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Generalized Bronze Leonardo sequence

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Abstract: In this study, we define the Bronze Leonardo, Bronze Leonardo–Lucas, and Modified Bronze Leonardo sequences, and some terms of these sequences are given. Then, we give special summation formulas, special generating functions, etc. Also, we obtain the Binet formulas in three different ways. The first is in the known classical way, the second is with the help of the sequence's generating functions, and the third is with the help of the matrices. In addition, we find the special relations between the terms of the Bronze Leonardo, Bronze Leonardo–Lucas, and Modified Bronze Leonardo sequences. Moreover, we examine the relationships among the Bronze Fibonacci and Bronze Lucas sequences of these sequences. Finally, we associate these sequences with the matrices.
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1 Introduction

The Fibonacci and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci sequences have been applied in various fields such as cryptology [5], phylotaxis [18], biomathematics [4], chemistry [16], engineering [10], etc.

Many generalizations of the Fibonacci sequence have been given. The known examples of such sequences are the Fermat, Fermat–Lucas, Oresme, Pell, Jacobsthal sequences, etc. (see for details in [9, 14, 15, 17]).

For $n \in \mathbb{N}$, the Fibonacci numbers F_n, Lucas numbers L_n, and Leonardo numbers l_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n, \; L_{n+2} = L_{n+1} + L_n, \; l_{n+2} = l_{n+1} + l_n + 1$$

with the initial conditions $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, and $l_0 = 1$, $l_1 = 1$.

For F_n, L_n, and l_n the Binet formulas are given by the following relations, respectively,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \; L_n = \alpha^n + \beta^n, \text{ and } l_n = 2\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - 1,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $t^2 - t - 1 = 0$. Here the α number is the known golden ratio.

In [6], Catarino and Borges defined Leonardo numbers. In addition, they found the properties of this sequence. Alp and Koçer studied the properties of Leonardo numbers [3].

For $n \in \mathbb{N}$, the Bronze Fibonacci numbers BF_n, and Bronze Lucas numbers BL_n are defined by the recurrence relations, respectively,

$$BF_{n+2} = 3BF_{n+1} + BF_n, \text{ and } BL_{n+2} = 3BL_{n+1} + BL_n$$

with the initial conditions $BF_0 = 0$, $BF_1 = 1$, $BL_0 = 2$, $BF_0 = 3$. For BF_n, and BL_n the Binet formulas are given by relations, respectively,

$$BF_n = \frac{\lambda^n - \psi^n}{\lambda - \psi} \text{ and } BL_n = \lambda^n + \psi^n,$$

where $\lambda = \frac{3+\sqrt{13}}{2}$, and $\psi = \frac{3-\sqrt{13}}{2}$ are the roots of the characteristic equation $x^2 - 3x - 1 = 0$. Here the λ number is the known Bronze Ratio.

In [1], Akbiyik and Alo defined third-order Bronze Fibonacci Bf_n, Bronze Lucas Bl_n, and Modified Bronze Fibonacci MBf_n numbers.

For $n \in \mathbb{N}$, the third-order Bronze Fibonacci Bf_n, third-order Bronze Lucas Bl_n, and Modified third-order Bronze Fibonacci MBf_n defined by the recurrence relations, respectively,

$$Bf_{n+3} = 3Bf_{n+2} + Bf_{n+1} + Bf_n, \; Bl_{n+3} = 3Bl_{n+2} + Bl_{n+1} + Bl_n,$$

$$MBf_{n+3} = 3MBf_{n+2} + MBf_{n+1} + MBf_n$$

with the initial conditions $Bf_0 = 0$, $Bf_1 = Bf_0 = 1$, $Bf_2 = 3$, $Bl_0 = 3$, $Bl_1 = 3$, $Bl_2 = 11$, and $MBf_0 = 1$, $MBf_1 = 2$, $MBf_2 = 7$. In addition, Akbiyik and Alo found the following important features of these sequences [1].

