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A quantum calculus framework for Gaussian Fibonacci and Gaussian Lucas quaternion numbers

Bahar Kuloğlu

Department of Engineering Basic Sciences,
Sivas Science and Technology University
Sivas, Türkiye

e-mails: bahar_kuloglu@hotmail.com, bkuloglu@sivas.edu.tr

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Abstract: In order to investigate the relationship between Gaussian Fibonacci numbers and quantum numbers and to develop both a deeper theoretical understanding in this study, q-Gaussian Fibonacci, q-Gaussian Lucas quaternions and polynomials are taken with quantum integers by bringing a different perspective. Based on these definitions, the Binet formula of these number sequences is found, and some algebraic properties, important theorems, propositions and identities related to the formula are given. Thus, new perspectives are obtained in the analysis and applications of complex systems.
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1 Introduction

Quantum integers are a generalization of classical integers that incorporate concepts from quantum mechanics. They are used in various fields of mathematics and physics, including quantum number theory, quantum information theory and quantum algorithms. They can represent more information than classical integers. They can be used to solve problems that are difficult for classical computers. They can be used to develop new and more efficient quantum algorithms [10, 20].

If we talk about the relationship between the quantum numbers mentioned above and Gaussian Fibonacci numbers: Gaussian Fibonacci numbers can be used in the design of quantum computers and the analysis of their algorithms. Furthermore, those working on the security of quantum technologies, such as quantum cryptography, can investigate how Gaussian Fibonacci sequences can be used in this field.

Quaternions, discovered by William Rowan Hamilton, are the number system that deals with the generalization of complex number systems. A quaternion is defined as

$$p = a + bi + cj + dk,$$

where $a, b, c, d \in \mathbb{R}$ and i, j, k are quaternion units.

The conjugate of the quaternion is defined as follows:

$$p^* = a - bi - cj - dk.$$

Horadam, inspired by William's work, defined Fibonacci and Lucas quaternions in 1963. Fibonacci and Lucas numbers took the place of a, b, c, d in this study, respectively. Later, in the following years, many authors such as [3, 4, 6, 21] worked on Pell, Jacobsthal, Pell–Lucas quaternions in addition to Fibonacci and Lucas quaternions. Gauss Fibonacci and Lucas numbers were defined by Jordan in [8] with the recurrence relations given below,

$$GF_n = GF_{n-1} + GF_{n-2}, \ \ GF_0 = i, \ \ GF_1 = 1$$

and

$$GL_n = GL_{n-1} + GL_{n-2}, \ \ GL_0 = 2 - i, \ \ GL_1 = 1 + 2i$$

afterwards their more general forms were discussed by many authors in the following years [2, 15, 22, 23].

The general terms of these numbers can be obtained more easily by making use of their relationship with the Fibonacci and Lucas numbers [8]. Namely:

$$GF_n = F_n + iF_{n-1}$$

and

$$GL_n = L_n + iL_{n-1}.$$

The fact that quantum calculus has many fields of use such as number theory, combinatorics, orthogonal polynomials, fundamental hypergeometric functions, and quantum theory, which is seen as a connection between other sciences, mechanics and relativity theory, studies in the fields of mathematics and physics have led this field. Based on this effect, many authors have worked in this field [1, 5, 11, 13, 14, 16, 17, 19]. Also, Kızılateş *et al.* [12] introduced higher-order generalized Fibonacci quaternions with q-integer components, combining q-calculus and quaternion theory. Several special cases and properties of these quaternions are also explored.

In accordance with our purpose, quaternion sequences with component quantum integers for $a \in \mathbb{N}$ is defined [9]:

$$[a]_q = \sum_{n=0}^{a-1} q^n.$$

If $-a$ is also defined, it is expressed as

$$\left[-a\right]_q = -\sum_{n=0}^{a-1} q^{-n}.$$

In particular, let K be a ring with a unit with the property of associative and q is an element of K. If $1-q$ is invertible, then the definitions we discussed above can be more easily formulated. Namely in accordance with our purpose, the q-integer of the number a is defined by

$$\left[a\right]_q = \begin{cases} \dfrac{1-q^a}{1-q}, & \text{if } q \neq 1 \\ a, & \text{if } q = 1 \end{cases}$$

In the quantum calculus approach, we can express addition and multiplication operations as

$$\left[a+b\right]_q = \left[a\right]_q + q^a \left[b\right]_q,$$

$$\left[ab\right]_q = \left[a\right]_q \left[b\right]_{q^a}$$

where $a, b \in \mathbb{Z}$ and q is invertible element in K.

For $q = \dfrac{\alpha_2}{\alpha_1}$, the Binet formula for Gaussian Fibonacci and Gaussian Lucas number can be expressed in q-integer form as follows:

$$QGF_n = \left(\alpha_1^{n-1}\left[n\right]_q + \alpha_1^{n-2}\left[n-1\right]_q i\right) \text{ and } QGL_n = \left(\alpha_1^n \frac{[2n]_q}{\left[n\right]_q} + \alpha_1^{n-1}\frac{[2n-2]_q}{\left[n-1\right]_q} i\right),$$

where $i = \sqrt{-1} = \alpha_1\sqrt{q}$ and α_1, α_2 are roots of characteristic equation for Gaussian Fibonacci and Gaussian Lucas numbers.

