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Some fundamental Fibonacci number congruences

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Dedicated to the memory of Professor Peter Bunderschuh!

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Abstract: This paper investigates a number of congruence properties related to the coefficients of a generalized Fibonacci polynomial. This polynomial was defined to produce properties comparable with those of the standard polynomials of some special functions. Some of these properties are compared with known identities, while others are seemingly characteristic of arbitrary order recurrences. These include generalizations of, and analogies for, results of Appell,

Bernoulli, Euler, Hilton, Horadam and Williams. In turn, the theorems lead to conjectures for further development.

Keywords: Congruences, Recurrence relations, Fibonacci sequences, Lucas sequences, Special functions, Appell polynomials, Generalized Fibonacci polynomials.
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1 Introduction

The purpose of this paper is to consider some congruences associated with a generalized Fibonacci polynomial, which is structured to produce analogies with classical polynomials, and which is defined in Equation (2.1) below in relation to two generalized arbitrary order, *r*, Fibonacci sequences.

$$u_n^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_j u_{n-j}^{(r)}, \quad n > 0, \tag{1.1}$$

with initial conditions

$$u_n^{(r)} = 1, n = 0, \text{ and } u_n^{(r)} = 0, \ n < 0;$$

$$v_n^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_j v_{n-j}^{(r)}, n > r, \tag{1.2}$$

with initial conditions

$$v_n^{(r)} = \sum_{j=1}^{r} \alpha_j^n, 0 \le n < r, \text{ and } v_n^{(r)} = 0, \ n < 0,$$

where the P_j are arbitrary integers and the α_j are the roots, assumed distinct of the auxiliary equation of the recurrence relations (1.1) and (1.2). As examples, the first four terms of $u_n^{(3)}$ and $v_n^{(3)}$ are displayed in Table 1. Respectively, $u_n^{(2)}$ and $v_n^{(2)}$ correspond to the Lucas fundamental and primordial numbers, respectively, [18].

Table 1. First four terms of $u_n^{(3)}, v_n^{(3)}$

n	0	1	2	3
$\boldsymbol{u_n^{(3)}}$	1	P_1	$P_1^2 - P_2$	$P_1^3 - 2P_1P_2 + P_3$
$\boldsymbol{v_n^{(3)}}$	3	P_1	$P_1^2 - P_2$	$P_1^3 - 3P_1P_2 + 3P_3$

2 A generalized Fibonacci polynomial

There are many forms of the generalized Fibonacci polynomials (for example, [2, 8, 10, 17, 20]), but the one used here can be represented by

$$\sum_{n=0}^{\infty} u_n(x) \frac{t^n}{n!} = e^{xt} \sum_{n=0}^{\infty} u_n(0) \frac{t^n}{n!} \tag{2.1}$$

by analogy with the polynomials of Bernoulli, Euler and Hermite [3, 27] and which satisfy the recurrence relation

$$u_{n+1}(x) = xu_n(x) + \sum_{j=0}^{n} n^{(j)_n} v_{j+1} u_{n-j}(x) \tag{2.2}$$

in which $(j)_n$ represents the falling factorial coefficient.

On equating coefficients of t in (2.1), we can obtain

$$\begin{aligned} u_n(x) &= \sum_{k=0}^{n} \frac{n!}{k!} u_{n-k}^{(r)} x^k = \sum_{k=0}^{n} \frac{n!}{k!} \frac{u_{n-k}(0)}{(n-k)!} x^k \\ &= \sum\nolimits_{k=0}^{n} \binom{n}{k} u_{n-k}(0) x^k = \sum\nolimits_{k=0}^{n} \binom{n}{k} (n-k)! u_{n-k} \\ &= \sum\nolimits_{k=0}^{n} \frac{n!}{k!} u_{n-k} x^k \end{aligned} \tag{2.3}$$

That $u_n^r(0) = n! u_n^{(r)}$ was, in effect, established by Asveld [4] whose polynomials for $r = 2$ can be re-written in terms of a triangle of their coefficients as in Table 2 for (2.3).

