\mathcal{G}_{X,s} = (\bigcup_{n=0}^{\infty} \mathcal{G}_{s,n}).$$

Lemma 3.3. *Suppose that we start with a position,*

$$(x, y) \in \mathcal{G}_{X,s}.$$

Then,

$$\text{move}_X(x, y) \cap \mathcal{G}_{X,s} = \varnothing. \tag{46}$$

Proof. By Lemma 2.5,

$$\text{move}(x, y) \cap \mathcal{G}_s = \varnothing. \tag{47}$$

By Lemma 3.1, $\text{move}_X(x, y) = \text{move}(x, y) \cap X$. Since $\mathcal{G}_{X,s} \subset \mathcal{G}_s$, we have (46). □

Lemma 3.4. *For any* $(x, y) \in X - \cup_{i=0}^{s} \mathcal{G}_{X,i}$,

$$\text{move}_X(x, y) \cap \mathcal{G}_{X,s} \neq \varnothing. \tag{48}$$

Proof. Suppose that

$$(x, y) \in X - \cup_{i=0}^{s} \mathcal{G}_{X,i}. \tag{49}$$

If $x \le 2s - 1$, then there exists k such that $0 \le k \le s - 1$ and $x = 2k$ or $x = 2k + 1$.
Since $(x, y) \in X$, by (ii) of Lemma 2.4,

$$(x, y) \in \{(2k, t) : 0 \le t \le k\} \subset \cup_{i=0}^{k} \cup_{n=0}^{\infty} \mathcal{G}_{i,n} \subset \cup_{i=0}^{s-1} \mathcal{G}_{X,i} \tag{50}$$

and

$$(x, y) \in \{(2k + 1, t) : 0 \le t \le k\} \subset \cup_{i=0}^{k} \cup_{n=0}^{\infty} \mathcal{G}_{i,n} \subset \cup_{i=0}^{s-1} \mathcal{G}_{X,i}. \tag{51}$$

Relations (50) and (51) contradict (49). Therefore, $x \ge 2s$. Then, by Lemma 2.6, there exists $n \in \mathbb{Z}_{\ge 0}$ such that

$$\text{move}(x, y) \cap \mathcal{G}_{s,n} \ne \varnothing. \tag{52}$$

By Lemma 3.1, Lemma 3.2 and (52), we obtain (48). □

Theorem 3.1. *$\mathcal{G}_{X,s}$ is the set of positions with Grundy number s in the game of this section.*

Proof. This is directly derived from Definition 3.2, and Lemmas 3.3 and 3.4. □

4 Josephus problem

In this section, we study the Josephus problem and its relation to the games in the previous sections.

Definition 4.1. *We have a finite sequence $1, 2, 3, 4, \ldots, v$ arranged in a circle, and we start with 2 to remove every second number until only one remains. This is a well-known Josephus problem, and we denote the number removed in this order by $e_1 = 2, e_2 = 4, \ldots, e_{v-1}$, and we denote the number that remains in the elimination process by e_v. For any v, let $F_s(v) = e_{v-s}$ for $s = 0, 1, 2, \ldots, v - 1$. Note that $F_0(v) = e_v$ is the final number that remains.*

For the details of the Josephus problem, see [3].

Lemma 4.1. *The following equations are obtained:*

$$F_s(s + k) = 2k \tag{53}$$

for any $1 \le k \le s$ and

$$F_s(2s + 1) = 1. \tag{54}$$

Proof. Because $1 \le k \le s$, we have numbers $1, 2, \ldots, 2k, \ldots, s + k$. We remove numbers

$$2, 4, \ldots, 2k, \ldots . \tag{55}$$

We denote the numbers removed in this order as $e_1 = 2, e_2 = 4, \ldots, e_{s+k-1}$, where e_{s+k} is the last number that remains. $F_s(s + k) = e_{s+k-s} = e_k$ is the k-th number to be removed. Therefore, from (55), we have $F_s(s + k) = e_{s+k-s} = e_k = 2k$.

