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New properties of arithmetic functions related to gcd and lcm

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Abstract: This paper explores additional properties of the arithmetic functions $f_\alpha(n)$ and $g_\alpha(n)$, defined respectively by $f_\alpha(n) = \prod_{i=1}^{r} p_i^{(e_i,\alpha)}$ and $g_\alpha(n) = \prod_{i=1}^{r} p_i^{[e_i,\alpha]}$, where $n = \prod_{i=1}^{r} p_i^{e_i}$ is the prime factorization of a positive integer $n > 1$, (a, b) and $[a, b]$ denote, respectively the greatest common divisor and the least common multiple of any two integers a and b. These functions and some of their properties have been introduced and investigated in previous works. In this paper, we establish several new theorems that reveal deeper insights into the relationships between these functions.

Keywords: Arithmetic function, Greatest common divisor, Least common multiple.
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1 Introduction

Throughout this paper, we let $\mathbb{N}^*$ denote the set $\mathbb{N} \setminus \{0\}$ of positive integers, and we denote by (a, b) the greatest common divisor and by $[a, b]$ the least common multiple of any two integers a and b. Let

$$n = \prod_{i=1}^{r} p_i^{e_i}$$

be the prime factorization of a positive integer $n > 1$, where $r, e_1, e_2, \ldots, e_r$ are positive integers, and $p_1, p_2, \ldots, p_r$ are distinct primes.

The author together with Derbel [4] introduced and studied some properties of the following arithmetic function for a positive integer α:

$$f_\alpha(n) = \prod_{i=1}^{r} p_i^{(e_i,\alpha)}, \quad f_\alpha(1) = 1,$$

which can be viewed as a generalization of the core function (see [5]):

$$\gamma(n) = \prod_{i=1}^{r} p_i, \quad \gamma(1) = 1,$$

since $f_1(n) = \gamma(n)$ for all n. Additionally, the author [2] explored further properties of these functions and defined new integer sequences associated with them. Next, the author [3], using similar to the approaches in [2, 4], studied a new arithmetic function related to the least common multiple, defined by:

$$g_\alpha(n) = \prod_{i=1}^{r} p_i^{[e_i,\alpha]}, \quad g_\alpha(1) = 1.$$

In the present paper, we will discuss additional interesting properties that provide deeper insights into the relationships between the functions f_α and g_α, including inequalities and identities involving their compositions. We also define and study equivalence relations on the set of positive integers induced by these functions, leading to a classification of integers based on these relations. Finally, we discuss an associated algebraic structure, showing that the set of these functions forms a commutative monoid under composition, and examine the connection between the orbits of integers under this action and the set of exponential divisors.

2 Main results

Theorem 2.1. *Let α be a positive integer. Then, for any positive integer n, we have*

$$(f_\alpha \cdot g_\alpha)(n) \leq n^\alpha \cdot \gamma(n),$$

where the equality holds only if $(e_i, \alpha) = 1$ for all $1 \leq i \leq r$.

Proof. First, we observe that the inequality is true for $n = 1$. Next, for $n > 1$, we use the following well-known property

$$\forall a, b \in \mathbb{N}^* \quad a + b \leq ab + 1$$

by taking $a = (e_i, \alpha)$ and $b = [e_i, \alpha]$, to obtain

$$(e_i, \alpha) + [e_i, \alpha] \leq (e_i, \alpha)[e_i, \alpha] + 1 = \alpha e_i + 1 \quad (1 \leq i \leq r).$$

This yields the inequality:

$$\begin{aligned}(f_\alpha \cdot g_\alpha)(n) &= \prod_{i=1}^{r} p_i^{(e_i,\alpha)+[e_i,\alpha]} \\ &\le \prod_{i=1}^{r} p_i^{\alpha e_i+1} \\ &= n^\alpha \cdot \gamma(n).\end{aligned}$$

If $\alpha = 1$, then we have $(e_i, 1) = 1$ $(1 \le i \le r)$ and the equality holds, since both sides are equal to $n \cdot \gamma(n)$.

