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A note on Chebyshev's theorem

A. Bërdëllima

Faculty of Engineering, German International University in Berlin
Am Borsigturm 162, 13507, Berlin, Germany
e-mail: berdellima@gmail.com

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Abstract: We revisit a classical theorem of Chebyshev about distribution of primes on intervals $(n, 2n)$, $n \in \mathbb{N}$, and prove a generalization of it. Extending Erdős' arithmetical-combinatorial argument, we show that for all $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that the intervals $(kn, (k+1)n)$ contain a prime for all $n \geq n_k$. A quantitative lower bound is derived for the number of primes on such intervals. We also give numerical upper bounds for n_k for $k \leq 20$, and we draw comparisons with existing results in the literature.
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1 Introduction

In 1845, J. Bertrand [3] conjectured that between n and $2n$ there is always a prime number for every $n \in \mathbb{N}$. This conjecture was solved completely by Chebyshev [19] in 1852, and it is commonly known as Chebyshev's Theorem.

Chebyshev's Theorem. *For every* $n > 1, n \in \mathbb{N}$ *there is a prime on the interval* $(n, 2n)$.

Since then, other proofs of Chebyshev's Theorem appeared in the literature, most notably the proof by Ramanujan [12], who used properties of the gamma function, and the proof by Erdős [9], employing the prime factorization of binomial coefficients. Chebyshev's Theorem can be obtained as a direct corollary of [18, Sylvester's Theorem], which states that the product of k consecutive integers greater than k is divisible by a prime greater than k. It follows from this by taking $k = n$, and considering the k numbers $n+1, n+2, \cdots, n+k = 2n$, where $n > 1$. It is natural to consider the question of distribution of primes on intervals of the kind $(kn, (k+1)n)$ for $k \in \mathbb{N}$. In [10], Hanson has shown that there is a prime between $3n$ and $4n$, while El Bachraoui [2] has shown that there is a prime between $2n$ and $3n$ for every $n > 1$. In a relatively recent work, [17], Shevelev et al. demonstrate that such a strong result as Chebyshev's theorem does not, in general, hold. They prove that the list of integers k for which $(kn, (k+1)n)$ contains a prime for all $n > 1$ includes $k = 1, 2, 3, 5, 9, 14$, and no other, at least for $k \leq 10^8$.

Motivated by Erdős' approach, we prove that for every $k \in \mathbb{N}$ and for all large enough $n \in \mathbb{N}$, there is a prime number on the intervals $(kn, (k+1)n)$ (Theorem 2.1). While such an asymptotic result is an immediate consequence of the prime number theorem, the proof of Theorem 2.1 presents perhaps an interesting extension of Erdős' arithmetical-combinatorial argument for the general case $k > 1$. We also get a lower estimate for the number of primes on such intervals (Theorem 2.2), that is roughly $c_k\, n/\log n$, with c_k being a constant depending only on k. However, from the asymptotic nature of the prime number theorem, one cannot determine an $n_k \in \mathbb{N}$ with the property that $(kn, (k+1)n)$ contains a prime for all $n \geq n_k$. Rosser and Schoenfeld provide in [14, Theorem 1] the following non-asymptotic variant of the prime number theorem

$$\frac{x}{\log x}\Big(1 + \frac{1}{2\log x}\Big) < \pi(x) \quad \text{for } x \geq 59, \tag{1}$$

$$\pi(x) < \frac{x}{\log x}\Big(1 + \frac{3}{2\log x}\Big) \quad \text{for } x > 1, \tag{2}$$

where $\pi(x)$ denotes the number of primes $\leq x$ for a given $x > 0$. Inequalities (1) and (2) give impressive numerical upper bounds for n_k, at least for $k \leq 10$, as Table 1 shows. Better estimates for n_k may be obtained from more recent variants of the inequalities (1) and (2) due to Dusart; e.g. see [7, Corollary 5.2]. For other refinements, but in terms of the Chebyshev functions, we refer to [4] by Broadbent *et al.* However, as k becomes sufficiently large, these estimates become less effective. It is thus desirable to have a method for determining n_k for any $k \geq 1$, such that, as k grows, the estimates for n_k comparably retain their effectiveness. It turns out that Erdős' strategy in the proof of our theorems helps us devise the numerical method for our computations. Note that while for $k \leq 10$ our numerical results, see Table 3, are not as impressive as the ones in Table 1, they get substantially better for $k > 10$, see Table 4.

