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Correction notice to "Leonardo's bivariate and complex polynomials" [Notes on Number Theory and Discrete Mathematics, 2022, Volume 28, Number 1, Pages 115–123]

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1 Correction

The present notice provides the corrected forms of the generating functions for Leonardo polynomials, Leonardo bivariate polynomials, and Leonardo complex bivariate polynomials, which were previously presented in [1] by Mangueira, Vieira, Alves, and Catarino.

We first recall the definitions of these polynomials given in [1].

The n-th Leonardo's polynomial $l_n(x)$ is given by

$$l_n(x) = 2xl_{n-1}(x) - l_{n-2}(x), \quad (n \geq 3) \tag{1}$$

with initial conditions $l_0(x) = l_1(x) = 1$ and $l_2(x) = 3$.

The n-th Leonardo's bivariate polynomial $l_n(x, y)$ is defined by

$$l_n(x, y) = 2xl_{n-1}(x, y) - yl_{n-3}(x, y), \quad (n \geq 3)$$

with initial conditions $l_0(x, y) = l_1(x, y) = 1$ and $l_2(x, y) = 3$.

The n-th Leonardo's complex bivariate polynomial $l_n(ix, y)$ has the recurrence relation

$$l_n(ix, y) = 2ixl_{n-1}(ix, y) - yl_{n-3}(ix, y), \quad (n \geq 3)$$

with initial conditions $l_0(ix, y) = l_1(ix, y) = 1, l_2(ix, y) = 3$ and $i^2 = -1$.

Following the definitions, we present the corrected forms of the generating functions of these polynomials, which were previously given in [1].

Theorem 2.4. *The generating function for Leonardo's polynomial sequence for $n \in \mathbb{N}$ is*

$$g(l_n(x), t) = \sum_{n=0}^{\infty} l_n(x)t^n = \frac{1 + (1 - 2x)t + (3 - 2x)t^2}{1 - 2xt + t^3}.$$

Proof. Let $g(l_n(x), t)$ be the generating function for Leonardo's polynomial sequence. Then,

$$\begin{aligned} g(l_n(x), t) &= l_0(x) + l_1(x)t + l_2(x)t^2 + l_3(x)t^3 + \cdots, \\ -g(l_n(x), t)2xt &= -l_0(x)2xt - l_1(x)2xt^2 - l_2(x)2xt^3 - l_3(x)2xt^4 - \cdots, \\ g(l_n(x), t)t^3 &= l_0(x)t^3 + l_1(x)t^4 + l_2(x)t^5 + l_3(x)t^6 + \cdots. \end{aligned}$$

The sum of the above equalities yields

$$\begin{aligned} g(l_n(x), t)(1 - 2xt + t^3) = l_0(x) &+ (l_1(x) - l_0(x)2x)t + (l_2(x) - l_1(x)2x)t^2 \\ &+ (l_3(x) - l_2(x)2x + l_0(x))t^3 + \cdots. \end{aligned}$$

By considering equation (1), the desired result follows immediately. □

Note that by substituting $x = 1$ into the generating function in Theorem 2.4, the generating function for the Leonardo number sequence, as given in [2], is obtained.

Theorem 3.4. *The generating function for Leonardo's bivariate polynomial sequence for $n \in \mathbb{N}$ is*

$$g(l_n(x, y), t) = \sum_{n=0}^{\infty} l_n(x, y)t^n = \frac{1 + (1 - 2x)t + (3 - 2x)t^2}{1 - 2xt + yt^3}.$$

Proof. The desired result can be derived using the technique applied in the proof of Theorem 2.4. However, to avoid repetition, an alternative approach will be employed here. Let $g(l_n(x, y), t)$ denote the generating function for Leonardo's bivariate polynomial sequence. Then,

$$\begin{aligned} g(l_n(x, y), t) &= l_0(x, y) + l_1(x, y)t + l_2(x, y)t^2 + \sum_{n=3}^{\infty} l_n(x, y)t^n \\ &= 1 + t + 3t^2 + \sum_{n=3}^{\infty} l_n(x, y)t^n \\ &= 1 + t + 3t^2 + \sum_{n=3}^{\infty} (2xl_{n-1}(x, y) - yl_{n-3}(x, y))t^n \\ &= 1 + t + 3t^2 + 2xt \sum_{n=3}^{\infty} l_{n-1}(x, y)t^{n-1} - yt^3 \sum_{n=3}^{\infty} l_{n-3}(x, y)t^{n-3} \\ &= 1 + t + 3t^2 + 2xt \sum_{n=2}^{\infty} l_n(x, y)t^n - yt^3 \sum_{n=3}^{\infty} l_{n-3}(x, y)t^{n-3} \end{aligned}$$

$$
\begin{aligned}
&= 1 + t + 3t^2 + 2xt\left(\sum_{n=0}^{\infty} l_n(x,y)t^n - l_0(x,y) - l_1(x,y)t\right) \\
&\quad - yt^3 \sum_{n=0}^{\infty} l_n(x,y)t^n \\
&= 1 + t + 3t^2 + 2xt\left(\sum_{n=0}^{\infty} l_n(x,y)t^n - 1 - t\right) - yt^3 \sum_{n=0}^{\infty} l_n(x,y)t^n.
\end{aligned}
$$

Thus,

$$g(l_n(x,y),t)(1 - 2xt + yt^3) = 1 + (1-2x)t + (3-2x)t^2$$

provides the final step in completing the proof. □

Substituting $x = y = 1$ into the generating function in Theorem 3.4 yields the generating function for the Leonardo number sequence, as given in [2].

Theorem 4.4. *The generating function for Leonardo's complex bivariate polynomial sequence for $n \in \mathbb{N}$ is*

$$g(l_n(ix,y),t) = \sum_{n=0}^{\infty} l_n(ix,y)t^n = \frac{1 + (1-2ix)t + (3-2ix)t^2}{1 - 2ixt + yt^3}.$$

Proof. The desired result is obtained by applying either of the techniques used in the proofs of Theorem 2.4 or Theorem 3.4. □

References

[1] Mangueira, M. C. d. S., Vieira, R. P. M., Alves, F. R. V., & Catarino, P. M. M. C. (2022). Leonardo's bivariate and complex polynomials. *Notes on Number Theory and Discrete Mathematics*, 28(1), 115–123.

[2] Catarino, P., & Borges, A. (2019). On Leonardo numbers. *Acta Mathematica Universitatis Comenianae*, 89(1), 75–86.