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A note on the approximation of divisor functions

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Abstract: We offer an arithmetic proof of a result from the recent paper [1]. A more general result is provided, too.
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1 Introduction

For a positive integer $m \geq 1$, and a real number α, the divisor functions of m are defined by

$$\sigma_\alpha(m) = \sum_{d|m} d^\alpha,$$

where d runs through all positive divisors of m. In a recent paper [1], by using interesting, but complicated analytical arguments, the authors prove the following theorem:

Theorem 1. *One has the following limit:*

$$\lim_{m\to\infty} \frac{\sigma_\alpha(m!)}{(m!)^\alpha} = \zeta(\alpha), \tag{1}$$

where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad (s > 1)$$

is the Riemann zeta function of real arguments. As clearly

$$\sigma_\alpha(m) = \sum_{d|m} \left(\frac{m}{d}\right)^\alpha = m^\alpha \cdot \sum_{d|m} \frac{1}{d^\alpha} \le m^\alpha \cdot \zeta(\alpha),$$

one has $\sigma_\alpha(m)/m^\alpha \le \zeta(\alpha)$, *and by* (1) *we obtain a new proof of a classical result by Grönwall* [2], *that*

$$\lim_{m\to\infty} \sup \frac{\sigma_\alpha(m)}{m^\alpha} = \zeta(\alpha). \tag{2}$$

In what follows, we will obtain a simple proof of (1), based on Legendre's classical formula for the prime factorizations of $m!$, combined with some elementary inequalities of real analysis.

2 Auxiliary results

Lemma 1 (Legendre)**.** *Let* $\gamma_p(m)$ *be the exponent of the prime* p *in the prime factorization of* $m!$. *Then*

$$\gamma_p(m) = \sum_{j=1}^{\infty} \left[\frac{m}{p^j}\right],$$

where $[x]$ *denotes the integer part of* x.

For a proof of this classical result, see e.g. [3].

Lemma 2 (Euler)**.** *One has the identity*

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \tag{3}$$

This is another classical result, see [3].

Lemma 3. *For any real number* $0 < x \le \frac{1}{4}$ *one has*

$$\log(1-x) > -\frac{4}{3}x. \tag{4}$$

Proof. Let $f(x) = \log(1-x) + \frac{4}{3}x$, $x \in \left[0, \frac{1}{4}\right]$. Then $f'(x) = \frac{1-4x}{3(1-x)} \ge 0$, so the function f is strictly increasing, implying that $f(x) > f(0) = 0$, so relation (4) follows. □

Lemma 4. *Let* p *denote a prime number, and let* $a > 1$. *Then*

$$\sum_{a<p\le m} \frac{1}{p^2} < \frac{1}{a}. \tag{5}$$

Proof. One has

$$\sum_{a<p\le m} \frac{1}{p^2} < \sum_{p>a} \frac{1}{p^2} < \int_a^{+\infty} \frac{dt}{t^2} = \frac{1}{a},$$

so inequality (5) follows. □

Lemma 5.

$$\lim_{s\to\infty} \zeta(s) = 1. \tag{6}$$

Proof. Clearly

$$1 < \zeta(s) \le 1 + \frac{1}{2^s} + \int_2^\infty \frac{1}{t^s} dt = 1 + \frac{1}{2^s} + \frac{1}{2^{s-1}\cdot(s-1)},$$

so by letting $s \to \infty$, relation (6) follows. □

3 Proof of Theorem 1

For any prime power p^r, we have

$$\sigma_\alpha(p^r) = 1 + p^r + \cdots + p^{r\alpha} = \frac{p^{(r+1)\alpha} - 1}{p^\alpha - 1} = p^{r\alpha} \cdot \frac{1 - \frac{1}{p^{(r+1)\alpha}}}{1 - \frac{1}{p^\alpha}},$$