Table 2. Coefficients of $u_n(x)$ for $r = 2$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6
$u_0(x)$	1						
$u_1(x)$	1	1					
$u_2(x)$	4	2	1				
$u_3(x)$	18	12	3	1			
$u_4(x)$	120	72	24	4	1		
$u_5(x)$	960	600	180	40	5	1	
$u_6(x)$	9360	5760	1800	360	60	6	1

Table 3. Coefficients of $u_n(x)$ for an arbitrary r

	x^0	x^1	x^2	x^3	x^4	x^5	x^6
$u_0(x)$	1						
$u_1(x)$	u_1	1					
$u_2(x)$	$2u_2$	$2u_1$	1				
$u_3(x)$	$6u_3$	$6u_2$	$3u_1$	1			
$u_4(x)$	$24u_4$	$24u_3$	$12u_2$	$4u_1$	1		
$u_5(x)$	$120u_5$	$120u_4$	$60u_3$	$20u_2$	$5u_1$	1	
$u_6(x)$	$720u_6$	$720u_5$	$360u_4$	$120u_3$	$30u_2$	$6u_1$	1

Now in [24], it is also proved that

$$u_{n+tm}(x) \equiv u_n(x)\left(u_m(x)\right)^t \pmod{m}. \tag{2.4}$$

When $t = 1$ and $x = 0$, this becomes the known result

$$u_{n+m}^{(r)}(0) \equiv u_n^{(r)}(0)u_m^{(r)}(0) \pmod{m},$$

which has scope for further development. Thus, when $r = 2$ and $P_1 = -P_2 = 1$ in (1.1) and (1.2), (2.4) can be reduced to the known result for Fibonacci and Lucas numbers.

$$nF_{n+1} = \sum_{j=0}^{n-1} L_{j+1}F_{n-j}.$$

Next, let the monic polynomial elements of $\left\{u_0^{(r)}(x), u_1^{(r)}(x), \ldots, u_n^{(r)}(x)\right\}$ with coefficients integral modulo m be such that

$$u_s^{(r)}(x) = \sum_{j=0}^{s} a_{s,j}x^j, \left(a_{s,s} = 1, 0 \le s \le n\right)$$

and if $\sum_{s=0}^{n} A_s u_s^{(r)}(x) \equiv 0 \pmod{m}$, then $A_s \equiv 0 \pmod{m}$, $0 \le s \le n$. This is stated in order to prove that each coefficient of the matrix $\left[a_{s,j}\right]$ is triangular and each coefficient of $u_s^{(r)}(x)$ is divisible by m. We establish the result as follows.

$$\sum_{s=0}^{n} A_s u_s^{(r)}(x) = \sum_{s=0}^{n}\sum_{j=0}^{s} A_s a_{s,j}x^j \equiv 0 \pmod{m}.$$

Now

$$\sum_{s=0}^{n}\sum_{j=0}^{s} A_s a_{s,j}x^j = A_0\left(a_{0,0}\right) + A_1\left(a_{1,0} + a_{1,1}x\right)$$
$$+ A_2\left(a_{2,0} + a_{2,1}x + a_{2,2}x^2\right) + \cdots + A_n(a_{n,0} + a_{n,1}x + \cdots + a_{n,n}x^n$$

so that

$$\sum_{j=0}^{n} x^j \sum_{s=j}^{n} A_s a_{s,j} \equiv 0 \pmod{m},$$

which implies that

$$\sum_{s=j}^{n} A_s a_{s,j} \equiv 0 \pmod{m},$$

since if $u_s^{(r)}(x)$ is a polynomial with integral coefficients, the statement

$$u_s^{(r)}(x) \equiv 0 \pmod{m}$$

means that each coefficient of $u_s^{(r)}(x)$ is divisible by m. Thus, $m \mid A_s$, $s = 0, 1, \ldots, n$, since the matrix $\left[a_{s,j}\right]$ is triangular. This completes the proof.