(i) $Bf_{n+3} - Bf_{n+1} = MBf_{n+3} + Bf_{n+2}$,

(ii) $MBf_{n+1} = Bf_{n+1} - Bf_n$,

(ii) $Bl_{n+2} = 2MBf_{n+2} - MBf_{n+1} - MBf_n$,

(iv) $MBf_{n+3} = 2Bf_{n+2} + Bf_{n+1} + Bf_n$.

With the help of the recurrence relation of the Fibonacci sequence, *k*-sequences were introduced and these sequences had an important place in number theory [11]. In [7], Catarino and Ricardo defined the *k*-Bronze Fibonacci numbers. Additionally, in this study, they examined the relationship of this sequence with matrices. Moreover, they presented on quaternion Gaussian Bronze Fibonacci numbers [8]. In [2], Alo defined third-order Bronze Fibonacci quaternions and found many properties related to these quaternions. Karaaslan worked on Gaussian Bronze Lucas numbers [12].

As seen above, many generalizations of Fibonacci and Lucas sequences have been given so far. In this study, we give new generalizations inspired by the Bronze Fibonacci sequence. We call these sequences the Bronze Leonardo, Bronze Leonardo–Lucas, and Modified Bronze Leonardo sequences and denote them as BE_n, BC_n, and BM_n, respectively.

We separate the article into three parts.

In Section 2, we define the Bronze Leonardo, Bronze Leonardo–Lucas, and Modified Bronze Leonardo sequences. Also, we give the special generating functions, special summation formulas for these sequences. In addition, we obtain the Binet formulas in two different ways. The first is in the known classical way, the second is with the help of the sequence's generating functions.

In Section 3, we find the relations between the terms of the Bronze Leonardo, Bronze Leonardo–Lucas, and Modified Bronze Leonardo sequences. Then, we examine the relationships among the Bronze Fibonacci and Bronze Lucas sequences of these sequences. In addition, we associate these sequences with the matrices. Finally, we obtain the Binet formulas of these sequences with the help of the matrices.

2 Generalized Bronze Leonardo sequences

For $n \in \mathbb{N}$, the Bronze Leonardo BE_n, Bronze Leonardo–Lucas BC_n, and Modified Bronze Leonardo BM_n sequences are defined by, respectively,

$$BE_{n+2} = 3BE_{n+1} + BE_n + 1 \tag{1}$$

with $BE_0 = 1$ and $BE_1 = 1$,

$$BC_{n+2} = 3BC_{n+1} + BC_n - 3 \tag{2}$$

with $BC_0 = 3$ and $BC_1 = 4$,

$$BM_{n+2} = 3BM_{n+1} + BM_n + 1 \tag{3}$$

with $BM_0 = 0$ and $BM_1 = 1$.

Also, the third-order recurrence relations of the BE_n, BC_n, and BM_n sequences are as follows, respectively,

$$BE_{n+3} = 4BE_{n+2} - 2BE_{n+1} - BE_n, \text{ with } BE_0 = 1, BE_1 = 1, \text{ and } BE_2 = 5,$$

$$BC_{n+3} = 4BC_{n+2} - 2BC_{n+1} - BC_n, \text{ with } BC_0 = 3, BC_1 = 4, \text{ and } BC_2 = 12,$$

$$BM_{n+3} = 4BM_{n+2} - 2BM_{n+1} - BM_n, \text{ with } BM_0 = 0, BM_1 = 1, \text{ and } BM_2 = 4.$$

Then, let us give some information about the equations of these sequences.

The characteristic equation of the Bronze Leonardo, Bronze Leonardo–Lucas, and Modified Bronze Leonardo sequences is

$$x^3 - 4x^2 + 2x + 1 = 0. \tag{4}$$

The roots of the characteristic equation are as follows;

$$\lambda = \frac{3+\sqrt{13}}{2}, \psi = \frac{3-\sqrt{13}}{2}, \text{ and } \delta = 1.$$

Here the λ number is the known Bronze ratio.

Next, we give the relationships between these roots below:

$$\lambda + \psi = 3, \ \lambda\psi = -1, \ \lambda + \psi + \delta = 4, \ \lambda\psi + \lambda\delta + \psi\delta = 2, \text{ and } \lambda\psi\delta = -1.$$

The first few values of Bronze Leonardo, Bronze Leonardo–Lucas, and Modified Bronze Leonardo sequences are

1, 1, 5, 17, 57, 189,625, 2065, 6821, 22529, …

3, 4, 12, 37, 120, 394, 1299, 4288, 14160, 46765, …

and

0, 1, 4, 14, 47, 156, 516, 1705, 5632, 18602, …,

respectively.