2 Gaussian Fibonacci quaternions with quantum calculus approach

In this section, the previously defined quaternion is considered more generally. Throughout this section, $n \in \mathbb{N}$ and $1-q$ will be treated as nonzero complex numbers.

Definition 2.1. The n-th q-Gaussian Fibonacci quaternion and n-th q-Gaussian Lucas quaternion of the form are as follows, respectively.

$$\begin{aligned} QGF_n &= \left(\alpha_1^{n-1}\left[n\right]_q + \alpha_1^{n-2}\left[n-1\right]_q i\right) + \left(\alpha_1^{n}\left[n+1\right]_q + \alpha_1^{n-1}\left[n\right]_q i\right)i + \left(\alpha_1^{n+1}\left[n+2\right]_q + \alpha_1^{n}\left[n+1\right]_q i\right)j \\ &\quad + \left(\alpha_1^{n+2}\left[n+3\right]_q + \alpha_1^{n+1}\left[n+2\right]_q i\right)k \\ &= \left(\alpha_1^{n-2}\left[n-1\right]_q + \alpha_1^{n}\left[n+1\right]_q\right)i + \left(\alpha_1^{n}\left[n+1\right]_q + \alpha_1^{n+2}\left[n+3\right]_q\right)k \end{aligned}$$

and

$$QGL_n = \left(\alpha_1^n \frac{[2n]_q}{[n]_q} + \alpha_1^{n-1} \frac{[2n-2]_q}{[n-1]_q} i \right) + \left(\alpha_1^{n+1} \frac{[2n+2]_q}{[n+1]_q} + \alpha_1^n \frac{[2n]_q}{[n]_q} i \right) i$$

$$+ \left(\alpha_1^{n+2} \frac{[2n+4]_q}{[n+2]_q} + \alpha_1^{n+1} \frac{[2n+2]_q}{[n+1]_q} i \right) j + \left(\alpha_1^{n+3} \frac{[2n+6]_q}{[n+3]_q} + \alpha_1^{n+2} \frac{[2n+4]_q}{[n+2]_q} i \right) k$$

$$= \left(\alpha_1^{n-1} \frac{[2n-2]_q}{[n-1]_q} + \alpha_1^{n+1} \frac{[2n+2]_q}{[n+1]_q} \right) i + \left(\alpha_1^{n+1} \frac{[2n+2]_q}{[n+1]_q} + \alpha_1^{n+3} \frac{[2n+6]_q}{[n+3]_q} \right) k$$

Theorem 2.2. The Binet's formula of the q-Gaussian Fibonacci quaternion and q-Gaussian Lucas quaternion are as follows, respectively.

$$QGF_n = \alpha_1^{n-2} [n]_q \alpha_1^* + \frac{(\alpha_1 q)^n}{\alpha_1^2} \alpha_1^{**}$$

and

$$QGL_n = \alpha_1^{n-1} \alpha_1^* + (\alpha_1 q)^{n-1} \alpha_1^{***},$$

where

$$\alpha_1^* = i + \alpha_1^2 i + \alpha_1^2 k + \alpha_1^4 k,$$
$$\alpha_1^{**} = [-1]_q i + \alpha_1^2 i + \alpha_1^2 k + \alpha_1^4 [3]_q k,$$
$$\alpha_1^{***} = i + \alpha_1^2 q^2 i + \alpha_1^2 q^2 k + \alpha_1^4 q^4 k.$$

Proof. By Definition 2.1, we get

$$QGF_n = \left(\alpha_1^{n-2} [n-1]_q + \alpha_1^n [n+1]_q \right) i + \left(\alpha_1^n [n+1]_q + \alpha_1^{n+2} [n+3]_q \right) k$$

$$= \alpha_1^{n-2} \left([n]_q + q^n [-1]_q \right) i + \alpha_1^n \left([n]_q + q^n \right) i + \alpha_1^n \left([n]_q + q^n \right) k + \alpha_1^{n+2} \left([n]_q + [3]_q q^n \right) k$$

$$= \alpha_1^{n-2} [n]_q \left(i + \alpha_1^2 i + \alpha_1^2 k + \alpha_1^4 k \right) + \alpha_1^{n-2} q^n \left([-1]_q i + \alpha_1^2 i + \alpha_1^2 k + \alpha_1^4 [3]_q k \right).$$

That is,

$$QGF_n = \alpha_1^{n-2} [n]_q \alpha_1^* + \frac{(\alpha_1 q)^n}{\alpha_1^2} \alpha_1^{**}.$$