Table 1. First ten upper bound values for n_k,
using the Rosser–Schoenfeld estimates (1) and (2).

$\boldsymbol{k}$	1	2	3	4	5	6	7	8	9	10
$\boldsymbol{n_k} \leq$	59	59	59	59	63	137	311	726	1725	4163

To this end, we let $\log x$ denote the natural logarithm of a positive real number x, and by $\vartheta(x) = \sum_{p \leq x} \log p$, $\psi(x) = \sum_{p^\alpha \leq x} \log p$ the first and second Chebyshev functions, respectively, where p runs through the primes.

2 Main results

Theorem 2.1. *For every $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that the intervals $(kn, (k+1)n)$ contain at least one prime number for all $n \geq n_k$.*

Proof. For any $n \in \mathbb{N}$, we have that

$$\frac{2^{(k+1)n}}{(k+1)n} \leq \binom{(k+1)n}{\lfloor \frac{1}{2}(k+1)n \rfloor}, \tag{3}$$

because

$$2^{(k+1)n} = \sum_{j=0}^{(k+1)n} \binom{(k+1)n}{j} \leq 2 + \sum_{j=1}^{(k+1)n-1} \binom{(k+1)n}{\lfloor \frac{1}{2}(k+1)n \rfloor} \leq (k+1)n \binom{(k+1)n}{\lfloor \frac{1}{2}(k+1)n \rfloor}.$$

Denote by

$$a(k,n) := \binom{(k+1)n}{\lfloor \frac{1}{2}(k+1)n \rfloor}.$$

For a given prime p and a natural number n, let $R(p,n) := \max\{r \in \mathbb{N} : p^r | n\}$. In view of Legendre's identity,

$$R(p, n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor \quad n \in \mathbb{N},$$

it follows that

$$R(p, a(k,n)) = \sum_{j=1}^{\infty} \left\lfloor \frac{(k+1)n}{p^j} \right\rfloor - \sum_{j=1}^{\infty} \left\lfloor \frac{\lfloor \frac{1}{2}(k+1)n \rfloor}{p^j} \right\rfloor - \sum_{j=1}^{\infty} \left\lfloor \frac{(k+1)n - \lfloor \frac{1}{2}(k+1)n \rfloor}{p^j} \right\rfloor.$$

We can simplify this as:

$$\begin{aligned} R(p, a(k,n)) &\leq \sum_{j=1}^{\infty} \left\lfloor \frac{(k+1)n}{p^j} \right\rfloor - 2 \sum_{j=1}^{\infty} \left\lfloor \frac{\lfloor \frac{1}{2}(k+1)n \rfloor}{p^j} \right\rfloor \\ &= \sum_{j=1}^{\infty} \left(\left\lfloor \frac{(k+1)n}{p^j} \right\rfloor - 2 \left\lfloor \frac{(k+1)n}{2p^j} \right\rfloor \right). \end{aligned}$$

From the identity that $\lfloor 2x \rfloor - 2\lfloor x \rfloor$ vanishes if $\{x\} < 1/2$ and it is 1 if $\{x\} \geq 1/2$, it follows that the last sum is finite since all terms with $j > \log_p((k+1)n)$ vanish. Thus, it is bounded above by $R(p, a(k,n)) \leq \log_p((k+1)n)$, implying that

$$p^{R(p,a(k,n))} \leq (k+1)n. \tag{4}$$

Now, consider the intervals

$$\left(n, \frac{k+1}{k}n\right], \left(\frac{k+1}{k}n, \frac{k+1}{k-1}n\right], \ldots, \left(\frac{k+1}{3}n, \frac{k+1}{2}n\right], \left(\frac{k+1}{2}n, (k+1)n\right].$$