3 Fundamental number congruences

We extend these ideas by considering the more general extension of Horadam [12]

$$w_{n+r}^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{r,j} w_{n+r-j}^{(r)}, \quad n > 0, \tag{3.1}$$

with appropriate initial conditions. Shannon [23] has shown that

$$w_n^{(r)} = \sum_{j=0}^{r-1} \sum_{k=j}^{r-1} (-1)^{k-j} P_{r,k-j} u_{r,n-k}^{(r)} w_j^{(r)} \tag{3.2}$$

in which $P_{r,0} = 1$ for notational convenience. For example,

$$\begin{aligned} w_n^{(2)} &= w_1^{(2)} u_{2,n-1}^{(2)} + w_0^{(2)} \left(u_{2,n}^{(2)} - P_{2,1} u_{2,n-1}^{(2)} \right) \\ &= w_1^{(2)} u_{2,n-1}^{(2)} - P_{2,2} w_0^{(2)} u_{2,n-2}^{(2)} \end{aligned}$$

which agrees with Equation (3.14) of Horadam [13]. The genesis of Equation (3.2) can be seen when we reverse the order of summation:

$$\begin{aligned} w_n^{(r)} &= \sum_{j=0}^{r-1} \sum_{k=j}^{r-1} (-1)^{k-j} P_{r,k-j} u_{r,n-k}^{(r)} w_j^{(r)} \\ &= \sum_{j=0}^{r-1} w_j^{(r)} \left(\sum_{k=j}^{r-1} (-1)^{k-j} P_{r,k-j} u_{r,n-k}^{(r)} \right) \\ &= \sum_{k=o}^{r-1} u_{r,n-k}^{(r)} \left(\sum_{m=0}^{k} (-1)^{k-m} P_{r,k-m} w_m^{(r)} \right). \end{aligned}$$

Now, if a_n is the least positive residue of $w_n^{(r)}$ modulo m, then we may associate with $\{w_n^{(r)}\}$ a second sequence $\{a_n\}$, which we may call the reduced sequence corresponding to $w_n^{(r)}$ modulo m. The period of cycle or characteristic number modulo m is the smallest positive integer $k = C(m)$: $w_{k+n}^{(r)} \equiv w_n^{(r)} \pmod{m}$ or $a_{k+n} = a_n \; \forall \, n \geq 0$. The number of non-repeating terms in $\{a_n\}$ is called the numeric of $\left\{ w_n^{(r)} \right\} \pmod{m}$ [17, 19]. For example, for the Fibonacci numbers

- mod 5 [1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0,] 1, 1, 2, …, so C(5) = 20, and ν(5) = 0 (Bundschuh and Shiue [5], Kuipers and Shiue [15], Niederreiter [21]);
- mod 8 [1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0,] 1, 1, 2, …, so C(8) = 12, and ν(8) = 2, [27].

When $v(m) = 0$, $\{w_n^{(r)}\}$ is said to be purely periodic modulo m. The sequence of Fibonacci numbers is thus purely periodic modulo 5. If all the terms of $\{w_n^{(r)}\}$ after a certain point are divisible by m, so that the repeating part of $\{a_n\}$ consists of the single residue zero, then the sequence $\{w_n^{(r)}\}$ is said to be a null sequence modulo m. For example, if $w_n^{(1)} = 4w_n^{(1)}$, $n \geq 1$, $w_0^{(1)}$ =1, then $64 \mid w_n^{(1)}$ for $n \geq 3$, and so $\{w_n^{(r)}\}$ is a null sequence modulo 64. In fact, more generally, if $\{u_n^{(r)}\}$ satisfies the difference congruence

$$0 \equiv \sum_{j=0}^{r-1}\left(\sum_{k=0}^{j}(-1)^{j-k} P_{r,j-k} w_k^{(r)}\right) u_{n-j}^{(r)} \pmod{m},\ n \geq i,$$

then $\{w_n^{(r)}\}$ is called a null sequence with $v < i$. The proof follows because, if the congruence is satisfied, then

$$\{w_n^{(r)}\} \equiv 0 \pmod{m} \text{ for } n \geq i,$$

since, on rearranging the order of summation in (3.2) as before, we have that

$$w_n^{(r)} = \sum_{j=0}^{r-1}\left(\sum_{k=0}^{j}(-1)^{j-k} P_{r,j-k} w_k^{(r)}\right) u_{n-j}^{(r)}.$$

We remark that m is said to be a null divisor of $\{w_n^{(r)}\}$ and i is the numeric of $\{w_n^{(r)}\}$ modulo m. This result solves the problem of determining the numeric of $\{w_n^{(r)}\}$ given its r initial values and the fundamental sequence $\{u_n^{(r)}\}$. This is a more succinct result than that of Ward [29], though he also determined all the null divisors of $\{w_n^{(r)}\}$ [30].

