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A note on the Diophantine equation $(x^k - 1)(y^k - 1)^2 = z^k - 1$

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Abstract: We prove that, for $k \geq 10$, the Diophantine equation $(x^k - 1)(y^k - 1)^2 = z^k - 1$ in positive integers x, y, z, k with $z > 1$, has no solutions satisfying $1 < x \leq y$ or $1 < y < x \leq ((y^k - 1)^{k-2} + 1)^{\frac{1}{k}}$.

Keywords: Diophantine equation, Diophantine approximation.
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1 Introduction

The Greek mathematician Diophantus, of the third century, found the set of four positive rational numbers $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$ with the property: the product of any two of them increased by 1 is a perfect square. Fermat found firstly the set of four positive integers $\{1, 3, 8, 120\}$ with the above property (see [5]). Now, a set of m positive rational numbers $\{a_1, a_2, \dots, a_m\}$ is called a Diophantine m-tuple if $a_i a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$.

In 2003, Bugeaud and Dujella [4] considered an analogous problem: the existence of sets $\{a, b, c\}$ of positive integers such that the three numbers $ab + 1, ac + 1$ and $bc + 1$ are perfect k-th powers, for an integer $k \geq 3$. And they further investigated several related questions. In 2004, Bugeaud [3] showed that if $\{1, a, b\}$ is such a triple, then k cannot exceed 74. Equivalently, the

Diophantine equation

$$(x^k - 1)(y^k - 1) = z^k - 1 \tag{1}$$

has no solutions in positive integers with $z \geq 2$ and $k \geq 75$.

Another motivation of the work of Bugeaud [3] is that, for any integer $n \geq 2$, there is an identity

$$(n^2 - 1)((n + 1)^2 - 1) = (n^2 + n - 1)^2 - 1,$$

which implies that the Diophantine equation

$$(x^2 - 1)(y^2 - 1) = z^2 - 1 \tag{2}$$

has infinitely many positive integer solutions. Kashihara [9] described the set of all integer solutions of Eq. (2), which can be derived from the trivial solutions $(n, 1, 1)$ and $(1, n, 1)$.

In 2007, Bennett [2] showed that Eq. (1) has only the solutions $(x, y, z, k) = (-1, 4, -5, 3)$ and $(4, -1, -5, 3)$ in integers x, y, z and k with $|z| \geq 2$ and $k \geq 3$. In the same paper, Bennett [2] also proved that the Diophantine equation

$$(x^k - 1)(y^k - 1) = (z^k - 1)^2 \tag{3}$$

has no solutions in integers x, y, z and k with $x \neq \pm y$, $|z| \geq 2$ and $k \geq 4$. An interesting case of Eq. (3) is the case of $k = 2$, i.e.,

$$(x^2 - 1)(y^2 - 1) = (z^2 - 1)^2. \tag{4}$$

There are many studies on this case, and we can refer to D23 of [7].

In 2014, as a generalization of Eq. (1), Zhang [11] showed that the Diophantine equation

$$(ax^k - 1)(by^k - 1) = abz^k - 1$$

has no positive integer solutions with $a, b \in \mathbb{Z}^+$, $|x| > 1$, $|y| > 1$ and $k \geq 4$.

In 2015, Goedhart and Grundman [6] proved that the Diophantine equation

$$(a^2cx^k - 1)(b^2cy^k - 1) = (abcz^k - 1)^2$$

has no solutions in positive integers $x, y, z > 1$ and $k \geq 7$ with $a, b, c \in \mathbb{Z}^+$ and $a^2x^k \neq b^2y^k$, which is a modification of Eq. (3).

In D23 of [7], Bennett asked for the complete set of solutions in integers $x, y, z > 1$ to the Diophantine equation

$$\frac{x^2 - 1}{y^2 - 1} = (z^2 - 1)^2. \tag{5}$$

In 2010, Hai and Walsh [8] obtained the set of all integer solutions of Eq. (5).

