Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2024, Volume 30, Number 4, 857–868
DOI: 10.7546/nntdm.2024.30.4.857-868

Factors of alternating convolution of the Gessel numbers

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Received: 3 July 2024 **Revised:** 3 December 2024
Accepted: 11 December 2024 **Online First:** 16 December 2024

Abstract: The Gessel number $P(n, r)$ is the number of lattice paths in the plane with $(1, 0)$ and $(0, 1)$ steps from $(0, 0)$ to $(n + r, n + r - 1)$ that never touch any of the points from the set $\{(x, x) \in \mathbb{Z}^2 : x \geq r\}$. We show that there is a close relationship between Gessel numbers $P(n, r)$ and super Catalan numbers $T(n, r)$. A new class of binomial sums, so called M sums, is used. By using one form of the Pfaff–Saalschütz theorem, a new recurrence relation for M sums is proved. Finally, we prove that an alternating convolution of Gessel numbers $P(n, r)$ multiplied by a power of a binomial coefficient is always divisible by $\frac{1}{2}T(n, r)$.
Keywords: Gessel number, Super Catalan number, M sum, Catalan number, Pfaff–Saalschütz theorem.
2020 Mathematics Subject Classification: 05A10, 11B65.

1 Introduction

Let n be a non-negative integer, and let r be a fixed positive integer. Let the number $P(n, r)$ denote $\frac{r}{2(n+r)}\binom{2n}{n}\binom{2r}{r}$. We shall call $P(n, r)$ the n-th Gessel number of order r.

It is known that $P(n, r)$ is always an integer. It has an interesting combinatorial interpretation, since the Gessel number $P(n, r)$ [6, p. 191] counts lattice paths in the plane with unit horizontal and vertical steps from $(0,0)$ to $(n+r, n+r-1)$ that never touch any of the points (r,r), $(r+1, r+1), \ldots$.

Recently [16, Theorem 5, p. 2], it has been shown that $P(n,r)$ is also equal to the number of lattice paths in the plane with $(1,0)$ and $(0,1)$ steps from $(0,0)$ to $(n+r, n+r-1)$ that never touch any of the points from the set $\{(x,x) \in \mathbb{Z}^2 : 1 \le x \le n\}$, where n and r are natural numbers.

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote [14, Chapter 5, p. 103] the n-th Catalan number, and let $T(n,r) = \frac{\binom{2n}{n}\binom{2r}{r}}{\binom{n+r}{n}}$ denote the n-th super Catalan number of order r. Obviously, $P(n,1) = C_n$. Furthermore, it is readily verified that

$$P(n,r) = \binom{n+r-1}{n}\frac{1}{2}T(n,r). \tag{1}$$

It is known that $T(n,r)$ is always an even integer except for the case $n = r = 0$. See [1, Introduction] and [3, Eq. (1), p. 1]. For only a few values of r, there exist combinatorial interpretations of $T(n,r)$. See for example [1, 3, 4, 12, 22, 24]. The problem of finding a combinatorial interpretation for super Catalan numbers of an arbitrary order r is an intriguing open problem.

By using Eq. (1), it follows that $P(n,r)$ is an integer. Note that Gessel numbers $P(n,r)$ have a generalization [11, Eq. (1.10), p. 2]. Also, it is known [6, p. 191] that, for a fixed positive integer r, the smallest positive integer K_r such that $\frac{K_r}{n+r}\binom{2n}{n}$ is an integer for every n, is equal to $\frac{r}{2}\binom{2r}{r}$.

Let us consider the following sum:

$$\varphi(2n, m, r-1) = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m P(k,r)P(2n-k,r), \tag{2}$$

where m is a positive integer.

For $r = 1$, the sum in Eq. (2) reduces to

$$\varphi(2n, m, 0) = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m C_k C_{2n-k}. \tag{3}$$

Recently, by using a new method, it has been shown in [19, Cor. 4, p. 2] that the sum $\varphi(2n, m, 0)$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and for all positive integers m. In particular, $\varphi(2n, 1, 0) = C_n\binom{2n}{n}$. See for example, [20, Th. 1, Eq. (2)], [23], and [5]. Gessel numbers appear [19, Eq. (68), p. 17] in this proof.

