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Series associated with harmonic numbers, Fibonacci numbers and central binomial coefficients $\binom{2n}{n}$

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Abstract: We find various series that involve the central binomial coefficients $\binom{2n}{n}$, harmonic numbers and Fibonacci numbers. Contrary to the traditional hypergeometric function ${}_pF_q$ approach, our method utilizes a straightforward transformation to obtain new evaluations linked to Fibonacci numbers and the golden ratio. We also gave a new series representation for $\zeta(2)$.
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1 Introduction

Harmonic numbers H_α are defined by the recurrence relation

$$H_\alpha = H_{\alpha-1} + \frac{1}{\alpha}$$

for $\alpha \in \mathbb{C} \setminus \mathbb{Z}^- \cup \{0\}$ with $H_0 = 0$.

The recurrence relation imply that if $\alpha = n$ is a non-negative integer, then

$$H_n = \sum_{j=1}^{n} \frac{1}{j}.$$

In an article published in 2016, Chen [4], gave a generating function for the sequences $\binom{2n}{n} H_n$, $\binom{2n}{n}(H_{2n} - H_n)$, $C_n(H_{2n} - H_n)$ and few others, where C_n is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. In search of interesting series associated with central binomial coefficients and Harmonic numbers, Chen established several interesting sums as follows:

$$\sum_{n=1}^{\infty} \frac{1}{4^n(2n+1)}\binom{2n}{n}(H_{2n} - H_n) = \pi \ln 2 - 2G, \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{1}{4^n n(2n+1)}\binom{2n}{n}(H_{2n-1} - H_n) = 2 + 2\ln 2 + \ln^2 2 + 4G - \pi(1 + 2\ln 2), \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{4^n(2n+3)} C_n H_n = 2 + 4\ln 2 - 4G - \pi + \pi \ln 2, \tag{3}$$

where G is the Catalan's constant, which is defined in [1]

$$G := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

Later in this paper, we established the following results:

$$\sum_{n=1}^{\infty} \frac{n}{4^n(2n-1)^2(2n+1)(2n+3)}\binom{2n}{n} = \frac{3\pi}{256}, \sum_{n=1}^{\infty} \frac{n^2}{4^n(2n-1)^2(2n+1)}\binom{2n}{n} = \frac{3\pi}{32}.$$

From [4], we have the generating function:

$$\mathscr{M}(x) := \sum_{n=1}^{\infty} \binom{2n}{n} H_n x^n = \frac{2}{\sqrt{1-4x}} \ln\left(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}\right) \tag{4}$$

which converges on $[-1/4, 1/4)$.

Now, before we continue, let F_n and L_n denote the n-th Fibonacci and Lucas numbers, respectively, both satisfying the recurrence relation $\Gamma_n = \Gamma_{n-1} + \Gamma_{n-2}$, $n \geq 2$, with conditions $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$. Also, $L_{-m} = (-1)^m L_m$ and $F_{-m} = (-1)^{m-1} F_m$. Throughout this paper, we denote the golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$ and write its conjugate $\beta = \frac{1-\sqrt{5}}{2}$, so that $\alpha\beta = -1$ and $\alpha + \beta = 1$. We have the Binet formulas for the Fibonacci and Lucas numbers to be:

$$F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}, \quad L_m = \alpha^m + \beta^m$$

for any integer m.

We will require the following, which are consequences of the Binet formula and well-known identities that are valid for integers m and n.

$$\begin{aligned} \alpha^{2m} &= \alpha^m F_m \sqrt{5} - (-1)^{m+1}, \\ \alpha^{2m} &= \alpha^m L_m - (-1)^m, \\ \beta^{2m} &= \beta^m L_m - (-1)^m, \end{aligned} \tag{5}$$

$$F_n^2 + (-1)^{n+m-1} F_m^2 = F_{n-m} F_{n+m},$$
$$L_{n+m} + (-1)^m L_{n-m} = L_n L_m.$$