Eq. (5) can be rewritten as

$$(y^2 - 1)(z^2 - 1)^2 = x^2 - 1, \text{ for } |y| > 1$$

which is a variant of Eq. (2) or Eq. (4).

To be more general, we generalize Eqs. (1) and (3) to

$$(x^k - 1)^{t_1}(y^k - 1)^{t_2} = (z^k - 1)^{t_3}, \tag{6}$$

where $t_1, t_2, t_3 \in \mathbb{Z}^+$ and $\gcd(t_1, t_2, t_3) = 1$. If $\gcd(t_1, t_2, t_3) = t > 1$, then $t_1 = s_1 t, t_2 = s_2 t$, $t_3 = s_3 t$ with $\gcd(s_1, s_2, s_3) = 1$. In this case, Eq. (6) can be simplified to $(x^k - 1)^{s_1}(y^k - 1)^{s_2} = (z^k - 1)^{s_3}$ with $\gcd(s_1, s_2, s_3) = 1$.

When $(t_1, t_2, t_3) = (1, 1, 1)$, Eq. (6) deduces Eq. (1); when $(t_1, t_2, t_3) = (1, 1, 2)$, Eq. (6) deduces Eq. (3). When $t_1 = t_2$ and $t_3 = 1$, it is easy to see that Eq. (6) has no positive integer solutions. This is because it is a special case of the famous Catalan equation $x^n - y^m = 1$, which has only one solution $(x, y, n, m) = (3, 2, 2, 3)$.

If $t_1 = t_3 = 1$, $t_2 = 2$ and $k = 2$, Eq. (6) is equivalent to Eq. (5) for $x > 1$. In this paper, we consider the general case of Eq. (5), that is, the Diophantine equation

$$\frac{x^k - 1}{y^k - 1} = (z^k - 1)^2,$$

which is equivalent to

$$(x^k - 1)(y^k - 1)^2 = z^k - 1, \quad \text{for } x > 1.$$

By using the theory of Diophantine approximation, we prove

Theorem 1.1. *If $k \geq 10$, the Diophantine equation*

$$(x^k - 1)(y^k - 1)^2 = z^k - 1, \tag{7}$$

in positive integers x, y, z, k with $z > 1$, has no solutions satisfying $1 < x \leq y$ or $1 < y < x \leq ((y^k - 1)^{k-2} + 1)^{\frac{1}{k}}$.

To prove our theorem, we use the following lemma from [1], which provides a bound on how well one can approximate certain algebraic numbers by rational numbers. For $n \geq 2$, define

$$\mu_n = \prod_{\substack{p \mid n \\ p \text{ prime}}} p^{\frac{1}{p-1}}.$$

Lemma 1.1 (Bennett, Theorem 1.3 of [1]). *If k, u, p and q are positive integers with $k \geq 3$ and*

$$(\sqrt{u} + \sqrt{u+1})^{2(k-2)} > (k\mu_k)^k,$$

then

$$\left| \sqrt[k]{1 + \frac{1}{u}} - \frac{p}{q} \right| > (8k\mu_k u)^{-1} q^{-\lambda}$$

with

$$\lambda = 1 + \frac{\log(k\mu_k(\sqrt{u} + \sqrt{u+1})^2)}{\log(\frac{1}{k\mu_k}(\sqrt{u} + \sqrt{u+1})^2)}. \tag{8}$$

2 Proof of the Theorem

Proof of Theorem 1.1. We focus particularly on the case where k is a prime number in Eq. (7). This is because if $k = pq$, with p being a prime number, then Eq. (7) can be written as

$$((x^q)^p - 1)\left((y^q)^p - 1\right)^2 = (z^q)^p - 1.$$

The prime numbers less than 10 are only $2, 3, 5$ and 7. Therefore, any composite number greater than 9 either contains a prime factor greater than 10, or contains one of the following numbers as a factor.

$$2^4,\ 3^3,\ 5^2,\ 7^2,\ 2^2\cdot 3,\ 2\cdot 3^2,\ 2\cdot 5,\ 2\cdot 7,\ 3\cdot 5,\ 3\cdot 7,\ 5\cdot 7. \tag{9}$$

Thus, we only need to consider the case that k is a prime number greater than 10, or k takes one of the numbers in (9).