By using Eq. (1), the sum $\varphi(2n, m, r-1)$ can be rewritten, as follows:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m\binom{k+r-1}{k}\binom{2n-k+r-1}{2n-k}\frac{1}{2}T(k,r)\frac{1}{2}T(2n-k,r). \tag{4}$$

Let $\Psi(2n, m, r-1)$ denote the sum

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m T(k,r)\cdot T(2n-k,r). \tag{5}$$

Recently, by using a new method, it has been shown in [18, Th. 3, p. 3] that the sum $\Psi(2n, m, r-1)$ is divisible by $T(n, r)$ for all non-negative integers n and m. In particular, $\Psi(2n, 1, r-1) = T(n, r) \cdot T(n+r, n)$. See [18, Th. 1, Eq. (1), p. 2].

Also, it is known [17, Th. 12] that the sum $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m\binom{k+r-1}{k}\binom{2n-k+r-1}{2n-k}$ is divisible by $\operatorname{lcm}(\binom{2n}{n}, \binom{n+r-1}{n})$ for all non-negative integers n and all positive integers m and r.

Our main result is as follows.

Theorem 1. *The sum $\varphi(2n, m, r-1)$ is always divisible by $\frac{1}{2}T(n, r)$ for all non-negative integers n and for all positive integers m and r.*

In order to prove Theorem 1, we use a new class of binomial sums [17, Eqns. (27) and (28)] that we call M sums.

Definition 2. Let n and a be non-negative integers, and let m be a positive integer. Let $S(n, m, a) = \sum_{k=0}^{n}\binom{n}{k}^m F(n, k, a)$, where $F(n, k, a)$ is an integer-valued function. Then the M sums for the sum $S(n, m, a)$ are as follows:

$$M_S(n, j, t; a) = \binom{n-j}{j}\sum_{v=0}^{n-2j}\binom{n-2j}{v}\binom{n}{j+v}^t F(n, j+v, a), \tag{6}$$

where j and t are non-negative integers such that $j \le \lfloor \frac{n}{2} \rfloor$.

Obviously, equation [17, Eq. (29)]

$$S(n, m, a) = M_S(n, 0, m-1; a) \tag{7}$$

holds.

Let n, j, t, and a be the same as in Definition 2.

It is known [17, Th. 8] that M sums satisfy the following recurrence relation:

$$M_S(n, j, t+1; a) = \binom{n}{j}\sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor}\binom{n-j}{u} M_S(n, j+u, t; a). \tag{8}$$

Moreover, we shall prove that M sums satisfy another interesting recurrence relation.

Theorem 3. *Let n and a be non-negative integers, and let m be a positive integer. Let $R(n, m, a)$ denote*

$$\sum_{k=0}^{n}\binom{n}{k}^m G(n, k, a),$$

where $G(n, k, a)$ is an integer-valued function. Let $Q(n, m, a)$ denote

$$\sum_{k=0}^{n}\binom{n}{k}^m H(n, k, a),$$

where $H(n, k, a) = \binom{a+k}{a}\binom{a+n-k}{a}G(n, k, a)$. Then the following recurrence relation is true:

$$M_Q(n, j, 0; a) = \binom{a+j}{a}\sum_{l=0}^{a}\binom{n-j+l}{l}\binom{n-j}{a-l} M_R(n, j+a-l, 0; a). \tag{9}$$

Note that by using the substitution $k = v + j$, Eq. (6) becomes

$$M_S(n,j,t;a) = \binom{n-j}{j} \sum_{k=j}^{n-j} \binom{n-2j}{k-j} \binom{n}{k}^t F(n,k,a). \tag{10}$$

From now on, we use Eq. (10) instead of Eq. (6).

2 Background

In 1998, Calkin proved that the alternating binomial sum $S_1(2n,m) = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. In 2007, Guo, Jouhet, and Zeng proved, among other things, two generalizations of Calkin's result [10, Thm. 1.2, Thm. 1.3, p. 2]. In 2018, Calkin's result [2, Thm. 1] was proved by using D sums [15, Section 8]. Note that there is a close relationship between D sums and M sums [17, Eqns. (9) and (19)].