If $\{a_n^{(r)}\}$ is any reduced sequence of residues modulo p such that

$$a_{n+r}^{(r)} = \sum_{j=1}^{r}(-1)^{j+1} P_{r,j} a_{n+r-j}^{(r)},\ 0 \leq a_n \leq p-1,$$

then the τ residues $a_0, a_1, \ldots, a_{\tau-1}$ are said to form a cycle (a) which belongs to the auxiliary polynomial of the recurrence relation. Two such cycles are said to be equal if either can be obtained from the other by a cyclic permutation of its elements. For instance, if L is any residue, then L is called a multiplier of (a), and if $La_n \equiv a_{n+1} \pmod{m}$, $(n = 0, 1, 2, \ldots, \tau - 1)$, then 1 is here called the span of L [16]. Furthermore, if $\epsilon(M)$ is the exponent to which the multiplier M belongs modulo p, and if μ is its span, then $\tau = \epsilon(M)\mu$.

Since $(M, p) = 1$, M belongs to $\epsilon(M)$ modulo p if $\epsilon(M)$ is the smallest integer:

$$M^{\epsilon(M)} \equiv 1 \pmod{p}.$$

If $\epsilon(M) \not\equiv 0 \pmod 3$, then $(\mu, 3) = 1$, but it does not seem to be clear when $r \geq 3$, whether $\tau = 3\epsilon(M)\mu$ or $\tau = \epsilon(M)\mu$. Anthony Hilton and colleagues at the University of Reading [9] have used a method of combining the roots of an auxiliary equation to aid their studies of the structure of second order recursive sequences in finite fields and this may suggest generalizations to higher orders. Other possible gaps in the theory are referred to in the final section.

4 An interesting property

Since Shannon [22] proved the generalized Fibonacci polynomial is an Appell polynomial, we shall consider an interesting property of the Appell polynomials (See Theorem 1). It is well known that the Appell polynomials can be written in a compact form as follows (Aceto, Malonek, and Tomaz [1]):

$$p_n(x)=\sum_{k=0}^{n}\binom{n}{k}c_{n-k}x^k,\quad n=0,1,\ldots,c_0\neq 0.$$

For example,

$$p_0(x)=c_0,$$
$$p_1(x)=c_1+c_0x,$$
$$p_2(x)=c_2+2c_1x+c_0x^2,$$
$$p_3(x)=c_3+3c_2x+3c_1x^2+c_0x^3.$$

Some examples of Appell polynomials:

- The generalized Fibonacci polynomials $\{u_n(x)\}_{n\geq 0}$:

$$u_n(x)=\sum_{k=0}^{n}\frac{n!}{k!}u_{n-k}x^k.$$

- The Bernoulli polynomial $\{B_n(x)\}_{n\geq 0}$:

$$B_n(x)=\sum_{k=0}^{n}B_{n-k}x^k,$$

where B_n is a Bernoulli number.

- The Euler polynomial $\{E_n(x)\}_{n\geq 0}$:

$$E_n(x)=\sum_{k=0}^{n}E_{n-k}x^k,$$

where E_n is an Euler number.

Theorem 1. *Let* $p_n(x)$ *be an Appell polynomial, i.e.,*

$$p_n(x)=\sum_{k=0}^{n}\binom{n}{k}c_{n-k}x^k,\quad n=0,1,\ldots,\ p_0(x)=c_0\neq 0.$$

Denote $a_{n,k}=\binom{n}{k}c_{n-k}$. *Then, for* $n\geq 1$,

$$a_{n,k}=\frac{n}{k}a_{n-1,k-1},\quad n,k\geq 1.$$

Proof. Note that

$$\frac{n}{k}a_{n-1,k-1}=\frac{n}{k}\binom{n-1}{k-1}c_{n-1-(k-1)}=\binom{n}{k}c_{n-k}=a_{n,k}.$$

The proof is completed. □

Examples.

