Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 1, 41–53
DOI: 10.7546/nntdm.2025.31.1.41-53

On tridimensional Lucas-balancing numbers and some properties

J. Chimpanzo [1,2], **P. Catarino** [3,4] **and M. V. Otero-Espinar** [5,6]

[1] Department of Exact Sciences, Higher Polytechnic Institute of Soyo
Zaire, Angola

[2] CMAT-UTAD, Polo of CMAT, University of Minho
4710-057 Braga, Portugal
e-mail: jalchimpanzo@gmail.com

[3] Department of Mathematics, University of Trás-os-Montes e Alto Douro
5000-801 Vila Real, Portugal

[4] CMAT-UTAD, Polo of CMAT, University of Minho
4710-057 Braga, Portugal
e-mail: pcatarin@utad.pt

[5] Departamento de Estatística, Análise Matemática e Optimización,
Universidade de Santiago de Compostela
15782 Santiago de Compostela, Spain

[6] Galician Center for Mathematical Research and Technology
15782 Santiago de Compostela, Spain
e-mail: mvictoria.otero@usc.es

Received: 22 May 2024 **Revised:** 28 January 2025
Accepted: 28 March 2025 **Online First:** 2 April 2025

Abstract: In this article, we introduce the tridimensional version of the Lucas-balancing numbers based on the unidimensional version, and we also study some of their properties and sum identities.

Keywords: Balancing numbers, Lucas-balancing numbers, Tridimensional Lucas-balancing numbers.

2020 Mathematics Subject Classification: 11B39, 11B83.

1 Introduction

Behera and Panda [1] introduced the concept of balancing numbers. In particular, a natural number n is a balancing number if it is a solution of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

for some natural number r, called the balancer corresponding to n. If n is a balancing number, $8n^2+1$ is a perfect square and its square root is called a Lucas-balancing number.

In [7], Panda introduced the Lucas-balancing numbers C_n the form of $C_n = \sqrt{8B_n^2+1}$, where B_n is called the balancing number of order n. The recurrence relation for the balancing numbers is

$$B_{n+2} = 6B_{n+1} - B_n, \tag{1}$$

with initial conditions $B_0 = 0$ and $B_1 = 1$. In the case of Lucas-balancing numbers $\{C_n\}_{n\geq 0}$, the recurrence relation is

$$C_{n+2} = 6C_{n+1} - C_n, \tag{2}$$

with initial conditions $C_0 = 1$ and $C_1 = 3$.

Remark 1.1. *The recurrence relation of the balancing numbers is the same as that of the Lucas-balancing numbers, but differs in its initial conditions.*

These two numerical sequences defined above are in *The On-Line Encyclopedia of Integers Sequences*® (OEIS®) [11], namely, A001109 for balancing numbers and A001541 for Lucas-balancing numbers, and the following Table 1 gives us their first few elements. Some detailed studies on these numbers are also available at [4–6, 8–10].

Table 1. Some first element of the sequences $\{B_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$

n	0	1	2	3	4	5	6	7	8	9
B_n	0	1	6	35	204	1189	6930	40391	235416	1372105
C_n	1	3	17	99	577	3363	19601	114243	665857	3880899

In this paper, our purpose is to introduce and study tridimensional Lucas-balancing numbers, as well as some of their properties and sum identities.

2 Tridimensional Lucas-balancing numbers

In this section, we introduce the tridimensional version of these numbers, based on the studies in [3]. Consider following definition:

Definition 2.1. *For n, m and p any non-negative integers, the numbers $C_{(n,m,p)}$ represent the tridimensional Lucas-balancing numbers that satisfy the following recurrence relations:*

$$\begin{cases} C_{(n+1,m,p)} = 6C_{(n,m,p)} - C_{(n-1,m,p)}, \\ C_{(n,m+1,p)} = 6C_{(n,m,p)} - C_{(n,m-1,p)}, \\ C_{(n,m,p+1)} = 6C_{(n,m,p)} - C_{(n,m,p-1)}, \end{cases}$$

with the initial conditions $C_{(0,0,0)} = 1$, $C_{(1,0,0)} = 3$, $C_{(0,1,0)} = C_{(0,0,1)} = 1+i$, $C_{(1,1,0)} = C_{(1,0,1)} = 3+i$, $C_{(0,1,1)} = 1+2i$, $C_{(1,1,1)} = 3+2i$ and $i^2 = -1$.

In the following, we will present some properties related with tridimensional Lucas-balancing numbers.

2.1 Some properties of tridimensional Lucas-balancing numbers

In this subsection, we state some properties of tridimensional Lucas-balancing numbers involving balancing and Lucas-balancing numbers in unidimensional version.