Now setting $x = \frac{1}{4(\alpha^{2r} + (-1)^{r+1})}$ and using the identity $\alpha^{2r} = \alpha^r F_r \sqrt{5} - (-1)^{r+1}$ in (4) for $r \in \mathbb{N}$ we get that:

$$\mathscr{M}\left([4(\alpha^{2r} + (-1)^{r+1})]^{-1}\right) = \sum_{n=1}^{\infty} \frac{1}{4^n(\alpha^{2r} + (-1)^{r+1})} \binom{2n}{n} H_n = \sum_{n=1}^{\infty} \frac{1}{4^n(\alpha^r F_r \sqrt{5})^n} \binom{2n}{n} H_n$$
$$= \sum_{n=1}^{\infty} \frac{1}{(4\sqrt{5})^n \alpha^{rn} F_r^n} \binom{2n}{n} H_n.$$

Notice,

$$\sqrt{1 - \frac{1}{\alpha^{2r} + (-1)^{r+1}}} = \sqrt{\frac{\alpha^{2r} + (-1)^{r+1} - 1}{\alpha^{2r} + (-1)^{r+1}}} = \sqrt{\frac{\alpha^r F_r \sqrt{5} - 1}{\alpha^r F_r \sqrt{5}}}.$$

Thus, upon substitution we have

$$\sum_{n=1}^{\infty} \frac{1}{(4\sqrt{5})^n \alpha^{rn} F_r^n} \binom{2n}{n} H_n = \frac{2\sqrt{\alpha^r F_r \sqrt{5}}}{\sqrt{\alpha^r F_r \sqrt{5} - 1}} \ln\left(\frac{\sqrt{\alpha^r F_r \sqrt{5}} + \sqrt{\alpha^r F_r \sqrt{5} - 1}}{2\sqrt{\alpha^r F_r \sqrt{5} - 1}}\right) \tag{6}$$

Evaluation at $r = 1, 2, 3$ in (6) gives:

$$\sum_{n=1}^{\infty} \frac{1}{(2^2\sqrt{5})^n \alpha^n} \binom{2n}{n} H_n = \frac{2\sqrt{\alpha\sqrt{5}}}{\sqrt{\alpha\sqrt{5} - 1}} \ln\left(\frac{\sqrt{\alpha\sqrt{5}} + \sqrt{\alpha\sqrt{5} - 1}}{2\sqrt{\alpha\sqrt{5} - 1}}\right) \tag{7}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2^2\sqrt{5})^n \alpha^{2n}} \binom{2n}{n} H_n = \frac{2\alpha\sqrt[4]{5}}{\sqrt{\alpha^2\sqrt{5} - 1}} \ln\left(\frac{\alpha\sqrt[4]{5} + \sqrt{\alpha^2\sqrt{5} - 1}}{2\sqrt{\alpha^2\sqrt{5} - 1}}\right) \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2^3\sqrt{5})^n \alpha^{3n}} \binom{2n}{n} H_n = \frac{2\sqrt{2}\sqrt{\alpha^3\sqrt{5}}}{\sqrt{2\alpha^3\sqrt{5} - 1}} \ln\left(\frac{\sqrt{2\alpha^3\sqrt{5}} + \sqrt{2\alpha^3\sqrt{5} - 1}}{2\sqrt{2\alpha^3\sqrt{5} - 1}}\right) \tag{9}$$

Also from (6), by replacing r with $2r$ we get,

$$\sum_{n=1}^{\infty} \frac{1}{(4\sqrt{5})^n \alpha^{2rn} F_{2r}^n} \binom{2n}{n} H_n = \frac{2\alpha^r \sqrt{F_{2r}\sqrt{5}}}{\sqrt{\alpha^{2r} F_{2r} \sqrt{5} - 1}} \ln\left(\frac{\alpha^r \sqrt{F_{2r}\sqrt{5}} + \sqrt{\alpha^{2r} F_{2r} \sqrt{5} - 1}}{2\sqrt{\alpha^{2r} F_{2r} \sqrt{5} - 1}}\right) \tag{10}$$

In this paper, by exploiting $\mathscr{M}(x)$ in (4), we shall produce more interesting results. To ensure accuracy, all formulas appearing in this paper were numerically verified by *Mathematica* 13.3.