1. The generalized Fibonacci polynomials:

$$u_4(x) = 24u_4 + 24u_3x + 12u_2x^2 + 4u_1x^3 + x^4 \quad \text{and}$$
$$u_5(x) = 120u_5 + 120u_4x + 60u_3x^2 + 20u_2x^3 + 5u_1x^4 + x^5$$

give

$$a_{4,2} = 12u_2, \quad a_{5,3} = 20u_2.$$

Then

$$a_{5,3} = \frac{5}{3}a_{4,2} = \frac{5}{3}\cdot 12u_2 = 20u_2.$$

2. The Bernoulli polynomials:

$$B_3(x) = \frac{1}{2}x - \frac{3}{2}x^2 + x^3 \quad \text{and} \quad B_4(x) = -\frac{1}{30} + x^2 - 2x^3 + x^4$$

give

$$a_{3,2} = -\frac{3}{2} \quad \text{and} \quad a_{4,3} = -2.$$

Then

$$a_{4,3} = \frac{4}{3}a_{3,2} = \frac{4}{3}\left(-\frac{3}{2}\right) = -2.$$

3. The Euler's polynomials:

$$E_4(x) = x - 2x^3 + x^4 \quad \text{and} \quad E_5(x) = -\frac{1}{2} + \frac{5}{2}x^2 - \frac{5}{2}x^4 + x^5$$

give

$$a_{5,2} = \frac{5}{2}a_{4,1} = 1.$$

Theorem 2. *Let* $p_n(x)$ *be an Appell polynomial, i.e.,*

$$p_n(x) = \sum_{k=0}^{n}\binom{n}{k}c_{n-k}x^k, \quad n = 0, 1, \ldots, c_0 \neq 0.$$

Then

$$p'(x) = np_{n-1}(x), \quad n \geq 1.$$

Proof. Let $a_{n,k} = \binom{n}{k}c_{n-k}$. Then

$$p_n(x) = \sum_{k=0}^{n}\binom{n}{k}c_{n-k}x^k = \sum_{k=0}^{n}a_{n-k}x^k = a_{n,0} + \sum_{k=1}^{n}a_{n-k}x^k.$$

Next, by Theorem 1,

$$\begin{aligned} p'(x) &= \frac{d}{dx}\left(\sum_{k=1}^{n}a_{n-k}x^k\right) = \frac{d}{dx}\left(\sum_{k=1}^{n}\frac{n}{k}a_{n-1,k-1}x^k\right) \\ &= \sum_{k=1}^{n}\left(\frac{n}{k}\cdot ka_{n-1,k-1}\right)x^{k-1} = n\sum_{k=1}^{n}a_{n-1,k-1}x^{k-1} \\ &= n\cdot p_{n-1}(x) \end{aligned}$$

The proof is completed. □

Note. Shannon [22] proved this result in a different way.

Examples.

1. The generalized Fibonacci polynomial

$$u_4(x) = 24u_4 + 24u_3x + 12u_2x^2 + 4u_1x^3 + x^4$$

has

$$u_4'(x) = 24u_3 + 24u_2x + 12u_1x^2 + 4x^3 = 4\left(6u_3 + 6u_2x + 3u_1x^2 + x^3\right) = 4u_3(x).$$

2. The Bernoulli polynomial

$$B_4(x) = -\frac{1}{30} + x^2 - 2x^3 + x^4$$

has

$$B_4'(x) = 2x - 6x^2 + 4x^3 = 4\left(\frac{1}{2}x - \frac{3}{2}x^2 + x^3\right) = 4B_3(x).$$