Now, let us suppose that $\alpha > 1$. Then

$$\begin{aligned}(f_\alpha \cdot g_\alpha)(n) = n^\alpha \cdot \gamma(n) &\Leftrightarrow \prod_{i=1}^{r} p_i^{(e_i,\alpha)+[e_i,\alpha]} = \prod_{i=1}^{r} p_i^{\alpha e_i+1} \\ &\Leftrightarrow (e_i,\alpha) + [e_i,\alpha] = \alpha e_i + 1 && (1 \le i \le r) \\ &\Leftrightarrow (e_i,\alpha) + [e_i,\alpha] = (e_i,\alpha)[e_i,\alpha] + 1 && (1 \le i \le r) \\ &\Leftrightarrow \Big((e_i,\alpha) - 1\Big)\Big([e_i,\alpha] - 1\Big) = 0 && (1 \le i \le r) \\ &\Leftrightarrow (e_i,\alpha) = 1 \text{ or } [e_i,\alpha] = 1 && (1 \le i \le r).\end{aligned}$$

However, we have $[e_i, \alpha] > 1$ for all $1 \le i \le r$, because $\alpha > 1$. Therefore, $(e_i, \alpha) = 1$ for all $1 \le i \le r$ is the only possibility for the equality to hold, and this completes the proof. □

Clearly, if $\beta = 1$, then $f_\alpha(g_\beta(n)) = g_\beta(f_\alpha(n))$ for all α and for all n. Also, this is the case when $\alpha = \beta$ as the following theorem states.

Theorem 2.2. *Let α be a positive integer. Then for every n:*

$$f_\alpha(g_\alpha(n)) = g_\alpha(f_\alpha(n)) = \gamma(n)^\alpha.$$

Proof. For $n = 1$, the statement is obviously true. If $n > 1$, then one can use the following well known property:

$$\forall a, b \in \mathbb{N}^* \quad [a, (a, b)] = (a, [a, b]) = a$$

to get

$$\begin{aligned}f_\alpha(g_\alpha(n)) &= \prod_{i=1}^{r} p_i^{([e_i,\alpha],\alpha)} \\ &= \prod_{i=1}^{r} p_i^{[(e_i,\alpha),\alpha]} = g_\alpha(f_\alpha(n)) \\ &= \prod_{i=1}^{r} p_i^{\alpha} = \gamma(n)^\alpha.\end{aligned}$$

The proof is finished. □

This theorem is, in fact, a special case of the following more general result:

Theorem 2.3. *Let α and β be positive integers such that $\alpha \mid \beta$. Then for every n:*

$$f_\alpha(g_\beta(n)) = \gamma(n)^\alpha \text{ and } g_\beta(f_\alpha(n)) = \gamma(n)^\beta.$$

Proof. The equalities are true for $n = 1$. If $n > 1$, then we have

$$\begin{aligned} f_\alpha(g_\beta(n)) &= \prod_{i=1}^{r} p_i^{([e_i,\beta],\alpha)} \\ &= \prod_{i=1}^{r} p_i^{\alpha} \qquad \Big(\alpha \mid \beta \Rightarrow \alpha \mid [e_i,\beta] \text{ for all } i\Big) \\ &= \gamma(n)^\alpha. \end{aligned}$$

Similarly,

$$\begin{aligned} g_\beta(f_\alpha(n)) &= \prod_{i=1}^{r} p_i^{[(e_i,\alpha),\beta]} \\ &= \prod_{i=1}^{r} p_i^{\beta} \qquad \Big(\alpha \mid \beta \Rightarrow (e_i,\alpha) \mid \beta \text{ for all } i\Big) \\ &= \gamma(n)^\beta. \end{aligned}$$

This proves the theorem. □

Theorem 2.4. *Let α and β be positive integers. Then for every n:*

1. *If $e_i \mid \alpha$ $(1 \le i \le r)$, then $f_\alpha(g_\beta(n)) = g_{(\alpha,\beta)}(n)$.*
2. *If $\beta \mid e_i$ $(1 \le i \le r)$, then $g_\beta(f_\alpha(n)) = f_{[\alpha,\beta]}(n)$.*

In particular, if $\beta \mid e_i \mid \alpha$ $(1 \le i \le r)$, then $(\alpha,\beta) = \beta$ and $[\alpha,\beta] = \alpha$. Hence,

$$f_\alpha(g_\beta(n)) = g_\beta(n) \text{ and } g_\beta(f_\alpha(n)) = f_\alpha(n).$$

Proof. Both statements are true for $n = 1$. If $n > 1$, we use the following identities to prove the statements:

$$\forall a,b,c \in \mathbb{N}^* \quad (a,[b,c]) = [(a,b),(a,c)], \tag{1}$$

$$\forall a,b,c \in \mathbb{N}^* \quad [a,(b,c)] = ([a,b],[a,c]), \tag{2}$$

(see e.g., [1, p. 22]).

1. If we suppose that $e_i \mid \alpha$ $(1 \le i \le r)$, then it follows by using (1) that:

$$\begin{aligned} f_\alpha(g_\beta(n)) &= \prod_{i=1}^{r} p_i^{([e_i,\beta],\alpha)} \\ &= \prod_{i=1}^{r} p_i^{[(e_i,\alpha),(\beta,\alpha)]} = \prod_{i=1}^{r} p_i^{[e_i,(\beta,\alpha)]} \\ &= g_{(\alpha,\beta)}(n), \end{aligned}$$

as claimed.