2 Main theorems

Theorem 2.1. *If r is a natural number, then*

$$\sum_{n=1}^{\infty} \frac{1}{4^n \alpha^{rn} L_r^n} \binom{2n}{n} H_n = \frac{2\sqrt{\alpha^r L_r}}{\sqrt{\alpha^r L_r - 1}} \ln\left(\frac{\sqrt{\alpha^r L_r} + \sqrt{\alpha^r L_r - 1}}{2\sqrt{\alpha^r L_r - 1}}\right) \tag{11}$$

Proof. Setting $x = \frac{1}{4(\alpha^{2r} + (-1)^r)}$ in (4) and using (5) (the second identity from the list of consequences from Binet's formula in the Introduction), the result follows immediately. $\square$

Example 2.1. *Evaluation at $r = 1, 2, 3$ in (11), gives:*

$$\sum_{n=1}^{\infty} \frac{1}{4^n \alpha^n} \binom{2n}{n} H_n = \frac{2\sqrt{\alpha}}{\sqrt{\alpha - 1}} \ln\left(\frac{\sqrt{\alpha} + \sqrt{\alpha - 1}}{2\sqrt{\alpha - 1}}\right), \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{1}{12^n \alpha^{2n}} \binom{2n}{n} H_n = \frac{2\alpha\sqrt{3}}{\sqrt{3\alpha^2 - 1}} \ln\left(\frac{\alpha\sqrt{3} + \sqrt{3\alpha^2 - 1}}{2\sqrt{3\alpha^2 - 1}}\right), \tag{13}$$

$$\sum_{n=1}^{\infty} \frac{1}{16^n \alpha^{3n}} \binom{2n}{n} H_n = \frac{4\sqrt{\alpha^3}}{\sqrt{4\alpha^3 - 1}} \ln\left(\frac{2\sqrt{\alpha^3} + \sqrt{4\alpha^3 - 1}}{2\sqrt{4\alpha^3 - 1}}\right). \tag{14}$$

Corollary 2.1. *If r is a natural number, then*

$$\sum_{n=1}^{\infty} \frac{1}{4^n \alpha^{2rn} L_{2r}^n} \binom{2n}{n} H_n = \frac{2\alpha^r\sqrt{L_{2r}}}{\sqrt{\alpha^{2r} L_{2r} - 1}} \ln\left(\frac{\alpha^r\sqrt{L_{2r}} + \sqrt{\alpha^{2r} L_{2r} - 1}}{2\sqrt{\alpha^{2r} L_{2r} - 1}}\right). \tag{15}$$

Proof. Replace r with $2r$ in (11), and the proof follows. □

Theorem 2.2. *If $r > 0$ is a non-negative integer, then*

$$\sum_{n=1}^{\infty} \frac{1}{4^n \alpha^{rn} L_r^n} \binom{2n}{n} (H_{2n} - H_n) = -\frac{\sqrt{\alpha^r L_r}}{\sqrt{\alpha^r L_r - 1}} \ln\left(\frac{\sqrt{\alpha^r L_r} + \sqrt{\alpha^r L_r - 1}}{2\sqrt{\alpha^r L_r}}\right), \tag{16}$$

$$\sum_{n=1}^{\infty} \frac{1}{(4\sqrt{5})^n \alpha^{rn} F_r^n} \binom{2n}{n} (H_{2n} - H_n) = -\frac{\sqrt{\alpha^r F_r \sqrt{5}}}{\sqrt{\alpha^r F_r \sqrt{5} - 1}} \ln\left(\frac{\sqrt{\alpha^r F_r\sqrt{5}} + \sqrt{\alpha^r F_r \sqrt{5} - 1}}{2\sqrt{\alpha^r F_r \sqrt{5}}}\right). \tag{17}$$

Proof. Before we establish the identities above, we shall establish a generating function for the sequence $\binom{2n}{n}(H_{2n} - H_n)$ in the manner of [4]. Observe that:

$$\sum_{n=1}^{\infty} \binom{2n}{n} (H_{2n} - H_n) x^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{k} \binom{2n-k}{n} x^n = \sum_{k=1}^{\infty} \frac{1}{k} \left\{ \sum_{n=k}^{\infty} \binom{2n-k}{n} x^n \right\}$$

