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Some new arithmetic functions

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Abstract: We introduce and study some new arithmetic functions, connected with the classical functions φ (Euler's totient), ψ (Dedekind's function) and σ (sum of divisors function).
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1 Introduction

The present paper is devoted to the introduction of certain new arithmetic functions, and the study of some inequalities involving them. Among the papers in the area that are relevant to the topic and can draw the attention of the interested reader, are the authors' book [7], a useful survey by Dimitrov [3], some recent papers [1,2,5,6]. Many other similar results can be found in literature.

Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factorization of the natural number $n > 1$, where p_i are discinct primes and $\alpha_i \geq 1$ are natural numbers. Then the classical Euler's totient, Dedekind's function and sum of divisors function satisfy:

$$\varphi(n) = \prod_{i=1}^{r}(p_i^{\alpha_i} - p_i^{\alpha_i - 1}), \varphi(1) = 1,$$
$$\psi(n) = \prod_{i=1}^{r}(p_i^{\alpha_i} + p_i^{\alpha_i - 1}), \psi(1) = 1, \tag{1}$$
$$\sigma(n) = \prod_{i=1}^{r}(p_i^{\alpha_i} + p_i^{\alpha_i - 1} + \cdots + 1), \sigma(1) = 1,$$

In what follows, we will introduce and study the arithmetic functions $\varphi^-, \varphi^+; \psi^-, \psi^+; \sigma^-, \sigma^+$, defined for the natural number $n > 1$ by

$$\varphi^-(n) = \prod_{i=1}^{r}(\varphi(p_i^{\alpha_i}) - 1), \quad \varphi^+(n) = \prod_{i=1}^{r}(\varphi(p_i^{\alpha_i}) + 1), \tag{2}$$

$$\psi^-(n) = \prod_{i=1}^{r}(\psi(p_i^{\alpha_i}) - 1), \quad \psi^+(n) = \prod_{i=1}^{r}(\psi(p_i^{\alpha_i}) + 1), \tag{3}$$

$$\sigma^-(n) = \prod_{i=1}^{r}(\sigma(p_i^{\alpha_i}) - 1), \quad \sigma^+(n) = \prod_{i=1}^{r}(\sigma(p_i^{\alpha_i}) + 1). \tag{4}$$

Below, we will study some of their properties.

2 Main results

Let us assume that

$$\varphi^-(1) = \varphi^+(1) = \psi^-(1) = \psi^+(1) = \sigma^-(1) = \sigma^+(1) = 1.$$

Thus, e.g., when $r = \omega(n) = 1$, one has

$$\varphi^-(p^\alpha) = \varphi(p^\alpha) - 1 = p^\alpha - p^{\alpha-1} - 1,$$
$$\varphi^+(p^\alpha) = \varphi(p^\alpha) + 1 = p^\alpha - p^{\alpha-1} + 1,$$
$$\psi^-(p^\alpha) = \psi(p^\alpha) - 1 = p^\alpha + p^{\alpha-1} - 1,$$
$$\psi^+(p^\alpha) = \psi(p^\alpha) + 1 = p^\alpha + p^{\alpha-1} + 1,$$
$$\sigma^-(p^\alpha) = \sigma(p^\alpha) - 1 = p^\alpha + p^{\alpha-1} + \cdots + p,$$
$$\sigma^+(p^\alpha) = \sigma(p^\alpha) + 1 = p^\alpha + p^{\alpha-1} + \cdots + p + 2.$$

Theorem 1. *For $n > 1$ one has*

$$\varphi^-(n) + 1 \le \varphi(n) \le \varphi^+(n) - 1, \tag{5}$$
$$\psi^-(n) + 1 \le \psi(n) \le \psi^+(n) - 1, \tag{6}$$
$$\sigma^-(n) + 1 \le \sigma(n) \le \sigma^+(n) - 1. \tag{7}$$

Proof. We give here only the proof of (5). We will apply the well-known classical inequalities

$$\prod_{i=1}^{r}(x_i - 1) \le \prod_{i=1}^{r} x_i - 1 \ \text{ for } x_i > 1, \tag{8}$$

$$\prod_{i=1}^{r}(x_i + 1) \ge \prod_{i=1}^{r} x_i + 1 \ \text{ for } x_i > 0. \tag{9}$$

Let $x_i = \varphi(p_i^{\alpha_i})$ in (8). As the function φ is multiplicative, one has

$$\prod_{i=1}^{r} \varphi(p_i^{\alpha_i}) = \varphi(\prod_{i=1}^{r} p_i^{\alpha_i}) = \varphi(n).$$

Applying (9) to the same x_i, we get the two inequalities from (5). The other inequalities (6) and (7), can be deduced in the same way. □

Remark 1. As in (8) and (9) there is an equality only for $r = 1$, we get that there are equalities in (5) only for $r = \omega(n) = 1$, i.e., when $n = p^\alpha$ for p being a prime number.