Any prime $p > n$ which divides $((k+1)n)!$ falls in one of the above intervals. If $p \in \left(\frac{k+1}{2}n, (k+1)n\right]$, then $2p$ does not divide $((k+1)n)!$ as $2p > (k+1)n$. Similarly, if $p \in \left(\frac{k+1}{3}n, \frac{k+1}{2}n\right]$, then $3p$ does not divide $((k+1)n)!$ as $3p > (k+1)n$. Note, however, that $2p \in \left(\frac{k+1}{2}n, (k+1)n\right]$, and so this prime p would contribute an equal amount of times in the numerator and denominator of $a(k,n)$. Therefore, $R(p, a(k,n)) = 0$. If $p \in \left(\frac{k+1}{4}n, \frac{k+1}{3}n\right]$, then again $4p$ cannot divide $((k+1)n)!$, and $2p, 3p \in \left(\frac{k+1}{2}n, (k+1)n\right]$. Consequently, p would contribute one more time in the numerator than in the denominator of $a(k,n)$, hence $R(p, a(k,n)) = 1$. We observe that if $p \in \left(\frac{k+1}{j+1}n, \frac{k+1}{j}n\right]$ for $j = 1, 2, 3, \ldots, k$, then $R(p, a(k,n)) = 0$ if j is even, and $R(p, a(k,n)) = 1$ if j is odd. Assume that there exists some $k \in \mathbb{N}, k \geq 2$, such that for any $m \in \mathbb{N}$, there is $n_m > m$ with the property that the interval $(kn_m, (k+1)n_m)$ contains no prime number. In view of (4), if $p > \sqrt{(k+1)n_m}$, then $R(p, a(k, n_m)) \leq 1$. By virtue of the fundamental theorem of arithmetic, we have:

$$a(k, n_m) = \prod_{p \leq a(k, n_m)} p^{R(p, a(k, n_m))}.$$

This can be broken down as:

$$a(k, n_m) = \prod_{p \leq \sqrt{(k+1)n_m}} p^{R(p, a(k, n_m))} \cdot \prod_{\sqrt{(k+1)n_m} < p \leq kn_m} p^{R(p, a(k, n_m))}.$$

We then decompose the second product into smaller products over disjoint sets of primes

$$\prod_{\sqrt{(k+1)n_m} < p \leq kn_m} p^{R(p, a(k, n_m))} = \Big(\prod_{\sqrt{(k+1)n_m} < p \leq n_m} p^{R(p, a(k, n_m))} \Big) \cdot \Big(\prod_{j=2}^{k} \Big(\prod_{\frac{k+1}{j+1} n_m < p \leq \frac{k+1}{j} n_m} p^{R(p, a(k, n_m))} \Big) \Big)$$
$$\cdot \Big(\prod_{\frac{k+1}{2} n_m < p \leq kn_m} p^{R(p, a(k, n_m))} \Big).$$

Using the above observations for $R(p, a(k,n))$, the inequalities (3), (4), and the well-known identity $\log(\prod_{p \leq n} p) = \vartheta(n)$, we obtain:

$$\frac{2^{(k+1)n_m}}{(k+1)n_m} < ((k+1)n_m)^{\sqrt{(k+1)n_m}} \cdot e^{\vartheta(n_m) - \vartheta(\sqrt{(k+1)n_m})}$$
$$\cdot e^{\sum_{\substack{2 \leq j \leq k \\ j \text{ odd}}} \vartheta(\frac{k+1}{j} n_m) - \vartheta(\frac{k+1}{j+1} n_m)} \cdot e^{\vartheta(kn_m) - \vartheta(\frac{k+1}{2} n_m)}.$$

Rearranging terms and taking logarithm both sides yields:

$$(k+1)n_m \log 2 < (\sqrt{(k+1)n_m} + 1) \log((k+1)n_m) + \vartheta(n_m) - \vartheta(\sqrt{(k+1)n_m}) \tag{5}$$
$$+ \sum_{\substack{2 \leq j \leq k \\ j \text{ odd}}} \Big(\vartheta(\frac{k+1}{j} n_m) - \vartheta(\frac{k+1}{j+1} n_m) \Big) + \vartheta(kn_m) - \vartheta(\frac{k+1}{2} n_m).$$

Dividing both sides by $(k+1)n_m$ and taking limit as $m \to \infty$ and so $n_m \to \infty$, we get

$$\log 2 \leq \sum_{\substack{1 \leq j \leq k \\ j \text{ odd}}} \Big(\frac{1}{j} - \frac{1}{j+1} \Big),$$

which is impossible as

$$\sum_{\substack{1 \leq j \leq k \\ j- \text{ odd}}} \Big(\frac{1}{j} - \frac{1}{j+1} \Big) < \sum_{j=1}^{\infty} \Big(\frac{1}{2j-1} - \frac{1}{2j} \Big) = \log 2. \qquad \square$$