In the following theorem, we express the Binet formulas of the Bronze Leonardo BE_n, Bronze Leonardo–Lucas BC_n, and Modified Bronze Leonardo BM_n sequences.

Theorem 2.1. *Let* $n \in \mathbb{N}$. *We obtain*

(i) $$BE_n = \frac{\lambda^2 + \lambda - 1}{(\lambda-\psi)(\lambda-1)}\lambda^{n-1} + \frac{\psi^2 + \psi - 1}{(\psi-\lambda)(\psi-1)}\psi^{n-1} - \frac{1}{(\lambda-1)(\psi-1)},$$

(ii) $$BC_n = \lambda^n + \psi^n + 1,$$

(iii) $$BM_n = \frac{1}{(\lambda-\psi)(\lambda-1)}\lambda^n + \frac{1}{(\psi-\lambda)(\psi-1)}\psi^n - \frac{1}{(\lambda-1)(\psi-1)}.$$

Proof. (i) The Binet form of a sequence is as follows

$$BE_n = x\lambda^n + y\psi^n + z\delta^n.$$

For these n values, we obtain:

$$\begin{aligned} BE_0 &= x + y + z, \\ BE_1 &= x\lambda + y\psi + z\delta, \quad , \\ BE_2 &= x\lambda^2 + y\psi^2 + z\delta^2. \end{aligned}$$

We find

$$x = \frac{\lambda^2 + \lambda - 1}{\lambda(\lambda-\psi)(\lambda-1)}, \ y = \frac{\psi^2 + \psi - 1}{\psi(\psi-\lambda)(\psi-1)}, \text{ and } z = -\frac{1}{(\lambda-1)(\psi-1)}.$$

The other items (ii) and (iii) may be proven similarly. □

In the following theorems, we give the generating functions of the Bronze Leonardo BE_n, Bronze Leonardo–Lucas BC_n, and Modified Bronze Leonardo BM_n sequences. In addition, we obtain Binet formulas of BE_n, BC_n and BM_n BM_n sequences with the help of generating functions.

Theorem 2.2. *The generating functions for Bronze Leonardo BE_n, Bronze Leonardo–Lucas BC_n, and Modified Bronze Leonardo BM_n sequences are given as follows, respectively,*

(i) $$e(x)=\sum_{n=0}^{\infty}BE_n x^n=\frac{1-3x+x^2}{1-4x+2x^2+x^3},$$

(ii) $$c(x)=\sum_{n=0}^{\infty}BC_n x^n=\frac{3-8x+2x^2}{1-4x+2x^2+x^3},$$

(iii) $$m(x)=\sum_{n=0}^{\infty}BM_n x^n=\frac{x}{1-4x+2x^2+x^3}.$$

Proof. (i). For the Bronze Leonardo sequence, we have

$$\begin{aligned}e(x)=\sum_{n=0}^{\infty}BE_n x^n&=1+x+5x^2+\sum_{n=3}^{\infty}BE_n x^n\\&=1+x+5x^2+4\sum_{n=3}^{\infty}BE_{n-1}x^n-2\sum_{n=3}^{\infty}BE_{n-2}x^n-\sum_{n=3}^{\infty}BE_{n-3}x^n\\&=1+x+5x^2+4x\sum_{n=2}^{\infty}BE_n x^n-2x^2\sum_{n=1}^{\infty}BE_n x^n-x^3\sum_{n=0}^{\infty}BE_n x^n.\end{aligned}$$

Thus, we obtain

$$e(x)=\sum_{n=0}^{\infty}BE_n x^n=\frac{1-3x+x^2}{1-4x+2x^2+x^3}.$$

The other items (ii) and (iii) may be proven similarly. □

Theorem 2.3. *For BE_n, BC_n and BM_n sequences, the Binet formulas can be obtained with the help of the generating functions.*