We assume that Eq. (7) has a positive integer solution and thereby obtain a contradiction. Let $k \geq 4$ be an integer. Let us write

$$u + 1 = x^k, \quad v + 1 = y^k, \tag{10}$$

where u, v are positive integers. Then, from Eq. (7), we obtain $uv^2 + 1 = z^k$.

Since

$$x^k y^{2k} = (u+1)(v+1)^2 > uv^2 + 1 = z^k,$$

we have $xy^2 \geq z + 1$, then

$$(u+1)(v+1)^2 \geq \left((uv^2+1)^{\frac{1}{k}} + 1\right)^k.$$

Expanding this, we have

$$\begin{aligned} 2uv + u + v^2 + 2v > k(uv^2+1)^{\frac{k-1}{k}} + \binom{k}{2}(uv^2+1)^{\frac{k-2}{k}} \\ + \binom{k}{3}(uv^2+1)^{\frac{k-3}{k}} + \binom{k}{4}(uv^2+1)^{\frac{k-4}{k}}. \end{aligned} \tag{11}$$

- **Case 1.** Suppose that $x > y > 1$, we have $u > v$. It follows from $k \geq 10$ that

  $$(uv^2+1)^{k-3} > (uv^2)^{k-3} > v^{3(k-1)} > v^{2k},$$

  therefore,

  $$(uv^2+1)^{\frac{k-3}{k}} > v^2. \tag{12}$$

  Similarly, we get

  $$(uv^2+1)^{\frac{k-4}{k}} > v. \tag{13}$$

  Let $v < u \leq v^{k-2}$, i.e., $y < x \leq ((y^k-1)^{k-2}+1)^{\frac{1}{k}}$. It follows from $k \geq 10$ that

  $$(uv^2+1)^{k-1} > (uv^2)^{k-1} = u^{k-1}v^{2k-2} \geq u^k v^k,$$

  therefore,

  $$(uv^2+1)^{\frac{k-1}{k}} > uv. \tag{14}$$

  Similarly, we have

  $$(uv^2+1)^{\frac{k-2}{k}} > u. \tag{15}$$

  Substituting (12), (13), (14), and (15) into (11) will lead to a contradiction.