By Definition 2.1, the Binet's formula of QGL_n is

$$QGL_n = \left(\alpha_1^{n-1} \frac{[2n-2]_q}{[n-1]_q} + \alpha_1^{n+1} \frac{[2n+2]_q}{[n+1]_q} \right) i + \left(\alpha_1^{n+1} \frac{[2n+2]_q}{[n+1]_q} + \alpha_1^{n+3} \frac{[2n+6]_q}{[n+3]_q} \right) k$$

$$= \alpha_1^{n-1} \left(\frac{1-q^{2n-2}}{1-q^{n-1}} \right) i + \alpha_1^{n+1} \left(\frac{1-q^{2n+2}}{1-q^{n+1}} \right) i + \alpha_1^{n+1} \left(\frac{1-q^{2n+2}}{1-q^{n+1}} \right) k + \alpha_1^{n+3} \left(\frac{1-q^{2n+6}}{1-q^{n+3}} \right) k$$

$$= \alpha_1^{n-1} \left(1 + q^{n-1} \right) i + \alpha_1^{n+1} \left(1 + q^{n+1} \right) i + \alpha_1^{n+1} \left(1 + q^{n+1} \right) k + \alpha_1^{n+3} \left(1 + q^{n+3} \right) k$$

$$= \alpha_1^{n-1} \left(i + \alpha_1^2 i + \alpha_1^2 k + \alpha_1^4 k \right) + \alpha_1^{n-1} q^{n-1} \left(i + \alpha_1^2 q^2 i + \alpha_1^2 q^2 k + \alpha_1^4 q^4 k \right).$$

That is,

$$QGL_n = \alpha_1^{n-1} \alpha_1^* + (\alpha_1 q)^{n-1} \alpha_1^{***}.$$

This completes the proof. □

Remark 2.3. Binet's formula of the q-quaternions QGF_n is written in another forms.

$$\begin{aligned} QGF_n &= \left(\alpha_1^{n-2}[n-1]_q + \alpha_1^n[n+1]_q\right)i + \left(\alpha_1^n[n+1]_q + \alpha_1^{n+2}[n+3]_q\right)k \\ &= \alpha_1^{n-2}\left(\frac{1-q^{n-1}}{1-q}\right)i + \alpha_1^n\left(\frac{1-q^{n+1}}{1-q}\right)i + \alpha_1^n\left(\frac{1-q^{n+1}}{1-q}\right)k + \alpha_1^{n+2}\left(\frac{1-q^{n+3}}{1-q}\right)k \\ &= \frac{\alpha_1^n - \alpha_1^n q^{n-1}}{\alpha_1^2 - \alpha_1^2 q} i + \frac{\alpha_1^{n+2} - \alpha_1^{n+2} q^{n+1}}{\alpha_1^2 - \alpha_1^2 q} i + \frac{\alpha_1^{n+2} - \alpha_1^{n+2} q^{n+1}}{\alpha_1^2 - \alpha_1^2 q} k + \frac{\alpha_1^{n+4} - \alpha_1^{n+4} q^{n+3}}{\alpha_1^2 - \alpha_1^2 q} k \\ &= \frac{\alpha_1^n}{\alpha_1^2 - \alpha_1^2 q}\left(i + \alpha_1^2 i + \alpha_1^2 k + \alpha_1^4 k\right) - \frac{\alpha_1^n q^{n-1}}{\alpha_1^2 - \alpha_1^2 q}\left(i + \alpha_1^2 q^2 i + \alpha_1^2 q^2 k + \alpha_1^4 q^4 k\right) \\ &= \frac{\alpha_1^n \alpha_1^* - \alpha_1^n q^{n-1} \alpha_1^{***}}{\alpha_1^2(1-q)}. \end{aligned}$$

Theorem 2.4. The exponential generating function for the q-Gaussian Fibonacci quaternions and q-Gaussian Lucas quaternions are as follows, respectively,

$$QGF_n = \frac{e^{\alpha_1 x}\alpha_1^* - q^{-1}e^{(\alpha_1 q)x}\alpha_1^{***}}{\alpha_1^2(1-q)}$$

and

$$QGL_n = \frac{\alpha_1^* e^{\alpha_1 x} + \alpha_1^{***} q^{-1} e^{(\alpha_1 q)x}}{\alpha}.$$

Proof. By using Binet formula of QGF_n and QGL_n, we have

$$e^{\omega x} = \sum_{n=0}^{\infty} \omega^n \frac{x^n}{n!}.$$

So, the desired result is obtained. So that,

$$\begin{aligned} \sum_{n=0}^{\infty} QGF_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty}\left(\frac{\alpha_1^n\alpha_1^* - \alpha_1^n q^{n-1}\alpha_1^{***}}{\alpha_1^2(1-q)}\right)\frac{x^n}{n!} \\ &= \frac{\alpha_1^*}{\alpha_1^2(1-q)}\sum_{n=0}^{\infty}\frac{(\alpha_1 x)^n}{n!} - \frac{q^{-1}\alpha_1^{***}}{\alpha_1^2(1-q)}\sum_{n=0}^{\infty}\frac{(\alpha_1 q x)^n}{n!} \\ &= \frac{\alpha_1^*}{\alpha_1^2(1-q)}e^{\alpha_1 x} - \frac{q^{-1}\alpha_1^{***}}{\alpha_1^2(1-q)}e^{\alpha_1 q x} \\ &= \frac{e^{\alpha_1 x}\alpha_1^* - q^{-1}e^{(\alpha_1 q)x}\alpha_1^{***}}{\alpha_1^2(1-q)}. \end{aligned}$$