Proof. With the help of the roots of the characteristic equation of these sequences, the roots of the $1-4x+2x^2+x^3=0$ equation become $\frac{1}{\lambda}$, $\frac{1}{\psi}$, and $\frac{1}{\delta}$. For BE_n, we obtain

$$\begin{aligned}\frac{1-3x+x^2}{1-4x+2x^2+x^3}&=\frac{\lambda^2+\lambda-1}{\lambda(\lambda-\psi)(\lambda-1)}\frac{1}{1-\lambda x}+\frac{\psi^2+\psi-1}{\psi(\psi-\lambda)(\psi-1)}\frac{1}{1-\psi x}-\frac{1}{(\lambda-1)(\psi-1)}\frac{1}{1-\delta x}\\&=\frac{\lambda^2+\lambda-1}{\lambda(\lambda-\psi)(\lambda-1)}\sum_{n=0}^{\infty}\lambda^n x^n+\frac{\psi^2+\psi-1}{\psi(\psi-\lambda)(\psi-1)}\sum_{n=0}^{\infty}\psi^n x^n-\frac{1}{(\lambda-1)(\psi-1)}\sum_{n=0}^{\infty}\delta^n x^n\\&=\sum_{n=0}^{\infty}BE_n x^n.\end{aligned}$$

Similarly, the Binet formulas of the BC_n and BM_n sequences are found. □

Next, we give special sum formulas of the Bronze Leonardo BE_n, Bronze Leonardo–Lucas BC_n, and Modified Bronze Leonardo BM_n sequences.

Theorem 2.4. *Let* $n \in \mathbb{N}$. *We obtain*

(i) $\sum_{s=0}^{n} BE_s = \frac{4BE_n + BE_{n-1} - n + 2}{3}$, (ii) $\sum_{s=0}^{n} BC_s = \frac{4BC_n + BC_{n-1} + 3n - 1}{3}$,

(iii) $\sum_{s=0}^{n} BM_s = \frac{4BM_n + BM_{n-1} - n}{3}$.

Proof. (ii). From the definition of the Bronze Leonardo–Lucas sequence, we obtain

$$\begin{aligned} BC_2 &= 3BC_1 + BC_0 - 3, \\ BC_3 &= 3BC_2 + BC_1 - 3, \\ &\vdots \\ BC_n &= 3BC_{n-1} + BC_{n-2} - 3. \end{aligned}$$

So, we have

$$-7 + \sum_{s=0}^{n} BC_s = 3\sum_{s=1}^{n-1} BC_s + \sum_{s=0}^{n-2} BC_s - (n-1)3,$$
$$-7 + \sum_{s=0}^{n} BC_s = 3\left(-BC_n - 3 + \sum_{s=0}^{n} BC_s\right) + \left(-BC_n - BC_{n-1} + \sum_{s=0}^{n} BC_s\right) - (n-1)3.$$

Thus, we obtain

$$\sum_{s=0}^{n} BC_s = \frac{4BC_n + BC_{n-1} + 3n - 1}{3}.$$

The other items (i) and (iii) may be proved similarly. □

Theorem 2.5. *Let* $n \in \mathbb{N}$. *We obtain*

(i) $\sum_{s=0}^{n} BE_{2s} = \frac{3BE_{2n+1} - n + 2}{3}$, (ii) $\sum_{s=0}^{n} BE_{2s+1} = \frac{3BE_{2n+1} + B_{2n} - n - 1}{3}$,

(iii) $\sum_{s=0}^{n} BC_{2s} = \frac{BC_{2n+1} + 3n + 5}{3}$, (iv) $\sum_{s=0}^{n} BC_{2s+1} = \frac{3BC_{2n+1} + BC_{2n} + 3n - 3}{3}$,

(v) $\sum_{s=0}^{n} BM_{2s} = \frac{BM_{2n+1} - n - 1}{3}$, (vi) $\sum_{s=0}^{n} BM_{2s+1} = \frac{3BM_{2n+1} + BM_{2n} - n}{3}$.