Lemma 2.1. *Let B_j and C_j be the balancing and Lucas-balancing numbers of order j, respectively. Then the following properties are valid for tridimensional Lucas-balancing numbers:*

1. $C_{(n,0,0)} = C_n$;
2. $C_{(0,m,0)} = C_{(0,0,m)} = (B_m - B_{m-1}) + B_m i$;
3. $C_{(n,1,0)} = C_{(n,0,1)} = C_n + (B_n - B_{n-1})\, i$;
4. $C_{(n,1,1)} = C_n + 2\,(B_n - B_{n-1})\, i$;
5. $C_{(1,m,0)} = C_{(1,0,m)} = 3\,(B_m - B_{m-1}) + B_m i$;
6. $C_{(0,m,1)} = C_{(0,1,m)} = (B_m - B_{m-1}) + (2B_m - B_{m-1})\, i$;
7. $C_{(1,m,1)} = C_{(1,1,m)} = 3\,(B_m - B_{m-1}) + (2B_m - B_{m-1})\, i$;
8. $C_{(n,m,0)} = C_{(n,0,m)} = C_n\,(B_m - B_{m-1}) + (B_n - B_{n-1})\, B_m i$;
9. $C_{(0,m,p)} = (B_m - B_{m-1})\,(B_p - B_{p-1}) + \Big(B_m\,(B_p - B_{p-1}) + (B_m - B_{m-1})\, B_p\Big) i$;
10. $C_{(n,m,1)} = C_{(n,1,m)} = C_n\,(B_m - B_{m-1}) + (B_n - B_{n-1})\,(2B_m - B_{m-1})\, i$;
11. $C_{(1,m,p)} = 3\,(B_m - B_{m-1})\,(B_p - B_{p-1}) + \Big(B_m\,(B_p - B_{p-1}) + (B_m - B_{m-1})\, B_p\Big) i$.

Proof.

- 1. The proof is performed by induction on n.

  For $n = 0$ and given the value of C_0 in Table 1 and given one of the initial conditions of the sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(0,0,0)} = 1 = C_0$ and the Lemma is true.

  For $n = 1$ and considering that $C_1 = 3$ in Table 1 and taking into account one of the initial conditions of sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(1,0,0)} = 3 = C_1$ and the Lemma is also true.

  Suppose that the proposition is true for all integer $k \leq n$. Let us show that it remains true for $n+1$.

Then, by the first recurrence relation of Definition 2.1, the induction hypothesis and by (2), we get

$$\begin{aligned} C_{(n+1,0,0)} &= 6C_{(n,0,0)} - C_{(n-1,0,0)} \\ &= 6C_n - C_{n-1} \\ &= C_{n+1}, \end{aligned}$$

which completes the proof of item 1.

- 2. The proof is done once more by induction, now on m.

  Let us first prove that $C_{(0,m,0)} = C_{(0,0,m)}$.

  For $m = 0$ and taking into account the first initial conditional of the sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(0,0,0)} = 1$ and the equality is true.

  For $m = 1$ and taking into account the third initial conditional of the sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(0,1,0)} = 1 + i = C_{(0,0,1)}$ and the equality is also true.

  Suppose that $C_{(0,k,0)} = C_{(0,0,k)}$ for any integer $k \leq n$. We want to show that this is still true for $m + 1$.

  Then, by second and third recurrence relation of Definition 2.1, we get

$$\begin{aligned} C_{(0,m+1,0)} &= 6C_{(0,m,0)} - C_{(0,m-1,0)} \\ &= 6C_{(0,0,m)} - C_{(0,0,m-1)} \\ &= C_{(0,0,m+1)}, \end{aligned}$$

  which shows that $C_{(0,m,0)} = C_{(0,0,m)}$.

  We will now prove that $C_{(0,0,m)} = (B_m - B_{m-1}) + B_m i$.

  For $m = 0$ and taking into account the values of B_0 and $B_{-1} = -B_1$ in [3] and Table 1 and considering one of the initial conditions of the sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(0,0,0)} = 1 = (B_0 - B_{-1}) + B_0 i$ and the Lemma is valid.

  For $m = 1$ and, once again, considering the values of B_0 and B_1 in Table 1 and also taking into account one of the initial conditions of the sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(0,0,1)} = 1 + i = (B_1 - B_0) + B_1 i$ and the Lemma is also valid.

  Suppose that $C_{(0,0,m)} = (B_m - B_{m-1}) + B_m$ is valid for any integer less than or equal to m and let us prove that it remains true for $m + 1$.

  Then, using the third recurrence relation of Definition 2.1, the induction hypothesis, and by (1),

$$\begin{aligned} C_{(0,0,m+1)} &= 6C_{(0,0,m)} - C_{(0,0,m-1)} \\ &= 6\Big((B_m - B_{m-1}) + B_m i\Big) - \Big((B_{m-1} - B_{m-2}) + B_{m-1} i\Big) \\ &= 6\left(B_m - B_{m-1}\right) + 6B_m i - \left(B_{m-1} - B_{m-2}\right) - B_{m-1} i \\ &= \Big((6B_m - B_{m-1}) - (6B_{m-1} - B_{m-2})\Big) + \left(6B_m - B_{m-1}\right) i \\ &= (B_{m+1} - B_m) + B_{m+1} i, \end{aligned}$$

  which is true.

