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A note on a bivariate Leonardo sequence

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Abstract: Recently, quite a few generalizations of Leonardo numbers have emerged in the literature. In this short note, we propose a new bivariate extension and provide its generating function. We correct the generating function of another recently proposed bivariate generalization.

Keywords: Leonardo sequence, Generating function, Recurrence relations, Hessenberg matrices, Determinant.
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1 Introduction

The Leonardo sequence (Le_n) is defined by the recurrence relation

$$\mathrm{Le}_n = \mathrm{Le}_{n-1} + \mathrm{Le}_{n-2} + 1\,, \quad \text{for } n \geqslant 2,$$

with initial conditions $\mathrm{Le}_0 = \mathrm{Le}_1 = 1$. The first few terms of the Leonardo sequence are
$1,\ 1,\ 3,\ 5,\ 9,\ 15,\ 25,\ 41,\ 67,\ 109,\ 177,\ 287,\ 465,\ \ldots\ .$

This sequence can be seen as a possible inhomogeneous version of the Fibonacci sequence [13, 18]. In fact, the Leonardo sequence is a very particular case of other more general sequences, with applications by the Turing Laureate, Edsger Dijkstra, in adaptive comparison computer sorting [5], just as their related Fibonacci numbers have sorting applications [1].

Alternatively, an elementary algebraic manipulation provides the homogeneous recurrence relation

$$\mathrm{Le}_n = 2\,\mathrm{Le}_{n-1} - \mathrm{Le}_{n-3}\,, \quad \text{for } n \geqslant 3, \tag{1}$$

for Leonardo numbers.

Among the representations for Leonardo sequence, the one in terms of the determinant in terms of a Hessenberg matrix is perhaps the least explored. Recall that for any constants $b_1, \ldots, b_r$, the sequence (a_n) defined by the homogeneous recurrence relation

$$a_n = p_{n,n-1}\, a_{n-1} + \cdots + p_{n,n-r}\, a_{n-r}\,, \tag{2}$$

for $n > r$, with initial conditions

$$a_1 = b_1\,, \ldots\,, a_r = b_r\,, \tag{3}$$

can be given explicitly as

$$a_n = \det \begin{pmatrix} b_1 & b_2 & \cdots & b_r & & & \\ -1 & 0 & \cdots & 0 & p_{r+1,1} & & & \\ & -1 & \ddots & \vdots & \vdots & \ddots & & \\ & & \ddots & 0 & \vdots & & \ddots & \\ & & & -1 & p_{r+1,r} & & & p_{n,n-r} \\ & & & & -1 & \ddots & & \vdots \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & -1 & p_{n,n-1} \end{pmatrix}. \tag{4}$$

For details and general applications the reader is referred to [11, 14, 20]. A general result can be found for example in [19, Theorem 4.20]. For the the permanent, one just needs to replace the -1's of the subdiagonal by $1\acute{s}$ (cf. [4]).

Therefore, one can write

$$\mathrm{Le}_n = \det \begin{pmatrix} 1 & 1 & 3 & & & & \\ -1 & 0 & 0 & -1 & & & \\ & -1 & 0 & 0 & \ddots & & \\ & & -1 & 2 & \ddots & \ddots & \\ & & & -1 & \ddots & \ddots & -1 \\ & & & & \ddots & \ddots & 0 \\ & & & & & -1 & 2 \end{pmatrix} \tag{5}$$

or, equivalently,

$$\mathrm{Le}_n = \det \begin{pmatrix} 1 & -1 & 1 & & & & \\ -1 & 2 & 0 & -1 & & & \\ & -1 & 2 & 0 & \ddots & & \\ & & -1 & 2 & \ddots & \ddots & \\ & & & -1 & \ddots & \ddots & -1 \\ & & & & \ddots & \ddots & 0 \\ & & & & & -1 & 2 \end{pmatrix}. \tag{6}$$

On the other hand, a generating function constitutes an unavoidable element in the study of any sequence. In [6] we may find the following result:

Theorem 1.1. *The generating function of the sequence (d_n) defined by the determinants*

$$d_n = \det \begin{pmatrix} b_0 & b_1 & \cdots & b_k & & & \\ -1 & c_1 & c_2 & \cdots & c_\ell & & \\ & -1 & c_1 & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & c_\ell \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & -1 & c_1 & c_2 \\ & & & & & -1 & c_1 \end{pmatrix}_{(n+1)\times(n+1)}$$

is

$$\sum_{n=0}^{\infty} d_n z^n = \frac{b_0 + b_1 z + \cdots + b_k z^k}{1 - c_1 z - \cdots - c_\ell z^\ell}.$$

In the previous matrix, we are assuming that $k < \ell$. The reader may read [10] for an earlier look and historical context, and [3, 12] for some applications.