When we have $1, 2, \ldots, 2s + 1$, we remove numbers $2, 4, \ldots, 2s$ for the first time around the circle and the s-th removed number is $e_s = 2s$. In the second time around the circle, 1 is removed, and $e_{s+1} = 1$. $F_s(2s + 1) = e_{2s+1-s} = e_{s+1} = 1$. □

Lemma 4.2. *For $s \in \mathbb{Z}_{\geq 0}$ and $v \in \mathbb{N}$ such that $0 \leq s \leq v-1$, we obtain the following recursions:*

$$F_s(2v) = 2F_s(v) - 1 \tag{56}$$

and

$$F_s(2v+1) = 2F_s(v) + 1. \tag{57}$$

Proof. First, we prove the recursions for F_0. We assume that

$$F_0(v) = x. \tag{58}$$

Then, x is the last remaining number when we start with the numbers $1, 2, \ldots, v$. Suppose that we start with the numbers $1, 2, 3, \ldots, 2v$. When all even numbers around the circle are removed for the first time, v numbers $1, 3, \ldots, 2v-1$ remain. From (58), the x-th number among these $1, 3, \ldots, 2v-1$ is the last remaining number in this Josephus problem, and the x-th number is $2x-1$. Therefore, we have

$$F_0(2v) = 2F_0(v) - 1.$$

Suppose that we start with the numbers $1, 2, 3, \ldots, 2v+1$. When all even numbers are removed for the first time around the circle and the number 1 is removed at the beginning of the second time around the circle, v numbers $3, 5, \ldots, 2n+1$ remain. From (58), the x-th number among these $3, 5, \ldots, 2n+1$ will survive, and the x-th number is $2x+1$. Therefore, we have

$$F_0(2v+1) = 2F_0(v) + 1.$$

Using a method similar to that used for F_0, we prove (56) and (57). □

Theorem 4.1. *If $v = (2s+1)2^n + m$ such that $s, n, m \in \mathbb{Z}_{\geq 0}$ and $0 \leq m \leq (2s+1)2^n - 1$, then*

$$F_s(v) = 2m + 1. \tag{59}$$

Proof. We proved this through mathematical induction. Suppose that $m = 2k$ such that $m \leq (2s+1) - 1 = 2s$ and $k \in \mathbb{N}$. Then, $k \leq s$ and $s < s + k$. Hence, by Lemmas 4.1 and 4.2,

$$\begin{aligned} F_s((2s+1)+m) &= F_s(2s+2k+1) \\ &= 2F_s(s+k) + 1 \\ &= 2(2k) + 1 = 2m + 1. \end{aligned} \tag{60}$$

By Lemma 4.1, (60) is valid for $m = 0$. Next, we suppose that $m = 2k+1$ such that $m \leq (2s+1) - 1 = 2s$ and $k \in \mathbb{Z}_{\geq 0}$. Then, $k < s$ and $s < s+k+1$. Hence, by Lemmas 4.1 and 4.2,

$$\begin{aligned} F_s((2s+1)+m) &= F_s(2s+2k+2) \\ &= 2F_s(s+k+1) - 1 \\ &= 2(2(k+1)) - 1 \\ &= 2(2k+1) + 1 = 2m + 1. \end{aligned}$$

We assume that there exist n_0 and m_0 such that (59) is valid for any $n \leq n_0$ or $m \leq m_0$. If $m_0 + 1 = 2m + 1$ for $m \leq m_0$, from Lemma 4.2 and the assumption of mathematical induction,

$$\begin{aligned} F_s((2s+1)2^{n_0+1} + m_0 + 1) &= F_s((2s+1)2^{n_0+1} + 2m + 1) \\ &= 2F_s((2s+1)2^{n_0} + m) + 1 \\ &= 2(2m+1) + 1 \\ &= 2(m_0+1) + 1. \end{aligned}$$

If $m_0 + 1 = 2m$ for $m \leq m_0$, from Lemma 4.2 and the assumption of mathematical induction,

$$\begin{aligned} F_s((2s+1)2^{n_0+1} + m_0 + 1) &= F_s((2s+1)2^{n_0+1} + 2m) \\ &= 2F_s((2s+1)2^{n_0} + m) - 1 \\ &= 2(2m+1) - 1 \\ &= 2(2m) + 1 \\ &= 2(m_0+1) + 1. \end{aligned}$$

□

Theorem 4.2. *For* $(x, y) \in \mathcal{G}_{s,n}$,

$$F_s(x+1) = 2y + 1.$$

Proof. From Definition 2.5, for $(x, y) \in \mathcal{G}_{s,n}$, there exists $m \in \mathbb{Z}_{\geq 0}$ such that

$$0 \leq m \leq (2s+1) \times 2^n - 1$$

and

$$(x, y) = ((2s+1)2^n - 1 + m, m).$$

By Theorem 4.1,

$$F_s(x+1) = F_s((2s+1)2^n + m) = 2m + 1 = 2y + 1.$$

□

By Theorems 2.2, 4.1, and 4.2, simple relations exist between positions whose Grundy number is s and the numbers that will be the $(v - s)$-th removed number in the Josephus problem with v numbers.

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