Similarly,

$$\sum_{n=0}^{\infty} QG\frac{x^n}{n!} = \sum_{n=0}^{\infty}\left(\alpha_1^{n-1}\alpha_1^{*} + (\alpha_1 q)^{n-1}\alpha_1^{***}\right)\frac{x^n}{n!}$$
$$= \frac{\alpha_1^{*}}{\alpha_1}\sum_{n=0}^{\infty}\frac{(\alpha_1 x)^n}{n!} - \frac{\alpha_1^{***}}{\alpha_1 q}\sum_{n=0}^{\infty}\frac{(\alpha_1 q x)^n}{n!}$$
$$= \frac{\alpha_1^{*}}{\alpha_1}e^{\alpha_1 x} + \frac{\alpha_1^{***}}{\alpha_1 q}e^{\alpha_1 q x}$$
$$= \frac{\alpha_1^{*}e^{\alpha_1 x} + \alpha_1^{***}q^{-1}e^{(\alpha_1 q)x}}{\alpha}.$$

This completes the proof. □

Remark 2.5. Let $\alpha_1 = \frac{1+\sqrt{5}}{2}, q = \frac{-1}{\alpha_1^2}$ then Gaussian Fibonacci quaternions are obtained. Similarly, for $\alpha_1 = 1+\sqrt{2},\ q = \frac{-1}{\alpha_1^2}$, Gaussian Pell quaternions and for $\alpha_1 = 2, q = \frac{-1}{2}$, Gaussian Jacobsthal quaternions are obtained.

Remark 2.6. Let $\alpha_1^2 q = -1$. From the Binet formula of QGF_n and QGL_n, $n \geq 1, n \in \mathbb{Z}$,

$$QGF_n - (\alpha_1 q)QGF_{n-1} = \frac{\alpha_1^n\alpha_1^{*} - \alpha_1^n q^{n-1}\alpha_1^{***}}{\alpha_1^2(1-q)} - (\alpha_1 q)\frac{\alpha_1^{n-1}\alpha_1^{*} - \alpha_1^{n-1}q^{n-2}\alpha_1^{***}}{\alpha_1^2(1-q)}$$
$$= \frac{\alpha_1^n\alpha_1^{*} - \alpha_1^n q^{n-1}\left(i + \alpha_1^2 q^2 i + \alpha_1^2 q^2 k + \alpha_1^4 q^4 k\right)}{\alpha_1^2(1-q)}$$
$$+ \frac{-(\alpha_1 q)\alpha_1^{n-1}\alpha_1^{*} + \alpha_1^n q^{n-1}\left(i + \alpha_1^2 q^2 i + \alpha_1^2 q^2 k + \alpha_1^4 q^4 k\right)}{\alpha_1^2(1-q)} \tag{1}$$
$$= \frac{\alpha_1^n\alpha_1^{*} - \alpha_1^n q\alpha_1^{*}}{\alpha_1^2(1-q)}$$
$$= \alpha_1^{n-2}\alpha_1^{*}$$

and

$$QGF_n - (\alpha_1)QGF_{n-1} = \frac{\alpha_1^n\left(i + \alpha_1^2 i + \alpha_1^2 k + \alpha_1^4 k\right) - \alpha_1^n q^{n-1}\alpha_1^{***}}{\alpha_1^2(1-q)}$$
$$- \frac{\alpha_1^n\left(i + \alpha_1^2 i + \alpha_1^2 k + \alpha_1^4 k\right) + \alpha_1^n q^{n-2}\alpha_1^{***}}{\alpha_1^2(1-q)} \tag{2}$$
$$= \frac{\alpha_1^n\alpha_1^{***}q^{n-2}(1-q)}{\alpha_1^2(1-q)}$$
$$= (\alpha_1 q)^{n-2}\alpha_1^{***}$$

multiplying equation (1) and (2) by α_1 and $\alpha_1 q$, respectively, the linearization of QGF_n are obtained, so that

$$\alpha_1^{n-1}\alpha_1^{*} = \alpha_1 QGF_n + QGF_{n-1}$$
$$(\alpha_1 q)^{n-1}\alpha_1^{***} = (\alpha_1 q)QGF_n + QGF_{n-1}.$$

Theorem 2.7 The summation identities for QGF_n and QGL_n are as follows:

(i) $$\sum_{n=0}^{m}\binom{m}{n}(-\alpha_1^2 q)^{m-n} QGF_{2n+k} = \begin{cases} \left(\sqrt{\nabla}\right)^m QGF_{m+k}, & m \text{ is even} \\ \left(\sqrt{\nabla}\right)^{m-1} QGL_{m+k-1}, & m \text{ is odd} \end{cases}$$

(ii) $$\sum_{n=0}^{m}\binom{m}{n}(-\alpha_1^2 q)^{m-n} QGL_{2n+k} = \begin{cases} \left(\sqrt{\nabla}\right)^m QGL_{m+k}, & m \text{ is even} \\ \left(\sqrt{\nabla}\right)^{m+1} QGF_{m+k-1}, & m \text{ is odd} \end{cases}$$

where $\nabla = (\alpha_1(1-q))^2$.