Recently, Calkin's result [2, Thm. 1] has been proved by using M sums [17, Section 5]. In particular, it is known [17, Eqns. (22) and (25)] that

$$M_{S_1}(2n,j,0) = \begin{cases} 0, & \text{if } 0 \le j < n, \\ (-1)^n, & \text{if } j = n, \end{cases}$$

$$M_{S_1}(2n,j,1) = (-1)^n \binom{2n}{n}\binom{n}{j}.$$

Both M sums and D sums give an elementary proof of Dixon's formula. See for example [17, Section 5, Eq. (25)] for a proof using M sums. See also [9, Introduction].

By using M sums, it has been shown [17, Section 7] that the sum

$$S_2(2n,m,a) = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m\binom{a+k}{k}\binom{a+2n-k}{2n-k}$$

is divisible by $\operatorname{lcm}\left(\binom{a+n}{a},\binom{2n}{n}\right)$ for all non-negative integers n and a and for all positive integers m. In particular, it is known [17, Eqns. (52) and (57)] that

$$M_{S_2}(2n,j,0;a) = (-1)^n\binom{a+n}{a}\binom{a+j}{j}\binom{a}{n-j},$$

$$M_{S_2}(2n,j,1;a) = (-1)^n\binom{2n}{n}\sum_{u=0}^{n-j}\binom{n}{j+u}\binom{j+u}{u}\binom{a+j+u}{j+u}\binom{a+n}{2n-j-u}.$$

Remark 4. *Note that there is a minor error in [17, Eq. (52)]. The number $(-1)^{n-j}$ should be replaced by $(-1)^n$. Similarly, the number $(-1)^{n-j-u}$ in [17, Eq. (57)] should be replaced by $(-1)^n$.*

By using D sums, it has been shown in [19, Thm. 1] that the sum

$$S_3(2n,m) = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m\binom{2k}{k}\binom{2(2n-k)}{2n-k}$$

is divisible by $\binom{2n}{n}$ for all non-negative integers n and for all positive integers m. The same result can also be proved by using M sums. It can be shown that

$$M_{S_3}(2n,j,0) = (-1)^j \binom{2n}{n}\binom{2j}{j}\binom{2(n-j)}{n-j}. \tag{11}$$

Note that Eq. (11) is equivalent to [19, Eq. (12)].

Recently, it has been shown by using D sums [18, Th. 3, p. 3] that $\Psi(2n,m,l-1)$ is divisible by $T(n,l)$ for all non-negative integers n and m. The same result can also be proved by using M sums. Let l be a positive integer and let n and j be non-negative integers such that $j \le n$. It is readily verified [17, Eq. (91)] that

$$M_{\Psi}(2n,j,0;l-1) = (-1)^j \frac{\binom{2l}{l}\binom{2n}{n}\binom{2j}{j}\binom{2(n+l-j)}{n+l-j}\binom{2n-j}{n}}{\binom{n+l}{n}\binom{2n+l-j}{n}}. \tag{12}$$

Furthermore, by using [18, Eq. (103)] and [17, Eq. (33)] it can be shown that

$$M_{\Psi}(2n,j,1;l-1) = (-1)^j \cdot T(n,l)\binom{n}{j}\sum_{v=0}^{n-j}(-1)^v \cdot T(n+l-j-v,n)\binom{2(j+v)}{j+v}\binom{n-j}{v}. \tag{13}$$

The rest of the paper is structured as follows. In Section 3, we give a proof of Theorem 3 by using one variant of the Pfaff–Saalschütz theorem. In Section 4, we give a proof of Theorem 1. Our proof of Theorem 1 consists of two parts. In the first part, we prove that Theorem 1 is true for $m = 1$. In the second part, we prove that Theorem 1 is true for all positive integers m such that $m \ge 2$.

3 Proof of Theorem 3

We use three known binomial formulae.

Let a, b, and c be non-negative integers such that $a \ge b \ge c$. The first formula [14, Eq. (1.4), p. 5] is

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}. \tag{14}$$

Let a, b, m, and n be non-negative integers. The second formula is one variant of the Pfaff–Saalschütz theorem (see [25, p. 243] and [7, Introduction]):

$$\sum_{k=0}^{\min(m,n)} \binom{a}{m-k}\binom{b}{n-k}\binom{a+b+k}{k} = \binom{a+n}{m}\binom{b+m}{n}. \tag{15}$$

Remark 5. *Eq.* (15) *is equivalent to the triple binomial identity [8, Eq. (5.28), p. 171]; it was first proved by J. Pfaff [21]. See also [13, Ex. 31, p. 7] and [26, Problem 14, p. 4].*