  By the transitivity of the equality relation, item 2 is proven.

- 3. The proof is performed by induction on n.

  Let us start by proving that $C_{(n,1,0)} = C_{(n,0,1)}$.

  For $n = 0$ and given the third initial conditional of the sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(0,1,0)} = 1 + i = C_{(0,0,1)}$ and the equality holds.

  For $n = 1$ and considering the fourth initial conditional of the sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(1,1,0)} = 3 + i = C_{(1,0,1)}$ and equality also holds.

  Suppose that $C_{(k,1,0)} = C_{(k,0,1)}$ for all integers $k \leq n$. Then we want to show that this remains true for $n + 1$.

  Then, applying first recurrence relation of Definition 2.1 and by the induction hypothesis, we get

$$\begin{aligned} C_{(n+1,1,0)} &= 6C_{(n,1,0)} - C_{(n-1,1,0)} \\ &= 6C_{(n,0,1)} - C_{(n-1,0,1)} \\ &= C_{(n+1,0,1)}, \end{aligned}$$

  and the equality $C_{(n,1,0)} = C_{(n,0,1)}$ is verified.

  Let us now prove that $C_{(n,0,1)} = C_n + (B_n - B_{n-1})\, i$.

  For $n = 0$ and given that $B_0 = 0$ and $B_{-1} = -B_1$ in Table 1 and given one of the initial conditions of the sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(0,0,1)} = 1 + i = C_0 + (B_0 - B_{-1})\, i$, which is verified.

  For $n = 1$ and, once again, considering the values of C_1, B_0 and B_1 and also one of the initial conditions of sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(1,0,1)} = 3 + i = C_1 + (B_1 - B_0)\, i$ which is also verified.

  Suppose that the statement is true for all values less than or equal to n. Let us show that it is still true for $n + 1$:

  Then, by the first recurrence relation of Definition 2.1, the induction hypothesis, and by (1), we obtain

$$\begin{aligned} C_{(n+1,0,1)} &= 6C_{(n,0,1)} - C_{(n-1,0,1)} \\ &= 6\Big(C_n + (B_n - B_{n-1})\, i\Big) - \Big(C_{n-1} + (B_{n-1} - B_{n-2})\, i\Big) \\ &= 6C_n + 6\,(B_n - B_{n-1})\, i - C_{n-1} - (B_{n-1} - B_{n-2})\, i \\ &= (6C_n - C_{n-1}) + (6B_n - B_{n-1})\, i - (6B_{n-1} - B_{n-2})\, i \\ &= C_{n+1} + (B_{n+1} - B_n)\, i, \end{aligned}$$

  which is valid.

  Since the equality relation is transitive, the item 3 is proven.

- 4. The proof is carried out by induction on n.

  For $n = 0$ and given that $C_0 = 1$, $B_0 = 0$ and $B_{-1} = -B_1$ in [3] or Table 1 and one of the initial conditions of sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(0,1,1)} = 1 + 2i = C_0 + 2\,(B_0 - B_{-1})\, i$ and the proposition is verified.

For $n = 1$ and given the values of C_1 and B_1 in Table 1 and also one of the initial conditions of sequence $\{C_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that $C_{(1,1,1)} = 3 + 2i = C_1 + 2\,(B_1 - B_0)\,i$ and the proposition also is verified.

Suppose that the proposition is true for any integer $k \leq n$ and let us prove that it remains true for $n + 1$.

Then, by the first recurrence relation of Definition 2.1, by the induction hypothesis and by the recurrence relation (1), we get

$$\begin{aligned}
C_{(n+1,1,1)} &= 6C_{(n,1,1)} - C_{(n-1,1,1)} \\
&= 6\Big(C_n + 2\,(B_n - B_{n-1})\,i\Big) - \Big(C_{n-1} + 2\,(B_{n-1} - B_{n-2})\,i\Big) \\
&= 6C_n + 12\,(B_n - B_{n-1})\,i - C_{n-1} - 2\,(B_{n-1} - B_{n-2})\,i \\
&= (6C_n - C_{n-1}) + 2\Big((6B_n - B_{n-1})\,i - (6B_{n-1} - B_{n-2})\Big)i \\
&= C_{n+1} + 2\,(B_{n+1} - B_n)\,i,
\end{aligned}$$

so the property 4 is verified.

- As the results of items 5, 6 and 7 are similar to the previous ones (items 2 and 3), we have omitted the respective proofs.

- 8. The proof is first done by induction on m and n is fixed.

  First of all, let us prove that $C_{(n,m,0)} = C_{(n,0,m)}$.

  For $m = 0$ and considering item 1 of Lemma 2.1 we have that $C_{(n,0,0)} = C_n = C_{(n,0,0)}$, which is true.