Remark 2.1. *It follows from Theorem 2.1 that for any $k, \ell \in \mathbb{N}$ with $\ell < k$, the interval $(\ell n, kn)$ contains at least a prime number for all sufficiently large $n \in \mathbb{N}$. This results from the fact that the interval $((k-1)n, kn)$ is entirely included in the interval $(\ell n, kn)$ for all $\ell < k$.*

Theorem 2.2. *For any $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that for all $n \geq n_k$, it holds*

$$\pi((k+1)n) - \pi(kn) > c_k \frac{n}{\log n} \quad \textit{where } c_k = \log 2 - \sum_{\substack{1 \leq j \leq k \\ j \text{ odd}}} \Big(\frac{1}{j} - \frac{1}{j+1}\Big). \tag{6}$$

Proof. By Theorem 2.1, there is at least one prime in the interval $(kn, (k+1)n)$ for every $k \in \mathbb{N}$ and for all sufficiently large $n \geq n_k$, for some $n_k \in \mathbb{N}$. Therefore, on the right side of (5), we should add the product of prime factors in the interval $(kn, (k+1)n)$. Moreover, note that for any such prime p, we have that $R(p, a(k,n)) \leq 1$, since $p^2 > k^2n^2 \geq (k+1)n$ for any $n \geq 2$ and $k \geq 1$. Using $\log(\prod_{p \leq n} p) = \vartheta(n)$, we can write the new inequality as follows

$$\begin{aligned}(k+1)n\log 2 &< (\sqrt{(k+1)n}+1)\ \log((k+1)n) + \vartheta(n) - \vartheta(\sqrt{(k+1)n}) \\ &\quad + \sum_{\substack{2 \leq j \leq k \\ j \text{ odd}}} \Big(\vartheta(\frac{k+1}{j}n) - \vartheta(\frac{k+1}{j+1}n)\Big) + \vartheta(kn) - \vartheta(\frac{k+1}{2}n) + \vartheta((k+1)n) - \vartheta(kn)\end{aligned}$$

or, equivalently,

$$\begin{aligned}\vartheta((k+1)n) - \vartheta(kn) &> (k+1)n\ \log 2 - (\sqrt{(k+1)n}+1)\ \log((k+1)n) \\ &\quad - (\vartheta(n) - \vartheta(\sqrt{(k+1)n}) - \Big(\sum_{\substack{2 \leq j \leq k \\ j \text{ odd}}} \vartheta(\frac{k+1}{j}n) - \vartheta(\frac{k+1}{j+1}n)\Big) \\ &\quad - (\vartheta(kn) - \vartheta(\frac{k+1}{2}n)).\end{aligned}$$

We employ the following estimate, e.g. see [7, Theorem 4.2], for all $n \geq 2$:

$$|\vartheta(n) - n| < \frac{5}{4} \cdot \frac{n}{\log n}. \tag{7}$$

Using (7), we obtain the following lower estimate for all $k, n \geq 2$:

$$\begin{aligned}\vartheta((k+1)n) - \vartheta(kn) &> (k+1)n\ \log 2 - (\sqrt{(k+1)n}+1)\ \log((k+1)n) \\ &\quad - \Big((1 + \frac{5}{4\log n})n - (1 - \frac{5}{2\log((k+1)n)})\sqrt{(k+1)n}\Big) \\ &\quad - \Big(\sum_{\substack{2 \leq j \leq k \\ j \text{ odd}}} (1 + \frac{5}{4\log(\frac{(k+1)n}{j})})\frac{(k+1)n}{j} - (1 - \frac{5}{4\log(\frac{(k+1)n}{j+1})})\frac{(k+1)n}{j+1}\Big) \\ &\quad - \Big((1 + \frac{5}{4\log(kn)})kn - (1 - \frac{5}{4\log(\frac{(k+1)n}{2})})\frac{(k+1)n}{2}\Big) \\ &> \frac{(k+1)n}{2}\,(\log 4 - A(n,k)),\end{aligned}$$

where

$$
\begin{aligned}
A(n,k) = \frac{2}{(k+1)n} \Big[& (\sqrt{(k+1)n}+1)\ \log((k+1)n) \\
& + \Big((1+\frac{5}{4\log n})n - (1-\frac{5}{2\log((k+1)n)})\sqrt{(k+1)n} \Big) \\
& \quad \Big(\sum_{\substack{2\le j\le k \\ j \text{ odd}}} (1+\frac{5}{4\log(\frac{(k+1)n}{j})})\frac{(k+1)n}{j} - (1-\frac{5}{4\log(\frac{(k+1)n}{j+1})})\frac{(k+1)n}{j+1} \Big) \\
& + \Big((1+\frac{5}{4\log(kn)})kn - (1-\frac{5}{4\log(\frac{(k+1)n}{2})})\frac{(k+1)n}{2} \Big) \Big].
\end{aligned}
$$