so by the multiplicativity of the function σ_α one has

$$\begin{aligned} \sigma_\alpha(m!) &= \prod_{p\le m} \left[p^{\gamma_p(m)\alpha} \cdot \frac{1 - \frac{1}{p^{(\gamma_p(m)+1)\alpha}}}{1 - \frac{1}{p^\alpha}} \right] \\ &= \frac{m!^\alpha}{\prod_{p\le m}(1 - \frac{1}{p^\alpha})} \cdot \prod_{p\le m} \left(1 - \frac{1}{p^{(\gamma_p(m)+1)\alpha}}\right), \end{aligned} \tag{7}$$

where we have used Lemma 1. Thus

$$\frac{\sigma_\alpha(m!)}{m!^\alpha} = \prod_{p\le m} \frac{1}{1 - p^{-\alpha}} \cdot A(m),$$

where

$$A(m) = \prod_{p\le m} \left(1 - \frac{1}{p^{(\gamma_p(m)+1)\alpha}}\right).$$

Thus by Lemma 2, in order to prove Theorem 1, one has to show that

$$\lim_{m\to\infty} A(m) = 1. \tag{8}$$

As

$$A(m) > \prod_{p\le m} \left(1 - \frac{1}{p^{\gamma_p(m)+1}}\right) = A_1(m),$$

and $A_1(m) < 1$, it will be sufficient to consider $A_1(m)$.

Let now write $A_1(m)$ as $A_1(m) = B(m).C(m)$, where

$$B(m) = \prod_{p\le a} \left(1 - \frac{1}{p^{\gamma_p(m)+1}}\right) \tag{9}$$

and

$$C(m) \prod_{a<p\le m} \left(1 - \frac{1}{p^{\gamma_p(m)+1}}\right) \tag{10}$$

We will select a so that, when $m \to \infty$, then $a \to \infty$. Put $a = \frac{m}{k}$. First, remark that in $B(m)$, since $\gamma_p(m) \geq \left[\frac{m}{p}\right] \geq k$ (by Lemma 1), one has $\gamma_p(m) + 1 \geq k + 1$, so

$$B(m) \geq \prod_{p \leq m/k} \left(1 - \frac{1}{p^{k+1}}\right) > \prod_{p \text{ prime}} \left(1 - \frac{1}{p^{k+1}}\right) = \frac{1}{\zeta(k+1)}.$$

Let e.g. $k = [\sqrt{m}] \to \infty$, when $m \to \infty$. Then clearly, by $1 > B(m) > \frac{1}{\zeta(k+1)}$, we get $\lim_{m\to\infty} B(m) = 1$, by Lemma 5.

Now, for $C(m)$, remark that

$$C(m) \geq \prod_{m/k < p \leq m} \left(1 - \frac{1}{p^2}\right),$$

only by Lemma 3 (applied to $x = \frac{1}{p^2} \leq \frac{1}{4}$),

$$C(m) \geq \exp\left\{ \sum_{m/k < p \leq m} \log\left(1 - \frac{1}{p^2}\right)\right\} > \exp\left\{ -\frac{4}{3} \cdot \sum_{m/k < p \leq m} \frac{1}{p^2}\right\} > \exp\left\{ -\frac{4}{3} \cdot \frac{k}{m}\right\},$$

by Lemma 4. Thus $\exp\left\{-\frac{4}{3}\cdot\frac{k}{m}\right\} < C(m) < 1$, and $\frac{k}{m} \to 0$, as $m \to \infty$, so we get $\lim_{m\to\infty} C(m) = 1$.

Thus, finally, relation (8) is proved.

The above proof shows that, the following generalization of Theorem 1 holds true.

Theorem 2.

$$\lim_{m\to\infty} \frac{\sigma_\alpha((m!)^t)}{(m!)^{t\alpha}} = \zeta(t\alpha), \tag{11}$$

for any positive integer $t \geq 1$.

The proof of this generalization is similar, and we omit the details.

References

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