  For $m = 1$ and taking into account Lemma 2.1, item 3, we have that $C_{(n,1,0)} = C_n + (B_n - B_{n-1})\,i = C_{(n,0,1)}$, which is also true.

  Suppose that $C_{(n,k,0)} = C_{(n,0,k)}$ for all integers $k \leq m$ and we want to show that this remains true for $m + 1$.

  Hence, using the second recurrence relation from Definition 2.1 and the induction hypothesis, we get

$$\begin{aligned}
C_{(n,m+1,0)} &= 6C_{(n,m,0)} - C_{(n,m-1,0)} \\
&= 6C_{(n,0,m)} - C_{(n,0,m-1)} \\
&= C_{(n,0,m+1)},
\end{aligned}$$

  and the equality $C_{(n,m+1,0)} = C_{(n,0,m+1)}$ is valid.

  For the second equality, we are going to proceed again by induction on m with fixed n.

  For $m = 0$, we get $C_n(B_0 - B_{-1}) + (B_n - B_{n-1})B_0 i = C_n = C_{(n,0,0)}$, which is true.

  For $m = 1$, we obtain $C_n(B_1 - B_0) + (B_n - B_{n-1})B_1 i = C_n + (B_n - B_{n-1})i = C_{(n,1,0)} = C_{(n,0,1)}$, which is also true.

  Suppose that $C_{(n,0,k)} = C_n(B_k - B_{k-1}) + (B_n - B_{n-1})B_k i$ for all integers $k \leq m$ and we are going to show that the second identity is still true for $m + 1$.

Hence, applying the third recurrence relation from Definition 2.1 together with the induction hypothesis, we obtain

$$\begin{aligned}
C_{(n,0,m+1)} &= 6C_{(n,0,m)} - C_{(n,0,m-1)} \\
&= 6\Big(C_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i\Big) \\
&\quad - \Big(C_n(B_{m-1} - B_{m-2}) + (B_n - B_{n-1})B_{m-1} i\Big) \\
&= C_n\Big(6(B_m - B_{m-1}) + (B_{m-1} - B_{m-2})\Big) + (B_n - B_{n-1})(6B_m - B_{m-1})i \\
&= C_n(6B_m - 6B_{m-1} - B_{m-1} + B_{m-2}) + (B_n - B_{n-1})B_{m+1} i \\
&= C_n\Big((6B_m - B_{m-1}) - (6B_{m-1} - B_{m-2})\Big) + (B_n - B_{n-1})B_{m+1} i \\
&= C_n(B_{m+1} - B_m) + (B_n - B_{n-1})B_{m+1} i,
\end{aligned}$$

which holds true.

Now let us perform induction on n.

Let us prove that $C_{(n,0,m)} = C_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i$, for m fixed.

For $n = 0$, we have $C_{(0,0,m)} = C_0(B_m - B_{m-1}) + (B_0 - B_{-1})B_m i = 1(B_m - B_{m-1}) + (0 - (-1))\,B_m i = (B_m - B_{m-1}) + B_m i = C_{(0,m,0)}$, and this is true.

For $n = 1$, we obtain $C_{(1,0,m)} = C_1(B_m - B_{m-1}) + (B_1 - B_0)B_m i = 3(B_m - B_{m-1}) + (1 - 0)B_m i = 3(B_m - B_{m-1}) + B_m i = C_{(1,m,0)}$, and this is also true.

Suppose that $C_{(k,0,m)} = C_k(B_m - B_{m-1}) + (B_k - B_{k-1})B_m i$ is true for any integers $k \leq n$ and we will show that it will be valid for $n + 1$.

Therefore, by Definition 2.1, the first recurrence, and by the induction hypothesis, we get

$$\begin{aligned}
C_{(n+1,0,m)} &= 6C_{(n,0,m)} - C_{(n-1,0,m)} \\
&= 6\Big(C_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i\Big) \\
&\quad - \Big(C_{n-1}(B_m - B_{m-1}) + (B_{n-1} - B_{n-2})B_m i\Big) \\
&= (B_m - B_{m-1})(6C_n - C_{n-1}) + \Big((6B_n - B_{n-1}) - (B_{n-1} - B_{n-2})\Big)B_m i \\
&= (B_m - B_{m-1})C_{n+1} + \Big((6B_n - B_{n-1}) - (6B_{n-1} - B_{n-2})\Big)B_m i \\
&= C_{n+1}(B_m - B_{m-1}) + (B_{n+1} - B_n)B_m i,
\end{aligned}$$

which is true. Now, by the law of transitivity, item 8 is also proven.

- As the proof of items 9, 10 and 11 is also done like the proof of the previous result, we omitted the respective proofs. □

The following theorem allows us to determine the element $C_{(n,m,p)}$, for any non-negative integers n, m and p, in terms of Lucas-balancing numbers.