The third formula is the symmetry [14, Thm. 1.1, p. 4] of binomial coefficients

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof. By setting $S := Q$ and $t := 0$ in Eq. (10), it follows that

$$M_Q(n,j,0;a) = \sum_{k=j}^{n-j} \binom{n-j}{j}\binom{n-2j}{k-j}\binom{a+k}{a}\binom{a+n-k}{a} G(n,k,a). \tag{16}$$

By using the symmetry of binomial coefficients and Eq. (14), it follows that

$$\binom{n-j}{j}\binom{n-2j}{k-j} = \binom{n-j}{k-j}\binom{n-k}{n-k-j}. \tag{17}$$

By using Eq. (17), Eq. (16) becomes

$$\begin{aligned} M_Q(n,j,0;a) &= \sum_{k=j}^{n-j} \binom{n-j}{k-j}\binom{n-k}{n-k-j}\binom{a+k}{a}\binom{a+n-k}{a} G(n,k,a), \\ &= \sum_{k=j}^{n-j} \binom{n-j}{k-j}\binom{a+n-k}{a}\binom{n-k}{n-k-j}\binom{a+k}{a} G(n,k,a). \end{aligned} \tag{18}$$

By using the symmetry of binomial coefficients and Eq. (18), it follows that

$$\binom{a+n-k}{a}\binom{n-k}{n-k-j} = \binom{a+n-k}{a+j}\binom{a+j}{j}. \tag{19}$$

By using Eq. (19), Eq. (18) becomes

$$M_Q(n,j,0;a) = \binom{a+j}{j}\sum_{k=j}^{n-j} \binom{n-j}{k-j}\binom{a+n-k}{a+j}\binom{a+k}{a} G(n,k,a). \tag{20}$$

By setting $m := a+j$, $n := a$, $a := n-k$, $b := k-j$, and $k := l$ in Eq. (15), we obtain that

$$\binom{a+n-k}{a+j}\binom{a+k}{a} = \sum_{l=0}^{a} \binom{n-k}{a+j-l}\binom{k-j}{a-l}\binom{n-j+l}{l}. \tag{21}$$

By using Eqns. (20) and (21), it follows that $M_Q(n,j,0;a)$ is equal to

$$\binom{a+j}{j}\sum_{k=j}^{n-j} \binom{n-j}{k-j}\sum_{l=0}^{a} \binom{n-k}{a+j-l}\binom{k-j}{a-l}\binom{n-j+l}{l} G(n,k,a). \tag{22}$$

In order for any summand on the right side of Eq. (21) to be nonzero, the following inequality must hold:

$$\begin{aligned} &n-k \geq a+j-l, \text{ or} \\ &\quad k \leq n-j-a+l. \end{aligned} \tag{23}$$

Similarly, in order for any summand on the right side of Eq. (22) to be nonzero, the following inequality must hold:

$$\begin{aligned} &k-j \geq a-l, \text{ or} \\ &\quad k \geq a+j-l. \end{aligned} \tag{24}$$

By changing the order of summation in Eq. (22) and by using inequalities (23) and (24), it follows that $M_Q(n,j,0;a)$ is equal to

$$\binom{a+j}{j}\sum_{l=0}^{a}\binom{n-j+l}{l}\sum_{k=a+j-l}^{n-(a+j-l)}\binom{n-j}{k-j}\binom{n-k}{a+j-l}\binom{k-j}{a-l}G(n,k,a). \tag{25}$$

By using the symmetry of binomial coefficients and Eq. (14), it follows that

$$\binom{n-j}{k-j}\binom{n-k}{a+j-l}=\binom{n-j}{a+j-l}\binom{n-a-2j+l}{k-j}. \tag{26}$$

By using Eqns. (22) and (26), it follows that $M_Q(n,j,0;a)$ is equal to

$$\binom{a+j}{j}\sum_{l=0}^{a}\binom{n-j+l}{l}\binom{n-j}{a+j-l}\sum_{k=a+j-l}^{n-(a+j-l)}\binom{n-a-2j+l}{k-j}\binom{k-j}{a-l}G(n,k,a). \tag{27}$$

By using the symmetry of binomial coefficients and Eq. (14), it follows that

$$\binom{n-a-2j+l}{k-j}\binom{k-j}{a-l}=\binom{n-a-2j+l}{a-l}\binom{n-2a-2j+2l}{k-j-a+l}. \tag{28}$$