Notice that for each $k \in \mathbb{N}$, $A(n,k)$ approaches the value

$$
1 + 2 \sum_{\substack{2\le j\le k \\ j \text{ odd}}} \Big(\frac{1}{j} - \frac{1}{j+1} \Big) \quad \text{as } n \to \infty.
$$

On the other hand, the estimate

$$
1 + 2 \sum_{\substack{2\le j\le k \\ j \text{ odd}}} \Big(\frac{1}{j} - \frac{1}{j+1} \Big) = 2 \sum_{\substack{1\le j\le k \\ j \text{ odd}}} \Big(\frac{1}{j} - \frac{1}{j+1} \Big) < 2 \sum_{j=1}^{\infty} \Big(\frac{1}{2j-1} - \frac{1}{2j} \Big) = 2\log 2
$$

implies that for every $k \in \mathbb{N},\ k \geq 2$, there exists $\widetilde{n}_k \in \mathbb{N}$ with

$$
A(k,n) < \frac{1}{2}\Big(2\log 2 - 2 \sum_{\substack{1\le j\le k \\ j \text{ odd}}} \Big(\frac{1}{j} - \frac{1}{j+1} \Big) \Big) + 2 \sum_{\substack{1\le j\le k \\ j \text{ odd}}} \Big(\frac{1}{j} - \frac{1}{j+1} \Big) := \widetilde{c}_k
$$

for all $n \geq \widetilde{n}_k$. In particular, we obtain that

$$
\log 4 - A(k,n) = 2\log 2 - A(k,n) > \log 2 - \sum_{\substack{1\le j\le k \\ j \text{ odd}}} \Big(\frac{1}{j} - \frac{1}{j+1} \Big) = \sum_{\substack{j>k \\ j \text{ odd}}} \Big(\frac{1}{j} - \frac{1}{j+1} \Big) > 0.
$$

Therefore, for every $k \geq 2$, it holds that

$$
\vartheta((k+1)n) - \vartheta(kn) > \frac{(k+1)n}{2}(\log 4 - A(n,k)) > 0 \quad \text{for all } n \geq \widetilde{n}_k. \tag{8}
$$

The following relation holds, e.g. see [1, Theorem 4.3, p. 78]:

$$
\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(y)}{y\log^2 y}\, dy. \tag{9}
$$

Thus, we obtain the inequality

$$
\pi((k+1)n) - \pi(kn) > (\log 4 - A(n,k))\frac{(k+1)n}{2\log((k+1)n)} > \underbrace{(\log 4 - \widetilde{c}_k)}_{:=c_k} \frac{n}{\log n}
$$

for every $k \in \mathbb{N}$ and $n \geq \max\{n_k, \widetilde{n}_k\}$. This completes the proof. □

3 Numerical computations

The proof of Theorem 2.1 offers a possibility to obtain numerical results for an upper bound for the numbers n_k for every $k \in \mathbb{N}$. The inequality (5) is the tool which we will use. While (5) holds for $k \geq 2$, a version of it is valid also for the simple case $k = 1$, where the term $\vartheta(n) - \vartheta(\sqrt{2n})$ is replaced by $\vartheta(2n/3) - \vartheta(\sqrt{2n})$, since no primes in the interval $(2n/3, n)$ appear in $a(1, n)$. In this case, (5) reduces to the inequality

$$2n \log 2 < (\sqrt{2n} + 1) \log(2n) + \vartheta(2n/3) - \vartheta(\sqrt{2n}).$$