Theorem 2.1. *For the non-negative integers m, n and p, the tridimensional Lucas-balancing numbers are defined in the form:*

$$C_{(n,m,p)} = C_n\left(B_m - B_{m-1}\right)\left(B_p - B_{p-1}\right) + \left(B_n - B_{n-1}\right)\Big(B_m\left(B_p - B_{p-1}\right) + \left(B_m - B_{m-1}\right)B_p\Big)i.$$

Proof. Let us start doing the induction on p, fixing n and m.

For $p = 0$ and, given the values of B_0 and $B_{-1} = -B_1$ in Table 1 and in [3], and by item 8 of Lemma 2.1, we have that

$$\begin{aligned} C_{(n,m,0)} &= C_n\left(B_m - B_{m-1}\right) + \left(B_n - B_{n-1}\right)B_m i \\ &= C_n\left(B_m - B_{m-1}\right)\left(B_0 - B_{-1}\right) + \left(B_n - B_{n-1}\right)\Big(B_m\left(B_0 - B_{-1}\right) + \left(B_m - B_{m-1}\right)B_0\Big)i, \end{aligned}$$

and the equality holds.

For $p = 1$ and again, given the values of $B_0 = 0$ and $B_1 = 1$ in Table 1 and by Lemma 2.1, item 10, we have that

$$\begin{aligned} C_{(n,m,1)} &= C_n\left(B_m - B_{m-1}\right) + \left(B_n - B_{n-1}\right)\left(2B_m - B_{m-1}\right)i \\ &= C_n\left(B_m - B_{m-1}\right)\left(B_1 - B_0\right) + \left(B_n - B_{n-1}\right)\Big(B_m\left(B_1 - B_0\right) + \left(B_m - B_{m-1}\right)B_1\Big)i, \end{aligned}$$

which is true.

Suppose that the theorem is true for any non-negative integer $k \leq p$ and let us show that it is still true for $p + 1$. Then, applying the third recurrence relation of Definition 2.1, the induction hypothesis and by (1), we obtain

$$\begin{aligned} C_{(n,m,p+1)} &= 6C_{(n,m,p)} - C_{(n,m,p-1)} \\ &= 6\Big(C_n\left(B_m - B_{m-1}\right)\left(B_p - B_{p-1}\right) \\ &\quad + \left(B_n - B_{n-1}\right)\Big(B_m\left(B_p - B_{p-1}\right) + \left(B_m - B_{m-1}\right)B_p\Big)i\Big) \\ &\quad - \Big(C_n\left(B_m - B_{m-1}\right)\left(B_{p-1} - B_{p-2}\right) \\ &\quad + \left(B_n - B_{n-1}\right)\Big(B_m\left(B_{p-1} - B_{p-2}\right) + \left(B_m - B_{m-1}\right)B_{p-1}\Big)i\Big) \\ &= 6C_n\left(B_m - B_{m-1}\right)\left(B_p - B_{p-1}\right) \\ &\quad + \left(B_n - B_{n-1}\right)\Big(B_m\left(B_{p-1} - B_{p-2}\right) + \left(B_m - B_{m-1}\right)B_{p-1}\Big)i\Big) \\ &= 6C_n\left(B_m - B_{m-1}\right)\left(B_p - B_{p-1}\right) \\ &\quad + 6\left(B_n - B_{n-1}\right)\Big(B_m\left(B_p - B_{p-1}\right) + \left(B_m - B_{m-1}\right)B_p\Big)i \\ &\quad - C_n\left(B_m - B_{m-1}\right)\left(B_{p-1} - B_{p-2}\right) \\ &\quad - \left(B_n - B_{n-1}\right)\Big(B_m\left(B_{p-1} - B_{p-2}\right) + \left(B_m - B_{m-1}\right)B_{p-1}\Big)i \\ &= C_n\left(B_m - B_{m-1}\right)\Big(\left(6B_p - B_{p-1}\right) - \left(6B_{p-1} - B_{p-2}\right)\Big) \\ &\quad + \left(B_n - B_{n-1}\right)\Big(B_m\Big(\left(6B_p - B_{p-1}\right) - \left(6B_{p-1} - B_{p-2}\right)\Big) \\ &\quad + \left(B_m - B_{m-1}\right)\left(6B_p - B_{p-1}\right)\Big)i \\ &= C_n\left(B_m - B_{m-1}\right)\left(B_{p+1} - B_p\right) \\ &\quad + \left(B_n - B_{n-1}\right)\Big(B_m\left(B_{p+1} - B_p\right) + \left(B_m - B_{m-1}\right)B_{p+1}\Big)i, \end{aligned}$$

as we wanted to prove.

Following the same reasoning used for induction on p, the result also holds for induction on m and n, when n and p are fixed, and m and p are fixed, respectively. Therefore, the theorem is true. □

2.2 Some sum identities of tridimensional Lucas-balancing numbers

In this subsection, we will study some identities of this tridimensional Lucas-balancing numbers, using certain properties that are inherent to it.