By using Eqns. (27) and (28), it follows that $M_Q(n,j,0;a)$ is equal to

$$\binom{a+j}{j}\sum_{l=0}^{a}\binom{n-j+l}{l}\binom{n-j}{a+j-l}\binom{n-a-2j+l}{a-l}\sum_{k=a+j-l}^{n-(a+j-l)}\binom{n-2a-2j+2l}{k-j-a+l}G(n,k,a). \tag{29}$$

Again, by using the symmetry of binomial coefficients and Eq. (14), it follows that

$$\binom{n-j}{a+j-l}\binom{n-a-2j+l}{a-l}=\binom{n-j}{a-l}\binom{n-(j+a-l)}{j+a-l}. \tag{30}$$

By using Eqns. (29) and (30), it follows that $M_Q(n,j,0;a)$ is equal to

$$\binom{a+j}{j}\sum_{l=0}^{a}\binom{n-j+l}{l}\binom{n-j}{a-l}\binom{n-(j+a-l)}{j+a-l}\sum_{k=a+j-l}^{n-(a+j-l)}\binom{n-2(a+j-l)}{k-(j+a-l)}G(n,k,a). \tag{31}$$

Note that, by setting $S := R$, $F := G$, $j := j+a-l$, and $t := 0$ in Eq. (10), it follows that

$$M_R(n,j+a-l,0;a)=\binom{n-(j+a-l)}{j+a-l}\sum_{k=a+j-l}^{n-(a+j-l)}\binom{n-2(a+j-l)}{k-(j+a-l)}G(n,k,a). \tag{32}$$

Hence, by using Eqns. (31) and (32), it follows that

$$M_Q(n,j,0;a)=\binom{a+j}{j}\sum_{l=0}^{a}\binom{n-j+l}{l}\binom{n-j}{a-l}M_R(n,j+a-l,0;a).$$

This completes the proof of Theorem 3.

□

4 Proof of Theorem 1

Let the function $\varphi(2n,m,r-1)$ be defined as in Eq. (4).

Let us consider the following sum

$$\phi(2n,m,r-1)=\frac{1}{4}\Psi(2n,m,r-1). \tag{33}$$

By Eq. (5), Eq. (33) becomes

$$\phi(2n,m,r-1)=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m\cdot\frac{1}{2}T(k,r)\cdot\frac{1}{2}T(2n-k,r). \tag{34}$$

Note that, since r is a positive integer, both numbers $\frac{1}{2}T(k,r)$ and $\frac{1}{2}T(2n-k,r)$ are integers. Therefore, the sum $\phi(2n,m,r-1)$ is a sum from Definition 2.

Now we can apply Theorem 3. By setting $Q:=\varphi$, $n:=2n$, $a:=r-1$, $R:=\phi$ and $G(2n,k,r-1):=(-1)^k\cdot\frac{1}{2}T(k,r)\cdot\frac{1}{2}T(2n-k,r)$ in Theorem 3, it follows by Eq. (9) that $M_\varphi(2n,j,0;r-1)$ is equal to

$$\binom{j+r-1}{j}\sum_{l=0}^{r-1}\binom{2n-j+l}{l}\binom{2n-j}{r-1-l}M_\phi(2n,j+r-1-l,0;r-1). \tag{35}$$

By Eqns. (6) and (33), it follows that

$$M_\phi(2n,j,t;r-1)=\frac{1}{4}M_\Psi(2n,j,t;r-1). \tag{36}$$

By setting $t:=0$ and $l:=r$ in Eq. (12) and by using Eq. (36) and the definition of super Catalan numbers, we obtain that

$$M_\phi(2n,j,0;r-1)=(-1)^j\frac{1}{2}T(n,r)\frac{\binom{2j}{j}\binom{2(n+r-j)}{n+r-j}\binom{2n-j}{n}}{2\binom{2n+r-j}{n}}. \tag{37}$$

By using Eq. (37), it follows that the sum $M_\phi(2n,j+r-1-l,0;r-1)$ is equal to

$$(-1)^{j+r-1-l}\frac{1}{2}T(n,r)\frac{\binom{2(j+r-1-l)}{j+r-1-l}\binom{2(n-j+l+1)}{n-j+l+1}\binom{2n-j-r+l+1}{n}}{2\binom{2n-j+l+1}{n}}. \tag{38}$$