Proof. By using Binet formula for QGF_n and QGL_n, we have

$$T = \sum_{n=0}^{m}\binom{m}{n}(-\alpha_1^2 q)^{m-n} QGF_{2n+k} = \sum_{n=0}^{m}\binom{m}{n}(-\alpha_1^2 q)^{m-n}\left(\frac{\alpha_1^{2n+k}\alpha_1^{*} - \alpha_1^{2n+k} q^{2n+k-1}\alpha_1^{***}}{\alpha_1^2(1-q)}\right).$$

Observe that

$$\sum_{n=0}^{m}\binom{m}{n}(-\alpha_1^2 q)^{m-n}\left(\alpha_1^2\right)^n = (\alpha_1^2 - \alpha_1^2 q)^m,$$

$$\sum_{n=0}^{m}\binom{m}{n}(-\alpha_1^2 q)^{m-n}\left(\left(\alpha_1 q\right)^2\right)^n = (\left(\alpha_1 q\right)^2 - \alpha_1^2 q)^m,$$

since

$$\alpha_1^2 - \alpha_1^2 q = \alpha_1\alpha_1(1-q) = \alpha_1\sqrt{\nabla},$$

$$\left(\alpha_1 q\right)^2 - \alpha_1^2 q = -\alpha_1 q\sqrt{\nabla},$$

$$T\,\frac{(\alpha_1\sqrt{\nabla})^m \alpha_1^k \alpha_1^{*} - (-\alpha_1 q\sqrt{\nabla})^m \left(\alpha_1 q\right)^k q^{-1}\alpha_1^{***}}{\alpha_1^2(1-q)} = \left(\sqrt{\nabla}\right)^m\left(\frac{\alpha_1^{m+k}\alpha_1^{*} - (-\alpha_1 q)^m\left(\alpha_1 q\right)^k q^{-1}\alpha_1^{***}}{\alpha_1^2(1-q)}\right).$$

If m is even, then

$$T = \left(\sqrt{\nabla}\right)^m\left(\frac{\alpha_1^{m+k}\alpha_1^{*} - (\alpha_1 q)^{m+k} q^{-1}\alpha_1^{***}}{\alpha_1^2(1-q)}\right) = \left(\sqrt{\nabla}\right)^m QGF_{m+k}.$$

If m is odd, then

$$\begin{aligned} T &= \left(\sqrt{\nabla}\right)^m\left(\frac{\alpha_1^{m+k}\alpha_1^{*} + (\alpha_1 q)^{m+k} q^{-1}\alpha_1^{***}}{\alpha_1^2(1-q)}\right) \\ &= \left(\sqrt{\nabla}\right)^m\left(\frac{\alpha_1^{m+k}\alpha_1^{*} + (\alpha_1 q)^{m+k} q^{-1}\alpha_1^{***}}{\alpha_1\sqrt{\nabla}}\right) \\ &= \left(\sqrt{\nabla}\right)^{m-1}\left(\alpha_1^{m+k-1}\alpha_1^{*} + (\alpha_1 q)^{m+k-1}\alpha_1^{***}\right) \\ &= \left(\sqrt{\nabla}\right)^{m-1} QGL_{m+k-1}. \end{aligned}$$

The other proof is done similarly. □

Theorem 2.8. Let $m \in \mathbb{N}$. Then

(i) $\sum_{n=0}^{m} \binom{m}{n} \left[\alpha_1 (1+q)\right]^n (-\alpha_1^2 q)^{m-n} QGF_n = QGF_{2m}$,

(ii) $\sum_{n=0}^{m} \binom{m}{n} \left[\alpha_1 (1+q)\right]^n (-\alpha_1^2 q)^{m-n} QGL_n = QGL_{2m}$.

Proof. (i) By using Binet formula for QGF_n and QGL_n, we get

$$\begin{aligned}
T &= \sum_{n=0}^{m} \binom{m}{n} \left[\alpha_1 (1+q)\right]^n (-\alpha_1^2 q)^{m-n} QGF_n \\
&= \sum_{n=0}^{m} \binom{m}{n} \left[\alpha_1 (1+q)\right]^n (-\alpha_1^2 q)^{m-n} \left(\frac{\alpha_1^n \alpha_1^* - (\alpha_1 q)^n q^{-1} \alpha_1^{***}}{\alpha_1^2 (1-q)} \right) \\
&= \sum_{n=0}^{m} \binom{m}{n} \left[\alpha_1^2 (1+q)\right]^n (-\alpha_1^2 q)^{m-n} \left(\frac{\alpha_1^*}{\alpha_1^2 (1-q)} \right) - \sum_{n=0}^{m} \binom{m}{n} \left[\alpha_1^2 q (1+q)\right]^n (-\alpha_1^2 q)^{m-n} \left(\frac{q^{-1} \alpha_1^{***}}{\alpha_1^2 (1-q)} \right) \\
&= \left(\alpha_1^2\right)^m \left(\frac{\alpha_1^*}{\alpha_1^2 (1-q)} \right) - \frac{\left(\alpha_1^2 q^2\right)^m q^{-1} \alpha_1^{***}}{\alpha_1^2 (1-q)} = QGF_{2m}.
\end{aligned}$$

The other proof is done similarly. □

Theorem 2.9. Let $n \geq 2$ be integer. Then it holds that

$$\begin{pmatrix} QGF_n & QGF_{n-1} \\ QGF_{n+1} & QGF_n \end{pmatrix} = \begin{pmatrix} QGF_2 & QGF_1 \\ QGF_3 & QGF_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}.$$