Identity 2.2.1. *The sum of the first* p *numbers* $C_{(n,m,s)}$ *of an odd index* s *is given by:*

$$\sum_{q=1}^{p} C_{(n,m,2q-1)} = \Big(C_n\left(B_m - B_{m-1}\right) + \left(B_n - B_{n-1}\right) B_m i\Big)\left(B_p^2 - B_p B_{p+1} + B_{2p}\right) + \left(B_n - B_{n-1}\right)\left(B_m - B_{m-1}\right) B_p^2 i.$$

Proof. Using Theorem 2.1, we have

$$\sum_{q=1}^{p} C_{(n,m,2q-1)} = \sum_{q=1}^{p}\Big(C_n\left(B_m - B_{m-1}\right)\left(B_{2q-1} - B_{2q-2}\right) + \left(B_n - B_{n-1}\right)\left(B_m\left(B_{2q-1} - B_{2q-2}\right) + \left(B_m - B_{m-1}\right) B_{2q-1}\right) i\Big).$$

Thus,

$$\begin{aligned}
\sum_{q=1}^{p} B_{(n,m,2q-1)} &= C_n\left(B_m - B_{m-1}\right)\sum_{q=1}^{p}\left(B_{2q-1} - B_{2q-2}\right) \\
&\quad + \left(B_n - B_{n-1}\right)\left(B_m \sum_{q=1}^{p}\left(B_{2q-1} - B_{2q-2}\right) + \left(B_m - B_{m-1}\right)\sum_{q=1}^{p} B_{2q-1}\right) i \\
&= C_n\left(B_m - B_{m-1}\right)\left(\sum_{q=1}^{p} B_{2q-1} - \sum_{q=1}^{p} B_{2q-2}\right) \\
&\quad + \left(B_n - B_{n-1}\right)\left(B_m\left(\sum_{q=1}^{p} B_{2q-1} - \sum_{q=1}^{p} B_{2q-2}\right) + \left(B_m - B_{m-1}\right)\sum_{q=1}^{p} B_{2q-1}\right) i \\
&= C_n\left(B_m - B_{m-1}\right)\left(\sum_{q=1}^{p} B_{2q-1} - \left(B_0 + \sum_{q=1}^{p-1} B_{2q}\right)\right) \\
&\quad + \left(B_n - B_{n-1}\right)\left(B_m\left(\sum_{q=1}^{p} B_{2q-1} - \left(B_0 + \sum_{q=1}^{p-1} B_{2q}\right)\right)\right. \\
&\quad \left. + \left(B_m - B_{m-1}\right)\sum_{q=1}^{p} B_{2q-1}\right) i \\
&= \Big(C_n\left(B_m - B_{m-1}\right) + \left(B_n - B_{n-1}\right) B_m i\Big)\left(\sum_{q=1}^{p} B_{2q-1} - \left(B_0 + \sum_{q=1}^{p-1} B_{2q}\right)\right) \\
&\quad + \left(B_n - B_{n-1}\right)\left(B_m - B_{m-1}\right)\sum_{q=1}^{p} B_{2q-1} i.
\end{aligned}$$

Using items (a) and (b) of Corollary 2.3.6 from [10] and taking into account the value of B_0 in Table 1, the result follows. □

Identity 2.2.2. *The sum of the first* p *numbers* $C_{(n,m,s)}$ *of an even index* s *can be described as follows:*

$$\sum_{q=1}^{p} C_{(n,m,2q)} = \Big(C_n\left(B_m - B_{m-1}\right) + \left(B_n - B_{n-1}\right) B_m i\Big)\left(B_p B_{p+1} - B_p^2\right) + \left(B_n - B_{n-1}\right)\left(B_m - B_{m-1}\right) B_p B_{p+1} i.$$

Proof. Applying Theorem 2.1, we have

$$\sum_{q=1}^{p} C_{(n,m,2q)} = \sum_{q=1}^{p}\Big(C_n\left(B_m - B_{m-1}\right)\left(B_{2q} - B_{2q-1}\right) + \left(B_n - B_{n-1}\right)\left(B_m\left(B_{2q} - B_{2q-1}\right) + \left(B_m - B_{m-1}\right) B_{2q}\right) i\Big).$$

Thus,

$$\begin{aligned}\sum_{q=1}^{p} C_{(n,m,2q)} &= C_n\left(B_m - B_{m-1}\right)\sum_{q=1}^{p}\left(B_{2q} - B_{2q-1}\right) \\ &\quad + \left(B_n - B_{n-1}\right)\left(B_m \sum_{q=1}^{p}\left(B_{2q} - B_{2q-1}\right) + \left(B_m - B_{m-1}\right)\sum_{q=1}^{p} B_{2q}\right) i \\ &= C_n\left(B_m - B_{m-1}\right)\left(\sum_{q=1}^{p} B_{2q} - \sum_{q=1}^{p} B_{2q-1}\right) \\ &\quad + \left(B_n - B_{n-1}\right)\left(B_m\left(\sum_{q=1}^{p} B_{2q} - \sum_{q=1}^{p} B_{2q-1}\right) + \left(B_m - B_{m-1}\right)\sum_{q=1}^{p} B_{2q}\right) i.\end{aligned}$$