By setting $(n = m + k)$, we have that:

$$\sum_{k=1}^{\infty} \frac{1}{k} \left\{ \sum_{n=k}^{\infty} \binom{2n-k}{n} x^n \right\} = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \binom{2m+k}{m+k} x^{m+k}$$

$$= \sum_{k=1}^{\infty} \frac{x^k}{k} \left\{ \sum_{m=0}^{\infty} \binom{2m+k}{m} x^m \right\}$$

Since, (from [[6], p.203])

$$\sum_{m=0}^{\infty} \binom{2m+k}{m} \vartheta^m = \frac{1}{\sqrt{1 - 4\vartheta}} \left(\frac{1 - \sqrt{1 - 4\vartheta}}{2\vartheta}\right)^k, \quad |\vartheta| < 1/4$$

Thus, we have that

$$\sum_{n=1}^{\infty} \binom{2n}{n} (H_{2n} - H_n) x^n = \frac{1}{\sqrt{1-4x}} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{1 - \sqrt{1-4x}}{2}\right)^k = -\frac{1}{\sqrt{1-4x}} \ln\left(\frac{1 - \sqrt{1-4x}}{2}\right). \tag{18}$$

Now set $x = \dfrac{1}{4(\alpha^{2r} + (-1)^{r+1})}$ and $x = \dfrac{1}{4(\alpha^{2r} + (-1)^r)}$ in (18), then (17) and (16) follows directly. □

Example 2.2. *Evaluation at $r = 1, 2, 3$ in (16) and (17), respectively, gives*

$$\sum_{n=1}^{\infty} \frac{1}{4^n \alpha^n} \binom{2n}{n} (H_{2n} - H_n) = -\frac{\sqrt{\alpha}}{\sqrt{\alpha - 1}} \ln \left(\frac{\sqrt{\alpha} + \sqrt{\alpha - 1}}{2\sqrt{\alpha}} \right) \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{1}{12^n \alpha^{2n}} \binom{2n}{n} (H_{2n} - H_n) = -\frac{\alpha\sqrt{3}}{\sqrt{3\alpha^2 - 1}} \ln \left(\frac{\alpha\sqrt{3} + \sqrt{3\alpha^2 - 1}}{2\alpha\sqrt{3}} \right) \tag{20}$$

$$\sum_{n=1}^{\infty} \frac{1}{16^n \alpha^{3n}} \binom{2n}{n} (H_{2n} - H_n) = -\frac{2\sqrt{\alpha^3}}{\sqrt{4\alpha^3 - 1}} \ln \left(\frac{2\sqrt{\alpha^3} + \sqrt{4\alpha^3 - 1}}{4\sqrt{\alpha^3}} \right) \tag{21}$$

$$\sum_{n=1}^{\infty} \frac{1}{(4\sqrt{5})^n \alpha^n} \binom{2n}{n} (H_{2n} - H_n) = -\frac{\sqrt{\alpha\sqrt{5}}}{\sqrt{\alpha\sqrt{5} - 1}} \ln \left(\frac{\sqrt{\alpha\sqrt{5}} + \sqrt{\alpha\sqrt{5} - 1}}{2\sqrt{\alpha\sqrt{5}}} \right) \tag{22}$$

$$\sum_{n=1}^{\infty} \frac{1}{(4\sqrt{5})^n \alpha^{2n}} \binom{2n}{n} (H_{2n} - H_n) = -\frac{\alpha\sqrt[4]{5}}{\sqrt{\alpha^2\sqrt{5} - 1}} \ln \left(\frac{\alpha\sqrt[4]{5} + \sqrt{\alpha^2\sqrt{5} - 1}}{2\alpha\sqrt[4]{5}} \right) \tag{23}$$

$$\sum_{n=1}^{\infty} \frac{1}{(8\sqrt{5})^n \alpha^{3n}} \binom{2n}{n} (H_{2n} - H_n) = -\frac{\sqrt{2}\sqrt{\alpha^3\sqrt{5}}}{\sqrt{\alpha^3 2\sqrt{5} - 1}} \ln \left(\frac{\sqrt{2}\sqrt{\alpha^3\sqrt{5}} + \sqrt{\alpha^3 2\sqrt{5} - 1}}{2\sqrt{2}\sqrt{\alpha^3\sqrt{5}}} \right) \tag{24}$$

The following identity connects Catalan numbers, harmonic numbers, to Apery's constant.