Proof. The proof can be easily seen through induction. □

Theorem 2.10 (Catalan identity). Let $m, k \in \mathbb{Z}^+, m \geq k$ then we have

$$QGF_{m+k} QGF_{m-k} - \left(QGF_m\right)^2 = \frac{-\left(q^k - 1\right)^2 q^{m-k-1} \alpha_1^{2m-4} \alpha_1^* \alpha_1^{***}}{(1-q)^2}$$

and

$$QGL_{m+k} QGL_{m-k} - \left(QGL_m\right)^2 = \alpha_1^* \alpha_1^{***} \alpha_1^{2m-2} q^{m-k-1} \left(q^k - 1\right)^2.$$

Proof. The Binet formula for the QGF_n and after some algebraic operations, we get the following equation:

$$\begin{aligned}
&\left(\frac{\alpha_1^{m+k} \alpha_1^* + (\alpha_1 q)^{m+k} q^{-1} \alpha_1^{***}}{\alpha_1^2 (1-q)} \right) \left(\frac{\alpha_1^{m-k} \alpha_1^* + (\alpha_1 q)^{m-k} q^{-1} \alpha_1^{***}}{\alpha_1^2 (1-q)} \right) - \left(\frac{\alpha_1^m \alpha_1^* + (\alpha_1 q)^m q^{-1} \alpha_1^{***}}{\alpha_1^2 (1-q)} \right)^2 \\
&= \frac{-\alpha_1^{2m} q^{m-k-1} \alpha_1^* \alpha_1^{***} - \alpha_1^{2m} q^{m+k-1} \alpha_1^* \alpha_1^{***} + 2\alpha_1^{2m} q^{m-1} \alpha_1^* \alpha_1^{***}}{\left(\alpha_1^2 (1-q)\right)^2} \\
&= \frac{-q^{m-1} \alpha_1^{2m} \alpha_1^* \alpha_1^{***} \left(q^{-k} + q^k - 2\right)}{\left(\alpha_1^2 (1-q)\right)^2} = \frac{-\left(q^k - 1\right)^2 q^{m-k-1} \alpha_1^{2m-4} \alpha_1^* \alpha_1^{***}}{(1-q)^2}.
\end{aligned}$$

The other proof is done similarly. □

Theorem 2.11 (Cassini identity). For $m \geq 1$ the following equation hold.

$$QGF_{m+1}QGF_{m-1} - (QGF_m)^2 = \frac{-(q-1)^2 q^{m-2} \alpha_1^{2m-4} \alpha_1^* \alpha_1^{***}}{(1-q)^2}$$

and

$$QGL_{m+1}QGL_{m-1} - (QGL_m)^2 = \alpha_1^* \alpha_1^{***} \alpha_1^{2m-2} q^{m-2} (q-1)^2 .$$

Proof. A special case for $k = 1$ of Theorem 2.10 the proof is done. □

Theorem 2.12 (d'Ocagnes identity). Let $m, n \in \mathbb{Z}^+$, $m > n+1$, then we have

$$QGF_m QGF_{n+1} - QGF_{m+1} QGF_n = \frac{\alpha_1^* \alpha_1^{***} \alpha_1^{m+n-3} (1 - q^{m-n}) q^{n-1}}{(1-q)}$$

and

$$QGL_m QGL_{n+1} - QGL_{m+1} QGL_n = \alpha_1^* \alpha_1^{***} \alpha_1^{m+n-1} (1 - q^{m-n})(q^n - q^{n-1}).$$

Proof. The Binet formula for the QGF_n and after some algebraic operations we get the following equation

$$\left(\frac{\alpha_1^m \alpha_1^* + (\alpha_1 q)^m q^{-1} \alpha_1^{***}}{\alpha_1^2 (1-q)} \right) \left(\frac{\alpha_1^{n+1} \alpha_1^* + (\alpha_1 q)^{n+1} q^{-1} \alpha_1^{***}}{\alpha_1^2 (1-q)} \right) - \left(\frac{\alpha_1^{m+1} \alpha_1^* + (\alpha_1 q)^{m+1} q^{-1} \alpha_1^{***}}{\alpha_1^2 (1-q)} \right) \left(\frac{\alpha_1^n \alpha_1^* + (\alpha_1 q)^n q^{-1} \alpha_1^{***}}{\alpha_1^2 (1-q)} \right)$$

$$= \frac{\alpha_1^* \alpha_1^{***} \alpha_1^{m+n-1} (-q^n - q^{m-1} + q^{n-1} + q^m)}{(\alpha_1^2 (1-q))^2} = \frac{\alpha_1^* \alpha_1^{***} \alpha_1^{m+n-1} (1 - q^{m-n})(q^{n-1} - q^n)}{(\alpha_1^2 (1-q))^2}$$

$$= \frac{\alpha_1^* \alpha_1^{***} \alpha_1^{m+n-3} (1 - q^{m-n}) q^{n-1}}{(1-q)}$$

as desired. The other proof is done similarly. □

3 Gaussian Fibonacci quaternion polynomials with quantum calculus approach

In this section, we get *q*-Gaussian Fibonacci $QGF_n(x)$ and *q*-Gaussian Lucas polynomials $QGL_n(x)$. Based on the definitions, important results of these polynomials are obtained.