Consequently, from Theorem 1.1 and (6), we readily get the well-known generating function:

$$\sum_{n=0}^{\infty} \mathrm{Le}_n z^n = \frac{1 - z + z^2}{1 - 2x + z^3} = \frac{1 - z + z^2}{(1 - z)(1 - z - z^2)}.$$

The purpose of this note is to provide the correct generating function for a bivariate Leonardo sequence proposed recently and suggest a new extension.

2 A new bivariate Leonardo sequence

Just as the Fibonacci sequence motivated versions with several variables over time [7–9, 15–17], recently the recurrence relation (1) has taken different extensions (see, for example, [2, 13, 18]). In general, the first two terms of the extensions are 1's.

In [13], it is considered the bivariate Leonardo $(\mathrm{Le}_n^{(x,y)})$ sequence defined by the recurrence relation

$$\mathrm{Le}_n^{(x,y)} = 2x\,\mathrm{Le}_{n-1}^{(x,y)} - y\,\mathrm{Le}_{n-3}^{(x,y)}, \quad \text{for } n \geqslant 3, \tag{7}$$

with initial conditions as for the standard Leonardo numbers. Accordingly,

$$\begin{aligned}
\mathrm{Le}_n^{(x,y)} &= \det\begin{pmatrix}
1 & 1 & 3 & & & & \\
-1 & 0 & 0 & -y & & & \\
 & -1 & 0 & 0 & \ddots & & \\
 & & -1 & 2x & \ddots & \ddots & \\
 & & & -1 & \ddots & \ddots & -y \\
 & & & & \ddots & \ddots & 0 \\
 & & & & & -1 & 2x
\end{pmatrix} \\
&= \det\begin{pmatrix}
1 & 1-2x & 3-2x & & & & \\
-1 & 2x & 0 & -y & & & \\
 & -1 & 2x & 0 & \ddots & & \\
 & & -1 & 2x & \ddots & \ddots & \\
 & & & -1 & \ddots & \ddots & -y \\
 & & & & \ddots & \ddots & 0 \\
 & & & & & -1 & 2x
\end{pmatrix}.
\end{aligned}$$

Therefore, from Theorem 1.1, the generating function of the sequence $(\mathrm{Le}_n^{(x,y)})$ is

$$\sum_{n=0}^{\infty} \mathrm{Le}_n^{(x,y)} z^n = \frac{1+(1-2x)z+(3-2x)z^2}{1-2xz+yz^3}\,.$$

This means that the generating function provided in Theorem 3.4 of [13], namely,

$$\frac{1-5z+z^2}{1-2xz+yz^3}\,,$$

is not correct.

3 Conclusion

Our aim with this short note has been to reconsider the recurrence relation (7), but now with the initial conditions that we feel somewhat more appropriate:

$$\mathrm{Le}_0^{(x,y)} = 1\,, \quad \mathrm{Le}_1^{(x,y)} = x \quad \text{and} \quad \mathrm{Le}_2^{(x,y)} = 2x^2 + y\,. \tag{8}$$

The first few terms of this new sequence are given in the following table:

n	$\mathrm{Le}_n^{(x,y)}$
0	1
1	x
2	$2x^2+y$
3	$4x^3+2xy-y$
4	$8x^4+4x^2y-3xy$
5	$16x^5+8x^3y-8x^2y-y^2$
6	$32x^6+16x^4y-20x^3y-4xy^2+y^2$

which, of itself, is worth exploring further for various values of the two variables.

In terms of the matricial representation, we have

$$\mathrm{Le}_n^{(x,y)} = \det \begin{pmatrix} 1 & -x & y & & & & \\ -1 & 2x & 0 & -y & & & \\ & -1 & 2x & 0 & \ddots & & \\ & & -1 & 2x & \ddots & \ddots & \\ & & & -1 & \ddots & \ddots & -y \\ & & & & \ddots & \ddots & 0 \\ & & & & & -1 & 2x \end{pmatrix}.$$

Consequently, the generating function is

$$\sum_{n=0}^{\infty} \mathrm{Le}_n^{(x,y)} z^n = \frac{1 - xz + yz^2}{1 - 2xz + yz^3}.$$

Taking into account this function, we believe that this might be a natural way to define a bivariate extension for the Leonardo numbers. Furthermore, together with some specification of the values x and y, it can also give rise to a plethora of identities which we leave for the interested reader to explore.

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