The result follows, by Corollary 2.3.6, items (b) and (a) in [10]. □

Identity 2.2.3. *The sum of the first* p *numbers* $B_{(n,m,s)}$*, with index* s *being a non-negative integer, can be described as follows:*

$$\sum_{q=1}^{p} C_{(n,m,q)} = C_n\left(B_m - B_{m-1}\right) B_p + \frac{1}{4}\left(B_n - B_{n-1}\right)\Big(4 B_m B_p + \left(B_m - B_{m-1}\right)\left(B_{p+1} - B_p - 1\right)\Big) i.$$

Proof. By Theorem 2.1, we have

$$\sum_{q=1}^{p} C_{(n,m,q)} = \sum_{q=1}^{p}\Big(C_n\left(B_m - B_{m-1}\right)\left(B_q - B_{q-1}\right) + \left(B_n - B_{n-1}\right)\left(B_m\left(B_q - B_{q-1}\right) + \left(B_m - B_{m-1}\right) B_q\right) i\Big).$$

Thus,

$$\begin{aligned}
\sum_{q=1}^{p} C_{(n,m,q)} &= C_n \left(B_m - B_{m-1}\right) \sum_{q=1}^{p} \left(B_q - B_{q-1}\right) \\
&\quad + \left(B_n - B_{n-1}\right) \left(B_m \sum_{q=1}^{p} \left(B_q - B_{q-1}\right) + \left(B_m - B_{m-1}\right) \sum_{q=1}^{p} B_q\right) i \\
&= C_n \left(B_m - B_{m-1}\right) \left(\sum_{q=1}^{p} B_q - \sum_{q=1}^{p} B_{q-1}\right) \\
&\quad + \left(B_n - B_{n-1}\right) \left(B_m \left(\sum_{q=1}^{p} B_q - \sum_{q=1}^{p} B_{q-1}\right) + \left(B_m - B_{m-1}\right) \sum_{q=1}^{p} B_q\right) i \\
&= C_n \left(B_m - B_{m-1}\right) \left(\sum_{q=1}^{p} B_q - \left(B_0 + \sum_{q=1}^{p-1} B_q\right)\right) \\
&\quad + \left(B_n - B_{n-1}\right) \left(B_m \left(\sum_{q=1}^{p} B_q - \left(B_0 + \sum_{q=1}^{p-1} B_q\right)\right) + \left(B_m - B_{m-1}\right) \sum_{q=1}^{p} B_q\right) i.
\end{aligned}$$

The result follows by Proposition 2.6, item 6 in [2] and given the values of B_0 in Table 1. □

The following results in Identities 2.2.4 to 2.2.9 refer to the tridimensional versions of the Lucas-balancing numbers. Identities 2.2.4, 2.2.5, and 2.2.6 are done by induction on m, while identities 2.2.7 to 2.2.9 are done by induction on n. For the proofs of the next three identities, we can apply Proposition 2.3, items 5 and 4 from [3] and use the fact that $B_n^2 = ST_n$, where ST_n are triangular numbers, and also Proposition 2.6, item 7 from [2].

As the proofs are similar to those in the previous case, so we omitted them here.

Identity 2.2.4. *The sum of the first m numbers $C_{(n,s,p)}$ of an odd index s can be described by:*

$$\begin{aligned}
\sum_{l=1}^{m} C_{(n,2l-1,p)} =& \Big(C_n \left(B_p - B_{p-1}\right) + \left(B_n - B_{n-1}\right) B_p i\Big) \left(B_m^2 - B_m B_{m+1} - B_{2m}\right) \\
&+ \left(B_n - B_{n-1}\right) B_m^2 \left(B_p - B_{p-1}\right) i.
\end{aligned}$$

Identity 2.2.5. *The sum of the first m numbers $C_{(n,s,p)}$ of an even index s is defined by:*

$$\begin{aligned}
\sum_{l=1}^{m} C_{(n,2l,p)} =& \Big(C_n \left(B_p - B_{p-1}\right) + \left(B_n - B_{n-1}\right) B_p i\Big) \left(B_m B_{m+1} - B_m^2\right) \\
&+ \left(B_n - B_{n-1}\right) B_m B_{m+1} \left(B_p - B_{p-1}\right) i.
\end{aligned}$$

Identity 2.2.6. *The sum of the first m numbers $C_{(n,s,p)}$, with index s being a non-negative integer, is given as follows:*

$$\begin{aligned}
\sum_{l=1}^{m} C_{(n,2l,p)} =& B_n B_m \left(B_p - B_{p-1}\right) \\
&+ \frac{1}{4} \left(B_n - B_{n-1}\right) \Big(\left(B_{m+1} - B_m - 1\right) \left(B_p - B_{p-1}\right) + 4 B_m B_p \Big) i.
\end{aligned}$$