Definition 3.1. Let $u(x)$ and $v(x)$ be polynomials with complex coefficients. The *q*-Gaussian polynomials $QGF_n(x)$ and $QGL_n(x)$ are defined by the recurrence relation

$$\begin{aligned} QGF_{n+2}(x) &= u(x) QGF_{n+1}(x) - v(x) QGF_n(x), \\ QGL_{n+2}(x) &= u(x) QGL_{n+1}(x) - v(x) QGL_n(x) \end{aligned} \tag{3}$$

with initial conditions $QGF_1(x) = 1$, $QGF_2(x) = u(x) + i$ and $QGL_1(x) = u(x) + 2i$, $QGL_2(x) = u^2(x) + 2 + u(x)i$, respectively.

If we take $u(x)=x$ and $v(x)=-1$, we get Gaussian Fibonacci polynomials and Gaussian Lucas polynomials [18].

Similarly, if we take $u(x)=2x$ and $v(x)=-1$, we get Gaussian Pell polynomials [7].

Let the roots of the characteristic equation

$$t^2-u(x)t+v(x)=0$$

of the recurrence (3) be

$$\alpha_1(t)=\frac{u(x)+\sqrt{u^2(x)-4v(x)}}{2}, \quad \alpha_2(t)=\frac{u(x)-\sqrt{u^2(x)-4v(x)}}{2},$$

then Binet formula for q-Gaussian polynomials $QGF_n(x)$ and $QGL_n(x)$ are

$$QGF_n(x)=\frac{\alpha_1^n(t)+i\alpha_1^{n-1}(t)}{\alpha_1(t)-\alpha_2(t)}-\frac{\alpha_2^n(t)+i\alpha_2^{n-1}(t)}{\alpha_1(t)-\alpha_2(t)}$$

and

$$QGL_n(x)=\left(\alpha_1^n(t)+i\alpha_1^{n-1}(t)\right)+\left(\alpha_2^n(t)+i\alpha_2^{n-1}(t)\right).$$

Definition 3.2. The *q*-Gaussian quaternion polynomials $QGF_n(x)$ and $QGL_n(x)$ are defined by the recurrence relation, $n\geq 1$

$$QGF_n(x)=QGF_n(x)+iQGF_{n+1}(x)+jQGF_{n+2}(x)+kQGF_{n+3}(x),$$
$$QGL_n(x)=QGL_n(x)+iQGL_{n+1}(x)+jQGL_{n+2}(x)+kQGL_{n+3}(x).$$

The initial conditions of q-Gaussian quaternion polynomials sequence $QGF_n(x)$ are

$$\begin{aligned}QGF_1(x)&=QGF_1(x)+iQGF_2(x)+jQGF_3(x)+kQGF_4(x)\\&=u(x)i+\left(u^3(x)+u(x)-2u(x)v(x)\right)k,\\QGF_2(x)&=QGF_2(x)+iQGF_3(x)+jQGF_4(x)+kQGF_5(x)\\&=\left(u^2(x)-v(x)\right)i+\left(u^4(x)+u^2(x)-3u^2(x)v(x)+v^2(x)-v(x)\right)k.\end{aligned}$$

The initial conditions of q-Gaussian quaternion polynomials sequence $QGL_n(x)$ are

$$\begin{aligned}QGL_1(x)&=QGL_1(x)+iQGL_2(x)+jQGL_3(x)+kQGL_4(x)\\&=4i+u^2(x)i+\left(2u(x)-u^3(x)+2u(x)v(x)\right)j+\left(3u^2(x)-4v(x)+u^4(x)-2u^2(x)v(x)\right)k,\end{aligned}$$

$$\begin{aligned}QGL_2(x)&=QGL_2(x)+iQGL_3(x)+jQGL_4(x)+kQGL_5(x)\\&=2+2v(x)+\left(u^3(x)+3u(x)-u(x)v(x)\right)i+\left(2u^2(x)+2u^2(x)v(x)-2v(x)-2v^2(x)\right)j\\&\quad+\left(u^5(x)+2u^3(x)-3u^3(x)v(x)-7u(x)v(x)+u(x)v^2(x)\right)k.\end{aligned}$$

Theorem 3.3. The generating functions for the *q*-Gaussian quaternion polynomials $QGF_n(x)$ and $QGL_n(x)$ are as follows, respectively.

$$QGF_n^*(s)=\frac{QGF_1(x)s+\left[QGF_2(x)-u(x)QGF_1(x)\right]s^2}{1-u(x)s+v(x)s^2}$$

and

$$QGL_n^*(s)=\frac{QGL_1(x)s+\left[QGL_2(x)-u(x)QGL_1(x)\right]s^2}{1-u(x)s+v(x)s^2}.$$