3. The Euler polynomial

$$E_3(x) = \frac{1}{4} - \frac{3}{2}x^2 + x^3$$

has

$$E_3'(x) = -3x + 3x^2 = 3\left(-x + x^2\right) = 3E_2(x).$$

Theorem 3. *Let* $p_n(x)$ *be an Appell polynomial, i.e.,*

$$p_n(x) = \sum_{k=0}^{n}\binom{n}{k}c_{n-k}x^k, \quad n = 0, 1, \ldots, c_0 \neq 0.$$

Then

$$p_{n+1}(x) = c_{n+1} + (n+1)\int_0^x p_n(y)\,dy, \tag{4.1}$$

$$\int_x^{x+1} p_n(y)\,dy = \frac{p_n(x+1) - p_n(x)}{n+1}. \tag{4.2}$$

Proof. First, we prove (4.1). Let $a_{n,k} = \binom{n}{k}c_{n-k}$. Then

$$\begin{aligned}
\int_0^x p_n(y)\,dy &= \int_0^x \sum_{k=0}^{n}\binom{n}{k}c_{n-k}y^k\,dy = \int_0^x \sum_{k=0}^{n} a_{n,k}y^k\,dy \\
&= \sum_{k=0}^{n}\int_0^x \frac{k+1}{n+1}a_{n+1,k+1}y^k\,dy \qquad \text{(by Theorem 1)} \\
&= \sum_{k=0}^{n}\frac{k+1}{n+1}a_{n+1,k+1}\frac{1}{k+1}x^{k+1} = \frac{1}{n+1}\sum_{k=0}^{n}a_{n+1,k+1}x^{k+1} \\
&= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}c_{n+1-(k+1)}x^{k+1} = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}c_{n-k}x^{k+1}.
\end{aligned}$$

Now,

$$p_{n+1}(x) = \sum_{k=0}^{n+1}\binom{n+1}{k}c_{n+1-k} = c_{n+1} + \sum_{k=1}^{n+1}\binom{n+1}{k}c_{n+1-k}x^k.$$

Note that

$$\sum_{k=1}^{n+1}\binom{n+1}{k}c_{n+1-k}x^k=\sum_{k=0}^{n}\binom{n+1}{k+1}c_{n-k}x^{k+1}.$$

We obtain the first result:

$$p_{n+1}(x)=c_{n+1}+(n+1)\int_0^x p_n(y)\,dy.$$

Next, also by (4.1), we have

$$p_{n+1}(x+1)=c_{n+1}+(n+1)\int_0^{x+1} p_n(y)\,dy$$

and

$$p_{n+1}(x)=c_{n+1}+(n+1)\int_0^x p_n(y)\,dy.$$

This implies

$$p_{n+1}(x+1)-p_n(x)=(n+1)\left[\int_0^{x+1}p_n(y)\,dy-\int_0^x p_n(y)\,dy\right]=(n+1)\int_x^{x+1}p_n(y)\,dy.$$

The proof is completed. □

Remark. By (4.2), we have

$$u_{n+1}(x+1)-u_n(x)=(n+1)\int_x^{x+1}u_n(y)\,dy,\quad n=1,2,3,\ldots \tag{4.3}$$

and

$$B_{n+1}(x+1)-B_n(x)=(n+1)\int_x^{x+1}B_n(y)\,dy,\quad n=1,2,3,\ldots \tag{4.4}$$

Result (4.3) was established in [25] and (4.4) was the known result in [7].

Examples.

1. Generalized Fibonacci polynomials:
   Let

$$u_n(x)=\sum_{k=0}^{n}\frac{n!}{k!}u_{n-k}x^k=\sum_{k=0}^{n}\binom{n}{k}(n-k)!u_{n-k}x^k.$$

   Then $c_{n-k}=(n-k)!u_{n-k}$ and

$$\begin{aligned}u_{n+1}(x)&=c_{n+1}+(n+1)\int_0^x u_n(y)\,dy\\&=(n+1)!u_{n+1}+(n+1)\int_0^x u_n(y)\,dy.\end{aligned}$$