- **Case 2.** Assume that $1 < x \le y$, we get $u \le v$. It follows from $k \ge 10$ that
$$(uv^2+1)^{k-2} > (uv^2)^{k-2} = u^{k-2}v^{2k-4} \ge u^k v^{2k-6} > u^k v^k,$$
therefore,
$$(uv^2+1)^{\frac{k-2}{k}} > uv. \tag{16}$$
Similarly, we obtain
$$(uv^2+1)^{\frac{k-3}{k}} > v \ \text{ and } \ (uv^2+1)^{\frac{k-4}{k}} > u. \tag{17}$$
Substituting (16) and (17) into (11), we obtain
$$v^2 > k(uv^2+1)^{\frac{k-1}{k}} > k(uv^2)^{\frac{k-1}{k}},$$
which yields
$$v > k^{\frac{k}{2}} u^{\frac{k-1}{2}}. \tag{18}$$
Note that from (10), we have
$$\left| \left(\frac{xy^2}{z}\right)^k - \frac{u+1}{u} \right| = \frac{(u+1)(2uv+u-1)}{u(uv^2+1)} < \frac{(u+1)(2v+1)}{uv^2+1} \le \frac{Cuv}{z^k},$$
where $C = \frac{2096128}{1046529}$, thus
$$\left| \sqrt[k]{1+\frac{1}{u}} - \frac{xy^2}{z} \right| < \frac{Cuv}{kz^k}, \tag{19}$$
where we have used the formula $a^k - b^k = (a-b)\sum_{i=0}^{k-1} a^{k-1-i}b^i$.
On the other hand, if $k \ge 10, x \ge 2$, we check easily that
$$\frac{\sqrt{u}+\sqrt{u+1}}{\sqrt{u+1}} > 1.99,$$
where $u+1 = x^k$. We thus have
$$(\sqrt{u}+\sqrt{u+1})^{2(k-2)} > (1.99 \cdot \sqrt{u+1})^{2(k-2)} \ge (1.99 \cdot 2^{\frac{k}{2}})^{2(k-2)} > k^{2k} > (k\mu_k)^k.$$
Then (19) together with Lemma 1.1 leads to
$$z^{k-\lambda} < 8C\mu_k u^2 v.$$
Since $z^k = uv^2+1$, we further obtain
$$v^{k-2\lambda} < 8^k C^k \mu_k^k u^{k+\lambda}. \tag{20}$$
Next, we will show that conditions (18) and (20) are contradictory. We only need to establish the following inequality
$$k^{\frac{k}{2}} u^{\frac{k-1}{2}} > 8^{\frac{k}{k-2\lambda}} C^{\frac{k}{k-2\lambda}} \mu_k^{\frac{k}{k-2\lambda}} u^{\frac{k+\lambda}{k-2\lambda}}. \tag{21}$$
In fact, when k is a prime number, (21) is equivalent to
$$k^{\frac{k}{2}-\frac{k}{(k-1)(k-2\lambda)}} u^{\frac{k-1}{2}-\frac{k+\lambda}{k-2\lambda}} > (8C)^{\frac{k}{k-2\lambda}},$$
and thus we obtain
$$k^{\frac{k-2\lambda}{2}-\frac{1}{k-1}} u^{\frac{k-2\lambda-3}{2}} > 8C.$$

- Case 2.1. If $k \geq 11$ is prime, then $\mu_k = k^{\frac{1}{k-1}}$. Since $u = x^k - 1$, from (8), it is not hard to see that λ is monotone decreasing in $x \geq 2$ and $k \geq 11$, whereby $\lambda < 2.83$. It is easy to verify that $\frac{k-2\lambda}{2} - \frac{1}{k-1} > 2$ and $\frac{k-2\lambda-3}{2} > 1$. Thus,

$$k^{\frac{k-2\lambda}{2}-\frac{1}{k-1}} u^{\frac{k-2\lambda-3}{2}} > k^2 u > 8C.$$

- Case 2.2. If k takes one of the numbers in (9), with the help of software Maple [10], we can easily verify that condition (21) still holds. For example, if $k = 10 = 2 \cdot 5$, then $\mu_k = 2^{\frac{1}{2-1}} \cdot 5^{\frac{1}{5-1}}$. Since $x \geq 2$, then $\lambda < 3.39$. A quick calculation shows that condition (21) holds.

The establishment of (21) means that conditions (18) and (20) are contradictory.

This completes the proof. □

3 Some related questions

For the Diophantine equation $(x^k-1)(y^k-1)^2 = z^k-1$, we show that there is no positive integer solutions satisfying $1 < x \leq y$ or $1 < y < x \leq ((y^k-1)^{k-2}+1)^{\frac{1}{k}}$. Based on some numerical calculations (for cases $3 \leq k \leq 10$ and $2 \leq y < x \leq 10^3$), we have

Question 3.1. *If $k \geq 3$, then the Diophantine equation*

$$(x^k-1)(y^k-1)^2 = z^k-1$$

has no solutions in positive integers $x, y, z > 1$.

More generally,

Question 3.2. *If $k \geq 3$, then the Diophantine equation*

$$(x^k-1)^{t_1}(y^k-1)^{t_2} = (z^k-1)^{t_3}$$

has no solutions in positive integers $x, y, z > 1$, where $t_1, t_2, t_3 \in \mathbb{Z}^+$ and $\gcd(t_1, t_2, t_3) = 1$.

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