Theorem 2. *For $n > 1$ one has*

$$\varphi^-(n) + \varphi^+(n) \ge 2\varphi(n) + 2(\omega(n) - 1), \tag{10}$$

$$\psi^-(n) + \psi^+(n) \ge 2\psi(n) + 2(\omega(n) - 1), \tag{11}$$

$$\sigma^-(n) + \sigma^+(n) \ge 2\sigma(n) + 2(\omega(n) - 1). \tag{12}$$

Proof. In [4, Relation (12)], the following inequality is stated:

$$\prod_{i=1}^{r}(x_i + 1) + \prod_{i=1}^{r}(x_i - 1) \ge \prod_{i=1}^{r} x_i + 2(r - 1) \tag{13}$$

for $r \ge 1$ and $x_i \ge 2$. In fact, inequality (13) holds true for $x_i \ge 1$ $(i = 1, \dots, r)$. Indeed, (13) is true for $r = 1$ and $r = 2$ with equality. Now, assuming that for r, for $r + 1$ one has:

$$(x_1 + 1)\cdots(x_r + 1)(x_{r+1} + 1) + (x_1 - 1)\cdots(x_r - 1)(x_{r+1} - 1)$$

$$= x_{r+1}((x_1{+}1)\cdots(x_r{+}1) + (x_1{-}1)\cdots(x_r{-}1)) + (x_1{-}1)\cdots(x_r{-}1) + (x_1{+}1)\cdots(x_r{+}1).$$

Now, by the induction hypothesis, as $x_{r+1} \ge 1$, we have to prove that

$$(x_1 + 1)\cdots(x_r + 1) \ge 2 + (x_1 - 1)\cdots(x_r - 1).$$

Then (13) will follow for $r + 1$, as $2(r - 1) + 2 = 2r$. The above inequalities follow immediately, again by induction.

Now, let $x_i = \varphi(p_i^{\alpha_i})$ in (13). Then we get relation (10).

Relations (11) and (12) can be proved in the same manner. □

Theorem 3. *For $n > 1$ with $\omega(n) \geq 2$ one has*

$$\varphi^+(n) - \varphi^-(n) \geq 2(\omega(n) - 1) \sum_{p^a | n} \varphi(p^a) \text{ for odd } n, \tag{14}$$

$$\psi^+(n) - \psi^-(n) \geq 2(\omega(n) - 1) \sum_{p^a | n} \psi(p^a), \tag{15}$$

$$\sigma^+(n) - \sigma^-(n) \geq 2(\omega(n) - 1) \sum_{p^a | n} \sigma(p^a) \tag{16}$$

Proof. We will apply the following inequality (see [4], relation (9)):

$$\prod_{i=1}^{r} (x_i + 1) - \prod_{i=1}^{r} (x_i - 1) \geq 2(r - 1) \sum_{i=1}^{r} x_i \tag{17}$$

for $r \geq 2, x_i \geq 2$ $(i = 1, \dots, r)$.

Let $x_i = \varphi(p_i^{\alpha_i}) \geq 2$ when n is an odd, so (14) follows from (17).

Relations (15) and (16) can be proved in the same manner. □

Remark 2. Relations (14)–(16) imply the introduction of the following arithmetic functions:

$$F(n) = \sum_{i=1}^{r} \varphi(p_i^{\alpha_i}),$$

$$G(n) = \sum_{i=1}^{r} \psi(p_i^{\alpha_i}),$$

$$H(n) = \sum_{i=1}^{r} \sigma(p_i^{\alpha_i}),$$

which are the "additive analogues" of the arithmetic functions φ, ψ, σ. We note that the additive analogue of the function $E(n) = n$ for $n > 1$ is

$$B^1(n) = \sum_{i=1}^{r} p_i^{\alpha_i}$$

(see [8, pp. 147–149]).