2. If we suppose that $\beta \mid e_i$ $(1 \leq i \leq r)$, then it follows by using (2) that:

$$\begin{aligned} g_\beta(f_\alpha(n)) &= \prod_{i=1}^{r} p_i^{[(e_i,\alpha),\beta]} \\ &= \prod_{i=1}^{r} p_i^{([e_i,\beta],[\alpha,\beta])} = \prod_{i=1}^{r} p_i^{(e_i,[\alpha,\beta])} \\ &= f_{[\alpha,\beta]}(n), \end{aligned}$$

as required. The proof is achieved. □

Now, let us consider the following relation on $\mathbb{N}^*$ defined by:

$$n \sim m \Leftrightarrow \exists \alpha \in \mathbb{N}^*; f_\alpha(n) = f_\alpha(m).$$

Note that if $p \neq q$ are distinct prime numbers, then $f_\alpha(p) \neq f_\alpha(q)$ for any $\alpha \in \mathbb{N}^*$, which implies that $p \not\sim q$.

Theorem 2.5. *The relation $\sim$ is an equivalence relation.*

Proof. It is easy to show that $\sim$ is a reflexive and symmetric relation. We now proceed to prove that $\sim$ is also a transitive relation. Let n, m, and l be positive integers such that $n \sim m$ and $m \sim l$. We wish to prove that $n \sim l$.

Let $m = \prod_{i=1}^{s} q_i^{m_i}$ and $l = \prod_{i=1}^{t} \mathfrak{p}_i^{l_i}$ be the unique prime factorizations of m and l, respectively. Then we have:

$$\begin{aligned} \begin{cases} n \sim m \\ m \sim l \end{cases} &\Rightarrow \begin{cases} \exists \alpha \in \mathbb{N}^*;\ f_\alpha(n) = f_\alpha(m), \\ \exists \beta \in \mathbb{N}^*;\ f_\beta(m) = f_\beta(l). \end{cases} \\ &\Rightarrow \begin{cases} \exists \alpha \in \mathbb{N}^*;\ \prod_{i=1}^{r} p_i^{(e_i,\alpha)} = \prod_{i=1}^{s} q_i^{(m_i,\alpha)}, \\ \exists \beta \in \mathbb{N}^*;\ \prod_{i=1}^{s} q_i^{(m_i,\beta)} = \prod_{i=1}^{t} \mathfrak{p}_i^{(l_i,\beta)}. \end{cases} \\ &\Rightarrow \begin{cases} r = s = t, \\ p_i = q_i = \mathfrak{p}_i \text{ for any } 1 \leq i \leq r, \\ (e_i,\alpha) = (m_i,\alpha) \text{ and } (m_i,\beta) = (l_i,\beta) \text{ for any } 1 \leq i \leq r. \end{cases} \end{aligned}$$

On the other hand, for any $1 \leq i \leq r$, we have

$$\begin{aligned} (m_i,(\alpha,\beta)) &= ((m_i,\alpha),\beta) = ((e_i,\alpha),\beta) = (e_i,(\alpha,\beta)) \\ &= ((m_i,\beta),\alpha) = ((l_i,\beta),\alpha) = (l_i,(\alpha,\beta)), \end{aligned}$$

from which it follows that $f_{(\alpha,\beta)}(n) = f_{(\alpha,\beta)}(l)$, i.e., $n \sim l$. □

If p is prime, then

$$[p] = \{p^r;\ r \in \mathbb{N}^*\} \quad (\text{take } \alpha = r+1).$$

If p and q are primes, then

$$[pq] = \{p^r q^s;\ r,s \in \mathbb{N}^*\} \quad (\text{take } \alpha = rs+1).$$

For any square-free integer $n \in \mathbb{N}^*$, we have

$$[n] = \left\{ m = \prod_{i=1}^{s} q_i^{m_i} \in \mathbb{N}^*;\ \gamma(m) = n \right\} \quad \left(\text{take } \alpha = 1 + \prod_{i=1}^{s} m_i \right).$$

Thus, we have

$$\frac{\mathbb{N}^*}{\sim} = \{[n];\ n \text{ is a square-free positive integer}\}.$$

Remark 2.1. *If we define $\mathcal{R}$ to be the relation on $\mathbb{N}^*$ such that:*

$$n\mathcal{R}m \Leftrightarrow \exists \alpha \in \mathbb{N}^*; g_\alpha(n) = g_\alpha(m),$$

then the same reasoning as above allows us to prove that $\mathcal{R}$ is an equivalence relation and $\frac{\mathbb{N}^*}{\mathcal{R}} = \frac{\mathbb{N}^*}{\sim}$.