By using the symmetry of binomial coefficients and Eq. (14), it follows that

$$\binom{2n-j}{r-1-l}\binom{2n-j-r+l+1}{n}=\binom{2n-j}{n}\binom{n-j}{r-l-1}. \tag{39}$$

By using Eqns. (38) and (39), it follows that the sum $\binom{2n-j}{r-1-l}M_\phi(2n,j+r-1-l,0;r-1)$ is equal to

$$(-1)^{j+r-1-l}\frac{1}{2}T(n,r)\binom{2n-j}{n}\frac{\binom{2(j+r-1-l)}{j+r-1-l}\binom{2(n-j+l+1)}{n-j+l+1}\binom{n-j}{r-l-1}}{2\binom{2n-j+l+1}{n}}. \tag{40}$$

By using Eqns. (35) and (40), it follows that the sum $M_{\varphi}(2n,j,0;r-1)$ is equal to

$$(-1)^{j+r-1}\binom{j+r-1}{j}\frac{1}{2}T(n,r)\binom{2n-j}{n}\sum_{l=0}^{r-1}(-1)^{l}\binom{2n-j+l}{l}\frac{\binom{2(j+r-1-l)}{j+r-1-l}\binom{2(n-j+l+1)}{n-j+l+1}\binom{n-j}{r-l-1}}{2\binom{2n-j+l+1}{n}}. \tag{41}$$

By setting $j:=0$ in Eq. (41), we obtain that

$$M_{\varphi}(2n,0,0;r-1) \tag{42}$$
$$=(-1)^{r-1}\frac{1}{2}T(n,r)\binom{2n}{n}\sum_{l=0}^{r-1}(-1)^{l}\binom{2n+l}{l}\frac{\binom{2(r-1-l)}{r-1-l}\binom{2(n+l+1)}{n+l+1}\binom{n}{r-l-1}}{2\binom{2n+l+1}{n}}$$
$$=(-1)^{r-1}\frac{1}{2}T(n,r)\sum_{l=0}^{r-1}(-1)^{l}\binom{2n+l}{l}\binom{2(r-1-l)}{r-1-l}\binom{n}{r-l-1}\frac{\binom{2n}{n}\binom{2(n+l+1)}{n+l+1}}{2\binom{2n+l+1}{n}}$$
$$=(-1)^{r-1}\frac{1}{2}T(n,r)\sum_{l=0}^{r-1}(-1)^{l}\binom{2n+l}{l}\binom{2(r-1-l)}{r-1-l}\binom{n}{r-l-1}\frac{1}{2}T(n,n+l+1). \tag{43}$$

By using Eq. (41), it follows that the sum $M_{\varphi}(2n,0,0;r-1)$ is divisible by $\frac{1}{2}T(n,r)$. By setting $n:=2n$, $m:=1$, $S:=\varphi$, and $a:=r-1$ in Eq. (7), we obtain that

$$\varphi(2n,1,r-1)=M_{\varphi}(2n,0,0;r-1). \tag{44}$$

By using Eqns. (43) and (44), it follows that the sum $\varphi(2n,1,r-1)$ is divisible by $\frac{1}{2}T(n,r)$. This completes the proof of Theorem 1 for the case $m=1$.

Let us calculate the sum $M_{\varphi}(2n,j,1;r-1)$, where n and j are non-negative integers such that $j\le n$.

By setting $n:=2n$, $S:=\varphi$, $t:=0$, and $a:=r-1$ in Eq. (8), we obtain that

$$M_{\varphi}(2n,j,1;r-1)=\sum_{u=0}^{n-j}\binom{2n}{j}\binom{2n-j}{u}M_{\varphi}(2n,j+u,0;r-1). \tag{45}$$

By using the symmetry of binomial coefficients and Eq. (14), it follows that

$$\binom{2n}{j}\binom{2n-j}{u}=\binom{2n}{2n-j-u}\binom{j+u}{u}. \tag{46}$$

By using Eqns. (45) and (46), it follows that

$$M_{\varphi}(2n,j,1;r-1)=\sum_{u=0}^{n-j}\binom{j+u}{u}\binom{2n}{2n-j-u}M_{\varphi}(2n,j+u,0;r-1). \tag{47}$$