   Example: $u_3(x)=6u_3+6u_2x+3u_1x^2+x^3$ gives

$$\begin{aligned}u_4(x)&=4!u_4+4\int_0^x\left(6u_3+6u_2y+3u_1y^2+y^3\right)dy\\&=24u_4+24u_3x+12u_2x^2+4u_1x^3+x^4\end{aligned}$$

2. The Bernoulli polynomials:
   Let

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k, \quad n = 0, 1, 2, \ldots$$

   Then $c_{n-k} = B_{n-k}$ and

$$B_{n+1}(x) = B_{n+1} + (n+1)\int_0^x B_n(y)\,dy.$$

   Example: $B_4(x) = -\frac{1}{30} + x^2 - 2x^3 + x^4$ gives

$$\begin{aligned} B_5(x) &= B_5 + 5\int_0^x B_4(y)\,dy \\ &= 0 + 5\left(-\frac{1}{30}x + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right) \\ &= -\frac{1}{6}x + \frac{5}{3}x^3 - \frac{5}{2}x^4 + x^5 \end{aligned}$$

3. The Euler polynomials:
   Let

$$E_n(x) = \sum_{k=0}^{n} E_{n-k} x^k, \quad n = 1, 2, 3, \ldots$$

   Then $c_{n-k} = E_{n-k}$ and

$$E_{n+1}(x) = E_{n+1} + (n+1)\int_0^x E_n(y)\,dy.$$

   Example: $E_3(x) = \frac{1}{4} - \frac{3}{2}x^2 + x^3$ gives

$$\begin{aligned} E_4(x) &= E_4 + 4\int_0^x E_3(y)\,dy \\ &= 0 + 4\left(\frac{1}{4}x - \frac{1}{2}x^3 + \frac{1}{4}x^4\right) \\ &= x - 2x^3 + x^4. \end{aligned}$$

5 Concluding comments

By way of conclusion this section also suggests investigating related power sum divisibility properties of Fibonacci and Lucas analogues of arbitrary order [20, 26, 28]. For this the following can help, utilizing Williams' congruence properties of generalized Lucas numbers [31] defined by

$$L_{s,n}^{(r)} = d^{-s} \sum_{j=1}^{(r)} \alpha_{r,j}^{n} \zeta_r^{s(j-1)}, \quad s = 0, 1, 2, \ldots, r-1, \tag{5.1}$$

in which

$$\zeta_m = \exp\left(\frac{2\pi i}{m}\right)$$

is a modification of Carlitz [6], and d is some real number for $r > 2$; for $r = 2$, d is the difference between the roots of the characteristic polynomial as usual, so that we get the Lucas primordial sequence

$$L_{0,n}^{(2)} = \alpha_{2,1}^{n} + \alpha_{2,2}^{n}.$$

Horadam [13] permits these coefficients to be arbitrary integers as does Williams in his study of $\left\{H_{s,n}^{(r)}\right\}$ defined in effect by

$$H_{s,n}^{(r)} = \frac{1}{d}\sum_{j=1}^{r} c_{s,j}\varphi_{r,j}^{n-1}, \tag{5.2}$$

where $c_{s,j}$ is the cofactor of $\alpha_{r,j}^{s-1}$ in d, the Vandermonde determinant of the roots, and

$$\varphi_{r,i} = \sum_{j=0}^{r} a_j \alpha_{r,i}^{j} \tag{5.3}$$

as a type of representation of $\varphi_{r,i}$ in the scale of $\alpha_{r,i}$, where the latter is the base of each scale [14]. When $r = 2$,

$$d = \begin{bmatrix} 1 & \alpha_{2,1} \\ 1 & \alpha_{2,2} \end{bmatrix} = \alpha_{2,2} - \alpha_{2,1},$$

the difference between the roots which occurs in the usual Binet form of the general term of the Fibonacci sequence. The two sequences $\left\{H_{s,n}^{(r)}\right\}$ and $\left\{u_{s,n}^{(r)}\right\}$ are quite neatly related by [25]

$$H_{s,m}^{(r)} = \sum_{j=1}^{r} u_{s,j}^{(r)} H_{j,m}^{(r)} \tag{5.4}$$

and variations may be found in [11].

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