Theorem 2.3. *Let C_n be the n-th Catalan's number, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, provided $\Re(s) > 1$ and*

$$n!! := \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n - 2k) := \begin{cases} \prod_{k=1}^{\frac{n}{2}} (2k), \text{ if } n \text{ is even}, \\ \prod_{k=1}^{\frac{n+1}{2}} (2k - 1), \text{ if } n \text{ is odd}. \end{cases}$$

Then,

$$\sum_{n=1}^{\infty} \frac{C_n (H_{2n} - \frac{1}{2} H_n)}{4^n (2n + 1)} \left(\frac{\pi}{2} - \frac{(2n)!!}{(2n + 1)!!} \right) = 2\ln(2) + \frac{7}{8}\zeta(3) + \frac{\pi}{12}(-12 + \pi(-1 + \ln(8))).$$

Proof. From [4] we have that:

$$\sum_{n=1}^{\infty} \binom{2n}{n} (H_{2n} - \frac{1}{2} H_n) x^n = -\frac{1}{\sqrt{1 - 4x}} \ln \sqrt{1 - 4x}$$

Immediately we see that:

$$\sum_{n=1}^{\infty} C_n (H_{2n} - \frac{1}{2} H_n) x^n = \frac{1}{2x} (1 - \sqrt{1 - 4x} + \sqrt{1 - 4x} \ln \sqrt{1 - 4x}), \quad |x| < 1/4$$

For the interesting part, set $x = \frac{1}{4} \sin^2 t$ for $t \in (-\pi/2, \pi/2)$. Then we have that:

$$\sum_{n=1}^{\infty} \frac{C_n (H_{2n} - \frac{1}{2} H_n)}{4^n} \sin^{2n} t = \frac{2}{\sin^2 t} (1 - \cos t + \cos t \ln \cos t). \tag{25}$$

Now multiply (25) by $t\cos t$ and integrating both sides from 0 to $\pi/2$. We have:

$$\sum_{n=1}^{\infty}\frac{C_n(H_{2n}-\frac{1}{2}H_n)}{4^n}\int_0^{\frac{\pi}{2}} t\cos t\sin^{2n} t\,\mathrm{d}t = 2\int_0^{\frac{\pi}{2}}\frac{t\cos t}{\sin^2 t}(1-\cos t+\cos t\ln\cos t)\,\mathrm{d}x$$

Thus, we evaluated the integral on the right hand side to $\ln 2+\frac{\pi}{24}(-12+\pi(-1+\ln 8))+\frac{7}{16}\zeta(3)$ and using integration by part for the integral on the left we have that:

$$\int_0^{\frac{\pi}{2}}\frac{t\cos t}{\sin^2 t}(1-\cos t+\cos t\ln\cos t)\,dt = \ln 2+\frac{\pi}{24}(-12+\pi(-1+\ln 8))+\frac{7}{16}\zeta(3),$$

$$\int_0^{\frac{\pi}{2}} t\cos t\sin^{2n} t\,dt = \frac{1}{2n+1}\left(\frac{\pi}{2}-\frac{(2n)!!}{(2n+1)!!}\right).$$

So the result follows immediately. □

In the next theorem, we present two new Ramanujan-like series involving harmonic numbers.

Theorem 2.4. *For the Catalan's constant G,*

$$\sum_{n=1}^{\infty}\frac{C_n(H_{2n}-H_n)}{4^{2n}}\binom{2n+2}{n+1} = \frac{16}{\pi}\left(2G+\pi-2-\ln 2-\pi\ln 2\right),$$

$$\sum_{n=1}^{\infty}\frac{C_nH_{2n}}{16^n}\binom{2n}{n} = \frac{2}{\pi}\left(2+\pi-2\ln 8\right).$$

Proof. Integrating both sides of (18) with respect to x, we obtain the generating function for the sequence $C_n(H_{2n}-H_n)$ as follows:

$$\sum_{n=1}^{\infty}C_n(H_{2n}-H_n)x^n = \frac{1}{2x}\left[(1-\sqrt{1-4x})+(1+\sqrt{1-4x})\ln\left(\frac{1+\sqrt{1-4x}}{2}\right)\right] \tag{26}$$