Therefore, getting a good upper bound for n_1 reduces to obtaining, or using, already good bounds on the Chebyshev function $\vartheta(n)$. For example, the following estimates are known:

$$|\vartheta(n) - n| < 3.965 \frac{n}{\log^2 n} \quad \text{[7, p. 2]} \tag{10}$$

$$\vartheta(n) < n + \frac{n}{36260} \quad \text{[8, Theorem 4.2]} \tag{11}$$

$$|\psi(n) - n| < 1.66 \frac{n}{\log^2 n} \quad \text{for all } n \geq 2 \quad \text{[7, Theorem 3.3].} \tag{12}$$

In view of the estimates (10) and (11), we get the inequality

$$2n \log 2 < (\sqrt{2n} + 1) \log(2n) + \Big(\frac{2}{3} + \frac{2}{3 \cdot 36260}\Big)n - \sqrt{2n} + 3.965 \cdot \frac{\sqrt{2n}}{\log^2(\sqrt{2n})}$$

that holds true for $1 \leq n \leq 108$. Therefore, $n_1 \leq 109$. Using the estimates (10), (11), and the inequality (5), one can, in principle, find upper bounds for n_k for any value of $k \in \mathbb{N}$, though for large k, the bounds become increasingly larger. Moreover, to improve our numerical results, note that for every $k \in \mathbb{N}$, we can always take some odd number $j_k \in \mathbb{N}$ with $j_k > k$ and

$$\frac{(k+1)n}{j_k + 2} \geq \sqrt{(k+1)n}. \tag{13}$$

In this way, the sum in (5) over j extends up to j_k. Using the estimates (10), (11), (12), and the estimate $|\psi(x) - \vartheta(x)| < 1.42620\, x^{1/2}$ for $x > 0$ from [14, Theorem 13], we obtain the following immediate result.

Lemma 3.1. *The following inequalities hold true*

$$|\vartheta(x) - x| < \eta \frac{x}{\log^2 x} \quad \textit{for all } x > x_\eta. \tag{14}$$

Here, we use the following values:

Table 2. η and respective x_η.

η	3.66	3.06	3.00	2.96	2.86	2.76
x_η	1402	5897	6929	7735	10293	13939
η	2.66	2.56	2.46	2.36	2.26	2.00
x_η	19278	27363	40118	61298	98878	531531

Proof. For any $x > 0$, we have

$$|\vartheta(x) - x| \le |\vartheta(x) - \psi(x)| + |\psi(x) - x|.$$

Employing inequality (12) and $|\psi(x) - \vartheta(x)| < 1.42620\, x^{1/2}$ for $x > 0$, we obtain:

$$|\vartheta(x) - x| < 1.66 \frac{x}{\log^2 x} + 1.42620\, x^{1/2}.$$

On the other hand, we have the inequalities

$$1.42620\, x^{1/2} < \widetilde{\eta}\, \frac{x}{\log^2 x} \quad \text{for all } x \ge x_\eta,$$

where $\widetilde{\eta} := \eta - 1.66$ and x_η are derived using a basic version of WolframAlpha. □

We now use Lemma 3.1 to get our numerical results. For this, we solve the inequalities

$$\begin{aligned}2n \log 2 < (\sqrt{2n} + 1) \log(2n) + \Big(\frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{j_1} - \frac{1}{j_1 + 1} + \frac{1}{j_1 + 2}\Big) 2n \\ + \Big(\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{j_1} + \frac{1}{j_1 + 2}\Big) \frac{2n}{36260} + \eta \sum_{i=1}^{\frac{j_k+1}{2}} \frac{n}{i\, \log^2(n/i)} - \sqrt{2n} + \eta\, \frac{\sqrt{2n}}{\log^2 \sqrt{2n}}\end{aligned} \tag{15}$$

and for $k \ge 2$

$$\begin{aligned}(k+1)n \log 2 < (\sqrt{(k+1)n} + 1) \log((k+1)n) \\ + \Big(\frac{k}{k+1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{j_k} - \frac{1}{j_k + 1} + \frac{1}{j_k + 2}\Big) (k+1)n \\ + \Big(\frac{k}{k+1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{j_k} + \frac{1}{j_k + 2}\Big) \frac{(k+1)n}{36260} \\ + \eta \sum_{i=1}^{\frac{j_k+1}{2}} \frac{(k+1)n}{2i\, \log^2((k+1)n/2i)} - \sqrt{(k+1)n} + \eta\, \frac{\sqrt{(k+1)n}}{\log^2(\sqrt{(k+1)n})}.\end{aligned} \tag{16}$$

In Table 3, we present some values for an upper bound of n_k for $k \le 10$, for certain $j_k \ge 1$, chosen so that (13) is not violated by the obtained upper bound for n_k. Appropriate values for η are chosen according to Lemma 3.1.