Identity 2.2.7. *The sum of the first n numbers $B_{(s,m,p)}$ of an odd index s is defined by:*

$$\sum_{k=1}^{n} C_{(2k-1,m,p)} = \frac{1}{4}\left(C_{n+1} - C_n - 2B_{n+1}^2 + 2B_n^2 + 4\right)\left(B_m - B_{m-1}\right)\left(B_p - B_{p-1}\right) + \left(B_n^2 - B_n B_{n+1} - B_{2n}\right)\Big(B_m\left(B_p - B_{p-1}\right) + \left(B_m - B_{m-1}\right) B_p\Big) i.$$

Identity 2.2.8. *The sum of the first n numbers $C_{(s,m,p)}$ of an even index s can be described by:*

$$\sum_{k=1}^{n} C_{(2k,m,p)} = \frac{1}{2}\left(B_{n+1}^2 - B_n^2 - 1\right)\left(B_m - B_{m-1}\right)\left(B_p - B_{p-1}\right) + \left(B_n B_{n+1} - B_n^2\right)\Big(B_m\left(B_p - B_{p-1}\right) + \left(B_m - B_{m-1}\right) B_p\Big) i.$$

Identity 2.2.9. *The sum of the first n numbers $C_{(s,m,p)}$, with s being a non-negative integer, is described as follows:*

$$\sum_{k=1}^{n} C_{(k,m,p)} = \frac{1}{2}\left(C_{n+1} - C_n + 2\right)\left(B_m - B_{m-1}\right)\left(B_p - B_{p-1}\right) + B_n\Big(B_m\left(B_p - B_{p-1}\right) + \left(B_m - B_{m-1}\right) B_p\Big) i.$$

3 Conclusion

This paper is a continuation of the work on bidimensional versions of Lucas-balancing numbers. We introduce the tridimensional recurrence relations of Lucas-balancing number and study some of their properties, as well as some of their sum identities. The results presented in this manuscript are considered as a contribution to the field of mathematics and offer an opportunity for researchers interested in this topic of number sequences.

Acknowledgements

The first two authors are members of the CMAT Research Center - CMAT-UTAD. The last author is a member of the Mathematics Research Group of the University of Santiago de Compostela - USC GI 2136 and also of the Homology, Homotopy and Categorical Invariants in Groups and Non-Associative Algebras Project - PID2020 115155GB I00, within the scope of the Dynamical Systems, Genetic Algebras and Non-Stochastic Quadratic Operators objective. This research was partially funded by FCT - Foundation for Science and Technology, under the projects UID/00013: Mathematics Centre of the University of Minho (CMAT/UM) and this research was also partially funded by the Ministry of Science and Innovation, State Research Agency (Spain), under the grant PID2020-115155GB-I0 and by the Council of Education, University and Vocational Training (Council of Galicia), grant ED431C 2023/3.

References

[1] Behera, A., & Panda, G. K. (1999). On the square roots of triangular numbers. *The Fibonacci Quarterly*, 37(2), 98–105.

[2] Catarino, P., Campos, H., & Vasco, P. (2015). On some identities for balancing and cobalancing numbers. *Annales Mathematicae et Informaticae*, 45, 11–24.

[3] Chimpanzo, J., Catarino, P., Vasco, P., & Borges, A. (2024). Bidimensional extensions of balancing and Lucas-balancing numbers. *Journal of Discrete Mathematical Sciences and Cryptography*, 27(1), 95–115.

[4] Liptai, K. (2006). Lucas balancing numbers. *Acta Mathematica Universitatis Ostraviensis*, 14(1), 43–47.

[5] Olajos, P. (2010). Properties of balancing, cobalancing and generalized cobalancing numbers. *Annales Mathematicae et Informaticae*, 37, 125–138.

[6] Panda, G. K. (2007). Sequence balancing and cobalancing numbers. *The Fibonacci Quarterly*, 45(3), 265–271.

[7] Panda, G. K. (2009). Some fascinating properties of balancing numbers. In: *Proceedings of Eleventh International Conference on Fibonacci Numbers and Their Applications, Congressus Numerantium*, 194, 185–189. Braunschweig, Germany. Utilitas Mathematica.

[8] Panda, G. K., Komatsu, T., & Davala, R. K. (2018). Reciprocal sums of sequences involving balancing and Lucas-balancing numbers. *Mathematical Reports*, 20(2), 201–214.

[9] Patel, B. K., Irmak, N., & Ray, P. K. (2018). Incomplete balancing and Lucas-balancing numbers. *Mathematical Reports*, 20(1), 59–72.

[10] Ray, P. K. (2009). *Balancing and Cobalancing Numbers*. Doctoral Thesis, National Institute of Technology Rourkela, India.

[11] Sloane, N. J. A. (2023). *The On-Line Encyclopedia of Integer Sequences*. Available online at: `https://oeis.org`.