Likewise, from [1] we have that:

$$\sum_{n=1}^{\infty}\binom{2n}{n}H_{2n}x^n = \frac{1}{\sqrt{1-4x}}\left[\ln\left(\frac{1+\sqrt{1-4x}}{2}\right)-2\ln\sqrt{1-4x}\right] \tag{27}$$

For this reason, integrating both sides of (27) we have that,

$$\sum_{n=1}^{\infty}C_nH_{2n}x^n = \frac{1}{2x}[(1-\sqrt{1-4x})-(1+\sqrt{1-4x})\ln(1+\sqrt{1-4x})+\ln 2+\sqrt{1-4x}\ln(2-8x)] \tag{28}$$

Now set $x=\frac{\sin^2 t}{4}$ in (26) and (28) and integrating both sides with respect to x from 0 to $\pi/2$, while using the following results:

$$\int_0^{\frac{\pi}{2}}\left\{(1-\cos t)+(1+\cos t)\ln\left(\frac{1+\cos t}{2}\right)\right\}dt = 2G+\pi-2-\ln 2-\pi\ln 2,$$

$$\int_0^{\frac{\pi}{2}}\frac{2}{\sin^2 t}\left[(1-\cos t)-(1+\cos t)\ln(1+\cos t)+\ln 2+\cos t\ln(2\cos^2 t)\right]dt = 2+\pi-2\ln 8,$$

and

$$\int_0^{\frac{\pi}{2}}\sin^{2n} t\,dt = \frac{\pi}{2}\frac{(2n)!}{4^n(n!)^2}.$$

Then the series follows directly. □

The next result provides a new series representation for the Basel sum, whose value was first determined by Leonhard Euler in 1734.

Theorem 2.5. *For* $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, *provided* $\Re(s) > 1$, *we have that,*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1024n}{3(2n-1)^2(2n+1)(2n+3)^2} \frac{\binom{2n}{n}}{\binom{2n+2}{n+1}}.$$

Proof. From [3], we have that:

$$\sum_{n=1}^{\infty} \frac{nx^{2n}}{4^n(2n-1)^2(2n+1)} \binom{2n}{n} = \frac{1}{8}\left(\sqrt{1-x^2} + 2x\sin^{-1}x - \frac{\sin^{-1}x}{x}\right). \tag{29}$$

Next we multiply (29) by x^2 and integrate both sides with respect to x to get:

$$\sum_{n=1}^{\infty} \frac{nx^{2n+3}}{4^n(2n-1)^2(2n+1)(2n+3)} \binom{2n}{n} = \frac{(8x^4-8x^2+3)\sin^{-1}x + \sqrt{1-x^2}(6x^3-3x)}{128} \tag{30}$$

which converges for $x \in [-1, 1]$. Now, we set $x = \sin t$ in (30) and integrate over the interval 0 to $\pi/2$. The result follows. □

We present a Ramanujan-like series involving the ratio of the Catalan's constant G and π.

Theorem 2.6. *For the Catalan's constant* G *and* π, *we have that,*

$$\sum_{n=1}^{\infty} \frac{n^2}{16^n(2n-1)^2(2n+1)} \binom{2n}{n}^2 = \frac{G}{4\pi} + \frac{1}{8\pi}$$

Proof. We begin by differentiating (29) with respect to x to get,

$$\sum_{n=1}^{\infty} \frac{2n^2x^{2n-1}}{4^n(2n-1)^2(2n+1)} \binom{2n}{n} = \frac{1}{8x^2}\left((2x^2+1)\sin^{-1}x - x\sqrt{1-x^2}\right). \tag{31}$$