Table 3. The first ten upper bound values for n_k.

k	1	2	3	4	5	6	7	8	9	10
j_k	1	5	5	9	9	9	9	11	11	11
η_k	3.965	3.965	3.965	3.66	3.66	3.66	3.06	2.96	2.86	2.76
$n_k \le$	109	520	1135	1855	3213	4582	6763	8960	13031	18852

In Table 4, we compare our results with the ones from the estimates (1) and (2) for $11 \le k \le 20$. Here, we choose $j_k = 21$ for all k. Note that inequalities (15) and (16) are solved using R version 4.2.2.

Table 4. Comparison with Rosser–Schoenfeld estimates for $11 \le k \le 20$.

$\boldsymbol{k}$	11	12	13	14	15
$\boldsymbol{\eta_k}$	2.76	2.66	2.56	2.56	2.46
Rosser–Schoenfeld $\boldsymbol{n_k} \le$	10172	25105	62479	156585	394795
Current $\boldsymbol{n_k} \le$	18082	22807	28428	38580	47413
$\boldsymbol{k}$	16	17	18	19	20
$\boldsymbol{\eta_k}$	2.46	2.36	2.36	2.26	2.26
Rosser–Schoenfeld $\boldsymbol{n_k} \le$	1000560	2547270	6510820	16700500	42972300
Current $\boldsymbol{n_k} \le$	64394	78378	107141	129632	179351

Remark 3.1. *Note that inequality* (16) *is effective up to some $k_0 \in \mathbb{N}$, because for $k \ge k_0$, the sums in the error terms of $\vartheta(n)$ grow arbitrarily large. However, if a bound of the type $|\vartheta(n) - n| \le A\, n^{\theta}$ for some $\theta < 1$ and an absolute constant $A > 0$ is known, then we could obtain an inequality analogue to* (16) *that would work, in principle, for all $k \in \mathbb{N}$.*

4 Further results

4.1 An analytic upper estimate for n_k

Let $k \ge 2$; the special case $k = 1$ is dealt with accordingly. In inequality (16), denote

$$C_k = \frac{k}{k+1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{j_k} - \frac{1}{j_k+1} + \frac{1}{j_k+2},$$
$$D_k = \frac{1}{36260}\Big(\frac{k}{k+1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{j_k} + \frac{1}{j_k+2}\Big),$$
$$E_k = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{j_k-1} + \frac{1}{j_k+1}.$$

Dividing both sides of (16) by $(k+1)n$ and using the fact that

$$\log((k+1)n/2i) \le \log((k+1)n/(j_k+1))$$

for all $i = 1, 2, \ldots, (j_k+1)/2$, we get the inequality

$$\log 2 < C_k + D_k + (\sqrt{(k+1)n} + 1)\frac{\log((k+1)n)}{(k+1)n} + \eta\, E_k\, \frac{1}{\log^2((k+1)n/(j_k+1))}$$
$$- \frac{1}{\sqrt{(k+1)n}} + 4\eta\, \frac{1}{\sqrt{(k+1)n}\, \log^2((k+1)n)}.$$

Consider the function $\Psi_k(x)$ of the real variable $x > 0$ given by

$$\Psi_k(x) = C_k + D_k + (\sqrt{x_k} + 1)\frac{\log x_k}{x_k} + \eta\, E_k\, \frac{1}{\log^2(\gamma_k\, x_k)} - \frac{1}{\sqrt{x_k}} + 4\eta\, \frac{1}{\sqrt{x_k}\, \log^2(x_k)} - \log 2,$$

where $\gamma_k = 1/(j_k + 1)$ and $x_k = (k + 1)x$. It can be shown that

$$\Psi_k'(x) = (k+1)\left[\frac{3 - \log x_k}{2x_k^{3/2}} + \frac{1 - \log x_k}{x_k^2} - 2\eta E_k \frac{1}{x_k \log^3(\gamma_k x_k)} - 2\eta \frac{1}{x^{3/2}\log^2 x_k} - 8\eta \frac{1}{x^{3/2}\log^3 x_k}\right]$$