Proof. (i) From the definition of the Bronze Leonardo sequence, we have

$$\begin{aligned} BE_3 &= 3BE_2 + BE_1 + 1, \\ BE_5 &= 3BE_4 + BE_3 + 1, \\ &\vdots \\ BE_{2n+1} &= 3BE_{2n} + BE_{2n-1} + 1. \end{aligned}$$

So, we get

$$-1+\sum_{s=0}^{n}BE_{2s+1}=3\sum_{s=1}^{n}BE_{2s}+\sum_{s=0}^{n-1}BE_{2s+1}+n,$$
$$-1+\sum_{s=0}^{n}BE_{2s+1}=3\left(-BE_0+\sum_{s=0}^{n}BE_{2s}\right)+\left(-BE_{2n+1}+\sum_{s=0}^{n}BE_{2s+1}\right)+n.$$

Thus, we obtain

$$\sum_{s=0}^{n}BE_{2s}=\frac{3BE_{2n+1}-n+2}{3}.$$

The other items (ii)–(vi) may be proved similarly. □

3 Relations among special sequences

In this section, we examine the relationships among the Bronze Leonardo BE_n, Bronze Leonardo–Lucas BC_n, Modified Bronze Leonardo BM_n, Bronze Fibonacci BF_n, and Bronze Lucas BL_n. sequences. In addition, we associate the terms of these sequences with matrices. Finally, we obtain the Binet formulas of these sequences with the help of the matrices.

In the following theorem, we examine the relations among the Bronze Leonardo BE_n, Bronze Leonardo–Lucas BC_n and Modified Bronze Leonardo BM_n sequences.

Theorem 3.1. *Let* $n\in\mathbb{N}$. *The following equations are true:*

(i) $BE_n=\frac{41}{117}BC_{n+2}+\frac{-15}{13}BC_{n+1}+\frac{55}{117}BC_n$, (ii) $BE_n=-3BM_{n+2}+13BM_{n+1}-9BM_n$,

(iii) $BC_n=\frac{3}{2}BE_{n+2}-\frac{17}{4}BE_{n+1}-\frac{1}{4}BE_n$, (iv) $BC_n=-2BM_{n+2}+11BM_{n+1}-12BM_n$,

(v) $BM_n=-\frac{1}{4}BE_{n+2}+BE_{n+1}+\frac{1}{4}BE_n$, (vi) $BM_n=\frac{17}{117}BC_{n+2}+\frac{-4}{13}BC_{n+1}-\frac{20}{117}BC_n$.

Proof. (i) The following relation is used for proofs;

$$BE_n=a\,x\,BC_{n+2}+b\,x\,BC_{n+1}+c\,x\,BC_n.$$

For these n values, we obtain;

$$BE_0=a\,x\,BC_2+b\,x\,BC_1+c\,x\,BC_0,$$
$$BE_1=a\,x\,BC_3+b\,x\,BC_2+c\,x\,BC_1,$$
$$BE_2=a\,x\,BC_4+b\,x\,BC_3+c\,x\,BC_2.$$

We find

$$a=\frac{41}{117},\ b=\frac{-15}{13},\text{ and } c=\frac{55}{117}.$$

The other items (ii)–(vi) may be proved similarly. □

In the following theorem, we examine the relations among the Bronze Leonardo BE_n, Bronze Leonardo–Lucas BC_n, Modified Bronze Leonardo BM_n, Bronze Fibonacci BF_n, and Bronze Lucas BL_n sequences.

Theorem 3.2. *Let* $n \in \mathbb{N}$. *The following equations are satisfied:*

(i) $BE_n = \frac{4}{3}BF_{n+1} - \frac{8}{3}BF_n - \frac{1}{3}$, (ii) $BE_n = -\frac{2}{39}BL_{n+1} + \frac{32}{39}BL_n - \frac{1}{3}$,

(iii) $BC_n = 2BF_{n+1} - 3BF_n + 1$, (iv) $BC_n = BL_n + 1$,

(v) $BM_n = \frac{1}{3}\left(BF_{n+1} + BF_n - 1\right)$, (vi) $BM_n = \frac{5}{39}BL_{n+1} - \frac{1}{39}BL_n - \frac{1}{3}$.