We conclude this paper with the following discussion about an algebraic structure related to f_α. Since $f_\alpha \circ f_\beta = f_{(\alpha,\beta)}$ for all $\alpha, \beta \in \mathbb{N}$, it follows that the set $\mathcal{M} = \{f_\alpha \mid \alpha \in \mathbb{N}\}$ forms a commutative monoid with the identity element f_0 under the composition of functions.

The mapping

$$\begin{array}{rccc} * : & \mathcal{M} \times \mathbb{N}^* & \to & \mathbb{N}^* \\ & (f_\alpha, n) & \mapsto & f_\alpha * n = f_\alpha(n) \end{array}$$

is a monoid action of $\mathcal{M}$ on $\mathbb{N}^*$. Indeed, $f_0(n) = n$ for all $n \in \mathbb{N}^*$, and

$$f_\alpha * (f_\beta * n) = f_\alpha * (f_\beta(n)) = f_\alpha(f_\beta(n)) = (f_\alpha \circ f_\beta) * n \quad \text{for all } n \in \mathbb{N}^* \text{ and } \alpha, \beta \in \mathbb{N}.$$

Let O_n denote the orbit of $n \in \mathbb{N}^*$ under the action of $\mathcal{M}$, i.e., $O_n = \{f_\alpha(n) \mid f_\alpha \in \mathcal{M}\}$.

On the other hand, a positive divisor d of n is said to be an exponential divisor if $d = \prod_{i=1}^{r} p_i^{d_i}$, where $d_i \mid e_i$ for all $1 \leq i \leq r$. The definition, basic properties, and more details about exponential divisors can be found in [6]. If we denote the set of all exponential divisors of n by E_n, then $O_n \subseteq E_n$ and $|E_n| = \prod_{i=1}^{r} \tau(e_i)$, where $\tau(a)$ is the usual divisor function. For example,

$$E_{72} = O_{72} = \{6 = f_1(72), 18 = f_2(72), 24 = f_3(72), 72 = f_0(72)\},$$

$$E_{36} = \{6, 12, 18, 36\} \text{ while } O_{36} = \{6 = f_1(36), 36 = f_0(36)\}.$$

Note that 12 cannot be in O_{36}. If this were the case, there would exist some $\alpha \in \mathbb{N}$ such that $(\alpha, 2) = 2$ and $(\alpha, 2) = 1$, which is impossible. The following theorem provides a classification of $n \in \mathbb{N}^*$ such that $O_n = E_n$.

Theorem 2.6. *Let n be a positive integer such that $O_n = E_n$. Then exactly one of the following possibilities holds:*

1. *$n = p^k$ for some prime p and $k \in \mathbb{N}^*$.*
2. *$n = \prod_{i=1}^{r} p_i^{e_i}$ with $(e_i, e_j) = 1$ for all $1 \leq i \neq j \leq r$.*

Proof.

1. Suppose that $n = p^k$ for some prime p and $k \in \mathbb{N}^*$, and let $d \in E_n$. Then $d = p^l$ for some l such that $l \mid k$. If we take $\alpha = l$, then $f_l(n) = p^l = d$, which means $d \in O_n$, thus proving the first item.

2. For the second item, assume that $n = \prod_{i=1}^{r} p_i^{e_i}$ with $(e_i, e_j) = 1$ for all $1 \le i \ne j \le r$, and let $d \in E_n$. Then $d = \prod_{i=1}^{r} p_i^{d_i}$ where $d_i \mid e_i$. By taking $\alpha = \prod_{i=1}^{r} d_i$, we obtain

$$f_\alpha(n) = \prod_{i=1}^{r} p_i^{(e_i, \prod_{i=1}^{r} d_i)} = \prod_{i=1}^{r} p_i^{d_i} = d,$$

which implies $d \in O_n$.

Finally, suppose that there exist e_i and e_j such that $(e_i, e_j) = a > 1$, and consider

$$d = p_i p_j^a \prod_{\substack{k=1 \\ k \ne i,j}}^{r} p_k \in E_n.$$

By contradiction, assume that $d \in O_n$. Then there exists $\alpha \in \mathbb{N}$ such that $(e_i, \alpha) = 1$ and $(e_j, \alpha) = a$. However, since $a \mid \alpha$ and $a \mid e_i$, it follows that $(e_i, \alpha) \ge a > 1$, which is a contradiction.

The proof is complete. $\square$

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