and, consequently, $\Psi_k'(x) < 0$ for all $x > e^3/(k + 1)$. It can be shown that $\lim_{x\to\infty} \Psi_k(x) = C_k + D_k - \log 2 < 0$ for all $k \le 3000$. This range could become larger if in D_k we divide by a value greater than 36260, but at the expense of inequality (11), which would hold for $n \ge n_0$ for some $n_0 \gg 2$. Moreover, $\Psi_k(e^3/(k+1)) > 0$, and then, by the intermediate value theorem, there is $\bar{x} > e^3/(k + 1)$ such that $\Psi_k(\bar{x}) = 0$. Strict monotonicity of $\Psi_k(x)$ ensures that the equation $\Psi_k(\bar{x}) = 0$ is uniquely solved for $\bar{x}$, so we could formally write $\bar{x} = \Psi_k^{-1}(0)$. Consequently, $n_k \le \lfloor \Psi_k^{-1}(0) \rfloor$ could be regarded as an (implicit) analytic upper estimate for n_k.

4.2 Some corollaries

Corollary 4.1. *For every $k \in \mathbb{N}$, there is $\overline{n}_k$ such that the intervals $(n, (1 + \frac{1}{k})n)$ contain a prime number whenever $n \ge \overline{n}_k$. Note that $\overline{n}_k = kn_k$.*

In [11], Nagura proved that for all $n \ge 25$, there is a prime in the intervals $(n, (1 + \frac{1}{5})n)$. From Table 1, we obtain $\overline{n}_5 \le 5 \cdot 3213 = 16065$, which is not as good as Nagura's upper bound $\overline{n}_5 \le 2103$, but his approach is applicable only for values of $k \le 5$.

Corollary 4.2. *Let $\varphi : \mathbb{N} \to \mathbb{R}$ be a function such that $\liminf_{n\to\infty} \varphi(n)/n > 0$, then the intervals $(n, n + \varphi(n))$ contain a prime for all large enough n.*

Proof. Let $\liminf_{n\to\infty} \varphi(n)/n = \alpha$ for some $\alpha > 0$. Then, for every $\varepsilon > 0$, there is $n(\varepsilon) \in \mathbb{N}$ such that $\varphi(n)/n > \alpha - \varepsilon$ for all $n \ge n(\varepsilon)$. In particular, for $\varepsilon = \alpha/2$, we have $\varphi(n)/n > \alpha/2$ for all sufficiently large $n \in \mathbb{N}$. Since $\alpha > 0$, there is $k \in \mathbb{N}$ such that $\alpha/2 > 1/k$. Then, for all sufficiently large $n \in \mathbb{N}$, it holds that $(n, (1 + \frac{1}{k})n) \subset (n, (1 + \frac{\alpha}{2})n) \subset (n, n + \varphi(n))$. It follows by Corollary 4.1 that there is a prime in $(n, n + \varphi(n))$ for all large enough $n \in \mathbb{N}$. □

Another equivalent formulation of Corollary 4.1 is as follows:

Corollary 4.3. *For every $k \in \mathbb{N}, k \ge 2$, there is an $\overline{n}_k \in \mathbb{N}$ such that the intervals $((1 - \frac{1}{k})n, n)$ contain a prime number whenever $n \ge \overline{n}_k$.*

In [14, 15], Rosser and Schoenfeld introduced a technique using smoothing functions and information on the zeros of Riemann's zeta function $\zeta(s)$ to estimate an x_0 such that $\vartheta(x) - \vartheta(x(1 - \Delta^{-1})) > 0$ for all $x \ge x_0$, given a certain $\Delta > 0$. For instance, in [16] Schoenfeld gave a sharp result for the case $\Delta = 16597$. Their method was refined in [13] by Ramaré and Saouter, where it was proved that the interval $((1 - \Delta^{-1})n, n)$, with $\Delta = 28314000$, always contains a prime if $n > 10726905041$.

More recently, in the same spirit, Cully-Hugill and Lee [5, 6] provided numerical results for such intervals for certain very large constants Δ and respective x_0.

5 Conclusion

We extended Erdős' arithmetical-combinatorial argument in his proof of Chebyshev's theorem, to obtain a generalization of this result. Moreover, this approach offered us a quantitative lower bound on the number of primes on intervals $(kn, (k+1)n),\ k \in \mathbb{N}$, as well as a numerical method for computations. Several comparisons were made with existing results in the literature.

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