By using Eq. (41), it follows that the sum $\binom{2n}{2n-j-u}M_{\varphi}(2n,j+u,0;r-1)$ equals

$$\binom{2n}{2n-j-u}\binom{2n-j-u}{n}(-1)^{j+u+r-1}\binom{j+u+r-1}{j+u}\frac{1}{2}T(n,r)\cdot$$
$$\cdot\sum_{l=0}^{r-1}(-1)^{l}\binom{2n-j-u+l}{l}\frac{\binom{2(j+u+r-1-l)}{j+u+r-1-l}\binom{2(n-j-u+l+1)}{n-j-u+l+1}\binom{n-j-u}{r-l-1}}{2\binom{2n-j-u+l+1}{n}}. \tag{48}$$

By using the symmetry of binomial coefficients and Eq. (14), it follows that

$$\binom{2n}{2n-j-u}\binom{2n-j-u}{n}=\binom{2n}{n}\binom{n}{j+u}. \tag{49}$$

By using Eqns. (48) and (49), it follows that the sum $\binom{2n}{2n-j-u}M_{\varphi}(2n,j+u,0;r-1)$ equals

$$\begin{aligned}&\frac{1}{2}T(n,r)\binom{n}{j+u}(-1)^{j+u+r-1}\binom{j+u+r-1}{j+u}\sum_{l=0}^{r-1}(-1)^{l}\binom{2n-j-u+l}{l}\cdot\\&\cdot\binom{2(j+u+r-1-l)}{j+u+r-1-l}\binom{n-j-u}{r-l-1}\frac{1}{2}T(n,n-j-u+l+1).\end{aligned} \tag{50}$$

Hence, by using the Eq. (50), we obtain that

$$\binom{2n}{2n-j-u}M_{\varphi}(2n,j+u,0;r-1)=\frac{1}{2}T(n,r)\cdot c(n,j+u,r-1), \tag{51}$$

where $c(n,j+u,r-1)$ is always an integer.

By using Eq. (51), Eq. (47) becomes

$$M_{\varphi}(2n,j,1;r-1)=\frac{1}{2}T(n,r)\sum_{u=0}^{n-j}\binom{j+u}{u}c(n,j+u,r-1). \tag{52}$$

By Eq. (52), it follows that the sum $M_{\varphi}(2n,j,1;r-1)$ is divisible by $\frac{1}{2}T(n,r)$ for all non-negative integers n and j such that $j\leq n$, and for all positive integers r. By using Eq. (8) and the induction principle, it can be shown that the sum $M_{\varphi}(2n,j,t;r-1)$ is divisible by $\frac{1}{2}T(n,r)$ for all non-negative integers n and j such that $j\leq n$, and for all positive integers r and t.

By setting $S:=\varphi$, $n:=2n$, $m:=t+1$, and $a:=r-1$ in the Eq. (7), it follows that

$$\varphi(2n,t+1,r-1)=M_{\varphi}(2n,0,t;r-1). \tag{53}$$

Since $t\geq 1$, it follows that $t+1\geq 2$. By Eq. (52), it follows that the sum $\varphi(2n,m,r-1)$ is always divisible by $\frac{1}{2}T(n,r)$ for all non-negative integers n, and for all positive integers m and r such that $m\geq 2$. This completes the proof of Theorem.

Remark 6. *See also [17, Section 4] for an additional insight of how the method of M sums works. For $r=1$, by using Theorem 1 and the fact $\frac{1}{2}T(n,1)=C_n$, it follows that the sum $\varphi(2n,m,0)$ is divisible by C_n. Therefore, for $r=1$, the result of Theorem 1 is weaker than the result that sum $\varphi(2n,m,0)$ is divisible by $\binom{2n}{n}$ [19, Cor. 4, p. 2]. However, for $n=3$, $r=2$, and $m=1$, the sum $\varphi(2n,m,r-1)$ is neither divisible by $\binom{2n}{n}$ nor by $T(n,r)$.*

Acknowledgements

I want to thank Professor Tomislav Došlić for inviting me to the third Croatian Combinatorial Days in Zagreb. Also I want to thank the anonymous referees for valuable comments which improved this article. Finally, I am grateful to Professor David Dolžan for helping me to correct a few grammatical errors.