Proof. The form of the generating function $QGF_n^*(s)$ for the q-Gaussian quaternion polynomials $QGF_n(x)$ is $\sum_{n=1}^{\infty} QGF_n(x)s^n$. Then the power series expansion of $-u(x)s$ and $v(x)s^2$ will be $\sum_{n=1}^{\infty} -u(x)QGF_n(x)s^{n+1}$ and $\sum_{n=1}^{\infty} v(x)QGF_n(x)s^{n+2}$, respectively. Thus, we obtain that

$$\left(1-u(x)s+v(x)s^2\right)QGF_n^*(s)=QGF_1(x)s+\left[QGF_2(x)-u(x)QGF_1(x)\right]s^2$$

and so

$$QGF_n^*(s)=\frac{QGF_1(x)s+\left[QGF_2(x)-u(x)QGF_1(x)\right]s^2}{1-u(x)s+v(x)s^2}$$

The generating function of q-Gaussian quaternion polynomials $QGL_n(x)$ is

$$QGL_n^*(s)=\frac{QGL_1(x)s+\left[QGL_2(x)-u(x)QGL_1(x)\right]s^2}{1-u(x)s+v(x)s^2}.$$

Theorem 3.4. The Binet formulas for $QGF_n(x)$ and $QGL_n(x)$ are as follows, respectively.

$$QGF_n(x)=\frac{\alpha_1^{n-1}(t)\overline{\alpha_1(t)}-\alpha_2^{n-1}(t)\overline{\alpha_2(t)}}{\alpha_1(t)-\alpha_2(t)}$$

and

$$QGL_n(x)=\alpha_1^{n-1}(t)\overline{\alpha_1(t)}-\alpha_2^{n-1}(t)\overline{\alpha_2(t)},$$

where

$$\overline{\alpha_1(t)}=i+\alpha_1^2(t)i+\alpha_1^2(t)k+\alpha_1^4(t)k,\ \overline{\alpha_2(t)}=i+\alpha_2^2(t)i+\alpha_2^2(t)k+\alpha_2^4(t)k.$$

The following relations can be obtained:

$$\begin{aligned}
QGF_2(x)-\alpha_1(t)QGF_1(x)&=\overline{\alpha_2(t)}\\
QGF_2(x)-\alpha_2(t)QGF_1(x)&=\overline{\alpha_1(t)}\\
QGL_2(x)-\alpha_1(t)QGL_1(x)&=\overline{\alpha_2(t)}\left(\alpha_2(t)-\alpha_1(t)\right)\\
QGL_2(x)-\alpha_2(t)QGL_1(x)&=\overline{\alpha_1(t)}(\alpha_1(t)-\alpha_2(t)).
\end{aligned}$$

Theorem 3.5. For $QGF_n(x)$ and $QGL_n(x)$, $n \geq 1$, we have the following summation formula

(i) $\sum_{n=0}^{m} (-v(x))^{m-n} (u(x))^{n} QGF_n(x) = QGF_{2m}(x)$

(ii) $\sum_{n=0}^{m} (-v(x))^{m-n} (u(x))^{n} QGL_n(x) = QGL_{2m}(x)$.

Proof. (i) For the Binet formula for $QGF_n(x)$, we get

$$\sum_{n=0}^{m} (-v(x))^{m-n} (u(x))^{n} \frac{\alpha_1^{n-1}(t)\overline{\alpha_1(t)} - \alpha_2^{n-1}(t)\overline{\alpha_2(t)}}{\alpha_1(t) - \alpha_2(t)}$$

$$= \sum_{n=0}^{m} (-v(x))^{m-n} (u(x))^{n} \frac{\alpha_1^{n-1}(t)\overline{\alpha_1(t)}}{\alpha_1(t) - \alpha_2(t)} - \sum_{n=0}^{m} (-v(x))^{m-n} (u(x))^{n} \frac{\alpha_2^{n-1}(t)\overline{\alpha_2(t)}}{\alpha_1(t) - \alpha_2(t)}$$

$$= (-v(x) + u(x)\alpha_1(t))^{m} \frac{\alpha_1^{n-1}(t)\overline{\alpha_1(t)}}{\alpha_1(t) - \alpha_2(t)} - (-v(x) + u(x)\alpha_1(t))^{m} \frac{\alpha_2^{n-1}(t)\overline{\alpha_2(t)}}{\alpha_1(t) - \alpha_2(t)}$$

$$= \frac{\alpha_1^{2m-1}(t)\overline{\alpha_1(t)} - \alpha_2^{2m-1}(t)\overline{\alpha_2(t)}}{\alpha_1(t) - \alpha_2(t)} = QGF_{2m}(x).$$

The proof of (ii) is done similarly. □

3 Conclusion

In this study, Quantum calculus approach to Gaussian Fibonacci and Gaussian Lucas recurrences with polynomials were created. Then the corresponding Binet formula of these sequences and many related properties were obtained. Also, several additive formulas of these new sequences were obtained. This work can be applied to different number sequences, as well as to expanding the p-analogue part to create (*p*, *q*)-analogue number sequences. This work can be placed in the historical perspective going back to Leonard Carlitz of Duke University.

Research on the relationship between Gaussian Fibonacci numbers and quantum numbers can lead to both a deeper theoretical understanding and the development of new methods for practical applications. This could contribute to the further advancement of quantum technologies. Also, this work can contribute to the literature in number theory, mathematical physics, cryptography, signal processing and other fields. In particular, it can offer new perspectives in the analysis and applications of complex systems.

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