Now we multiply (31) by x and set $x = \sin t$, then we integrate both sides over the interval 0 to $\pi/2$. Finally, using the result:

$$\int_0^{\frac{\pi}{2}} \frac{t(2\sin^2 t+1) - \sin t\cos t}{\sin t}\,dt = \int_0^{\frac{\pi}{2}} (2t\sin t - \cos t)\,dt + \int_0^{\frac{\pi}{2}} \frac{t}{\sin t}\,dt = 1 + 2G,$$

the series follows immediately. □

3 Some interesting series

Observe that the series in (26) converges on $[-1/4, 1/4)$. Setting $x = -1/8, 1/16$ and $-1/16$, respectively, we obtain the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n C_n}{8^n}(H_{2n} - H_n) = -\frac{4}{\sqrt{2}}\left[(\sqrt{2}-\sqrt{3}) + (\sqrt{2}+\sqrt{3})\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right)\right], \quad (32)$$

$$\sum_{n=1}^{\infty} \frac{C_n}{16^n}(H_{2n} - H_n) = 4\left[(2-\sqrt{3}) + (2+\sqrt{3})\ln\left(\frac{2+\sqrt{3}}{4}\right)\right], \quad (33)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n C_n}{16^n}(H_{2n} - H_n) = -4\left[(2-\sqrt{5}) + (2+\sqrt{5})\ln\left(\frac{2+\sqrt{5}}{4}\right)\right]. \quad (34)$$

In a similar manner, recall from (30) that it converges on $[-1, 1]$. By setting $x = 1$ and $x = -1$, respectively, we obtain two interesting series:

$$\sum_{n=1}^{\infty} \frac{n}{4^n(2n-1)^2(2n+1)(2n+3)}\binom{2n}{n} = \frac{3\pi}{256}, \quad (35)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n+3} n}{4^n(2n-1)^2(2n+1)(2n+3)}\binom{2n}{n} = \frac{-3\pi}{256}. \quad (36)$$

From (31) we have:

$$\sum_{n=1}^{\infty} \frac{n^2}{4^n(2n-1)^2(2n+1)}\binom{2n}{n} = \frac{3\pi}{32}. \quad (37)$$

Also, from (18) we get that:

$$\sum_{n=1}^{\infty} \frac{H_{2n} - H_n}{8^n}\binom{2n}{n} = -\sqrt{2}\ln\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right), \quad (38)$$

$$\sum_{n=1}^{\infty} \frac{H_{2n} - H_n}{16^n}\binom{2n}{n} = -\frac{2}{\sqrt{3}}\ln\left(\frac{2-\sqrt{3}}{2\sqrt{2}}\right), \quad (39)$$

$$\sum_{n=1}^{\infty} \frac{C_n}{8^n}\left(H_{2n} - \frac{1}{2}H_n\right) = 4\left(1 - \frac{1}{\sqrt{2}} - \frac{\ln 2}{2\sqrt{2}}\right), \quad (40)$$

$$\sum_{n=1}^{\infty} \frac{C_n}{16^n}\left(H_{2n} - \frac{1}{2}H_n\right) = 8\left(1 - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\ln\frac{\sqrt{3}}{2}\right). \quad (41)$$

4 Conclusion

In this paper we presented new closed forms for some types of series involving the central binomial coefficients $\binom{2n}{n}$. To prove our results, we used some generating functions, combined with basic differentiation and integration. Using similar techniques, we established series evaluations involving Harmonic numbers with Fibonacci and Lucas sequences. In addition, readers can exploit (29), (30) and (31) to generate more exotic series. To assure accuracy of the results, we verified all the series via *Mathematica* 13.3.

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References

[1] Abramowitz, M., & Stegun, I. A. (1965). *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables* (Vol. 55). Courier Corporation.

[2] Adegoke, K., Frontczak, R., & Goy, T. (2024). On some series involving the binomial coefficients $C(3n, n)$.*Notes on Number Theory and Discrete Mathematics*, 30(2), 319–334.

[3] Bhandari, N. (2022). Binomial sums resulting to the ratio, G/π and $1/\pi$. *Archimede Mathematical Journal*, 9(1), 4–11.

[4] Chen, H. (2016). Interesting series associated with central binomial coefficients, Catalan numbers and harmonic numbers. *Journal of Integer Sequences*, 19(1), Article 16.1.5.

[5] Chen, H. (2022). Interesting Ramanujan-like series associated with powers of central binomial coefficients. *Journal of Integer Sequences*, 25(1), Article 22.1.8.

[6] Graham, R. L., Knuth, D. E., & Patashnik, O. (1994). *Concrete Mathematics*. (2nd ed.). Addison-Wesley Publishing Company, Inc.