References

[1] Allen, E., & Gheorghicius, I. (2014). A weighted interpretation for the super Catalan numbers. *Journal of Integer Sequences*, 17, Article 14.10.7.

[2] Calkin, N. J. (1998). Factors of sums of powers of binomial coefficients. *Acta Arithmetica*, 86, 17–26

[3] Callan, D. (2005). A combinatorial interpretation for a super-Catalan recurrence. *Journal of Integer Sequences*, 8, Article 05.1.8

[4] Chen, X., & Wang, J. (2012). *The super Catalan numbers* $S(m, m+s)$ *for* $s \leq 4$. Preprint. Available online at: `https://arxiv.org/abs/1208.4196.`

[5] Chu, W., & Kiliç, E. (2021). Binomial sums involving Catalan numbers. *Rocky Mountain Journal of Mathematics*, 4, 1221–1225.

[6] Gessel, I. M. (1992). Super ballot numbers. *Journal of Symbolic Computation*, 14, 179–194.

[7] Goulden, I. P. (1985). A bijective proof of the q-Saalschütz theorem. *Discrete Mathematics*, 57, 39–44.

[8] Graham, R. L., Knuth, D., & Patashnik, O. (1994). *Concrete Mathematics* (2nd ed.). Addison-Wesley.

[9] Guo, V. J. W. (2018). A new proof of the q-Dixon identity. *Czechoslovak Mathematical Journal*, 68, 577–580.

[10] Guo, V. J. W., Jouhet, F., & Zeng, J. (2007). Factors of alternating sums of products of binomial and q-binomial coefficients. *Acta Arithmetica*, 127, 17–31.

[11] Guo, V. J. W. (2014). Proof of two divisibility properties of binomial coefficients conjectured by Z. W. Sun. *Electronic Journal of Combinatorics*, 21, # P54.

[12] Killpatrick, K. (2023). *Super FiboCatalan numbers and generalized FiboCatalan numbers.* Preprint. Available online at: `https://www.researchgate.net/publication/373437785.`

[13] Knuth, D. E. (1968). *The Art of Computer Programming, Vol. 1* (3rd ed.). Addison-Wesley.

[14] Koshy, T. (2009). *Catalan Numbers with Applications*. Oxford University Press.

[15] Mikić, J. (2018). A method for examining divisibility properties of some binomial sums. *Journal of Integer Sequences*, 21, Article 18.8.7.

[16] Mikić, J. (2014). A note on the Gessel numbers. *JP Journal of Algebra, Number Theory and Applications*, 63, 225–235.

[17] Mikić, J. (2020). New class of binomial sums and their applications. *Proceedings of 3rd Croatian Combinatorial Days*, September 21-22, 2020, Zagreb. Available online at: https://www.grad.hr/crocodays/proc_ccd3/mikic_final.pdf.

[18] Mikić, J. (2021, March 15). On a new alternating convolution formula for the Super Catalan numbers. *Romanian Mathematical Magazine*. Available online at: https://www.ssmrmh.ro/2021/03/15/on-an-alternative-convolution-formula-for-the-catalan-super-numbers/

[19] Mikić, J. (2020). On a certain sums divisible by the central binomial coefficient. *Journal of Integer Sequences*, 23, Article 20.1.6.

[20] Mikić, J. (2019). Two new identities involving the Catalan numbers and sign-reversing involutions. *Journal of Integer Sequences*, 22, Article 19.7.7.

[21] Pfaff, J. (1797). Observationes analyticae ad L. Euler Institutiones Calculi Integralis, Vol. IV, Supplem. II et IV, Historia de 1793, *Nova Acta Academiae Scientiarum Imperialis Petropolitanae*, 11 (1797), 38–57.

[22] Pippenger, N., & Schleich, K. (2006). Topological characteristics of random triangulated surfaces. *Random Structures and Algorithms*, 28, 247–288.

[23] Prodinger, H. (2019). Two new identities involving the Catalan numbers: A classical approach. Preprint. Available online at: https://arxiv.org/abs/1911.07604.

[24] Schaeffer, G. (2003). *A combinatorial interpretation of super-Catalan numbers of order two*. Manuscript.

[25] Slater, L. J. (1966). *Generalized Hypergeometric Functions*. Cambridge University Press, Cambridge.

[26] Stanley, R. P. (2009). *Bijective proof problems*. Available online at: https://math.mit.edu/~rstan/bij.pdf.