Theorem 4. *For $n > 1$ one has*

$$(\varphi^-(n))^{\frac{1}{\omega(n)}} \leq (\varphi(n))^{\frac{1}{\omega(n)}} - 1 \tag{18}$$

$$(\psi^-(n))^{\frac{1}{\omega(n)}} \leq (\psi(n))^{\frac{1}{\omega(n)}} - 1 \tag{19}$$

$$(\sigma^-(n))^{\frac{1}{\omega(n)}} \leq (\sigma(n))^{\frac{1}{\omega(n)}} - 1 \tag{20}$$

$$(\varphi^+(n))^{\frac{1}{\omega(n)}} \geq (\varphi(n))^{\frac{1}{\omega(n)}} + 1 \tag{21}$$

$$(\psi^+(n))^{\frac{1}{\omega(n)}} \geq (\psi(n))^{\frac{1}{\omega(n)}} + 1 \tag{22}$$

$$(\sigma^+(n))^{\frac{1}{\omega(n)}} \geq (\sigma(n))^{\frac{1}{\omega(n)}} + 1 \tag{23}$$

Proof. We will apply the Minkowski's inequality

$$\sqrt[r]{\prod_{i=1}^{r}(x_i + y_i)} \geq \sqrt[r]{\prod_{i=1}^{r} x_i} + \sqrt[r]{\prod_{i=1}^{r} y_i} \tag{24}$$

for $x_i, y_i \geq 0$.

Let $y_i = 1$ in (24). Then we get the inequality

$$\sqrt[r]{\prod_{i=1}^{r}(x_i + 1)} \geq \sqrt[r]{\prod_{i=1}^{r} x_i} + 1. \tag{25}$$

Suppose that $z_i \geq 1$ and put $x_i = z_i - 1$ in (25). Then we get from (25):

$$\sqrt[r]{\prod_{i=1}^{r}(z_i - 1)} \leq \sqrt[r]{\prod_{i=1}^{r} z_i} - 1. \tag{26}$$

We apply (26) for $z_i = \varphi(p_i^{\alpha_i})$ and (25) for $x_i = \varphi(p_i^{\alpha_i})$. As $r = \omega(n)$, we get from (26) relation (18), and we get from (25) relation (21). The other inequalities can be proved in the same manner, so we omit the details. □

Theorem 5. *For $n > 1$ one has*

$$\psi^+(n) \leq \sigma(n), \tag{27}$$

with equality only for $n = \left(\prod_{i=1}^{r} p_i^{\alpha_i}\right)^2$, where $p_1, \ldots, p_r$ are distinct primes.

Proof. From (3) it follows

$$\begin{aligned} \psi^+(n) &= \prod_{i=1}^{r}(\psi_1(p_i^{\alpha_i}) + 1) \\ &= \prod_{i=1}^{r}(p_i^{\alpha_i} + p_i^{\alpha_i - 1} + 1) \\ &\leq \prod_{i=1}^{r}(p_i^{\alpha_i} + p_i^{\alpha_i - 1} + \cdots + 1) \\ &= \sigma(n). \end{aligned}$$

There is an equality only if

$$p_i^{\alpha_i} + p_i^{\alpha_i - 1} + 1 = p_i^{\alpha_i} + p_i^{\alpha_i - 1} + \cdots + 1,$$

i.e., $p_i^{\alpha_i - 1} = p_i$ and we see immediately that $a_i - 1 = 1$, i.e., $a_i = 2$ for each $i = 1, \ldots, r$. Then $n = \left(\prod_{i=1}^{r} p_i\right)^2$. □

3 Conclusion

In the present paper we introduced six new arithmetic functions and studied some of their properties. On can consider the more general functions

$$F^{-}(n) = \prod_{i=1}^{r}(F(p_i^{\alpha_i}) - 1)$$

and

$$F^{+}(n) = \prod_{i=1}^{r}(F(p_i^{\alpha_i}) + 1).$$

For $F(n) = n$ we get that the function F^{-} coincides with φ^* and F^{+} coincides with σ^*. Therefore the new functions coincide with E. Cohen's functions. When F is multiplicative and has some specific properties, some of the above theorems can be generalized.

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