Proof. (iii). If Binet formulas are used for proofs, we get

$$
\begin{aligned}
2BF_{n+1} - 3BF_n + 1 &= 2\left(\frac{\lambda^{n+1} - \psi^{n+1}}{\lambda - \psi}\right) - 3\left(\frac{\lambda^n - \psi^n}{\lambda - \psi}\right) + 1 \\
&= \frac{\lambda^n\left(2\lambda - 3\right) + \psi^n\left(2\psi - 3\right)}{\lambda - \psi} + 1 \\
&= \frac{\lambda^n\left(2\lambda - 3\right) + \psi^n\left(2\psi - 3\right)}{\lambda - \psi} + 1 \\
&= \lambda^n + \psi^n + 1 \\
&= BC_n.
\end{aligned}
$$

The other items (i), (ii), (iv)–(vi) may be proved similarly. □

Theorem 3.3. *Let* $n \in \mathbb{N}$. *The following equations are true:*

(i) $BF_n = \frac{1}{4}\left(BE_{n+1} - BE_n\right)$, (ii) $BL_n = \frac{1}{4}\left(BE_{n+1} + 5BE_n + 2\right)$,

(iii) $BF_n = \frac{1}{13}\left(2BC_{n+1} - 3BC_n + 1\right)$, (iv) $BL_n = BC_n - 1$,

(v) $BF_n = -BM_{n+1} + 4BM_n + 1$, (vi) $BL_n = 5BM_{n+1} - 14BM_n - 3$.

Proof. (iii) If Binet formulas are used for proofs, we obtain

$$
\begin{aligned}
\frac{1}{13}\left(2BC_{n+1} - 3BC_n + 1\right) &= \frac{1}{13}\left(2\left(\lambda^{n+1} + \psi^{n+1} + 1\right) - 3\left(\lambda^n + \psi^n + 1\right) + 1\right) \\
&= \frac{\lambda^n\left(2\lambda - 3\right) + \psi^n\left(2\psi - 3\right) + 2 - 3 + 1}{\left(\lambda - \psi\right)^2} \\
&= \frac{\lambda^n\left(\lambda - \psi\right) + \psi^n\left(\psi - \lambda\right)}{\left(\lambda - \psi\right)^2} \\
&= BF_n
\end{aligned}
$$

The other items (i), (ii), (iv)–(vi) may be proven similarly. □

In the following theorem, we find special relation for the Bronze Leonardo BE_n, and Bronze Leonardo–Lucas BC_n sequences.

Theorem 3.4. *Let $n \in \mathbb{N}$. The following equation is satisfied:*

$$BE_n BC_n = BE_{2n} + \frac{1}{2}\left(BC_n^2 - BC_{2n}\right) - (-1)^n BE_{-n}.$$

Proof. Let $BE_n = \frac{\lambda^2+\lambda-1}{\lambda(\lambda-\psi)(\lambda-1)}\lambda^n + \frac{\psi^2+\psi-1}{\psi(\psi-\lambda)(\psi-1)}\psi^n - \frac{1}{(\lambda-1)(\psi-1)} = X\lambda^n + Y\psi^n - Z$.

If Binet formulas are used for proofs, we obtain

$$(-1)^n BE_{-n} = \left(\lambda^n\psi^n\right)\left(X\lambda^{-n} + Y\psi^{-n} - Z\right) = X\psi^n + Y\lambda^n - Z\lambda^n\psi^n$$

and

$$\frac{1}{2}\left(BC_n^2 - BC_{2n}\right) = \frac{1}{2}\left[\left(\lambda^n+\psi^n+1\right)^2 - \left(\lambda^{2n}+\psi^{2n}+1\right)\right] = \lambda^n + \psi^n + \lambda^n\psi^n.$$

Thus,

$$\begin{aligned} BE_n BC_n &= \left(X\lambda^n + Y\psi^n - Z\right)\left(\lambda^n+\psi^n+1\right) = X\lambda^{2n} + Y\psi^{2n} - Z + (X+Y-Z)\lambda^n \\ &\quad + (X+Y-Z)\psi^n + (X+Y-Z)\psi^n\lambda^n + \left(-X\psi^n - Y\lambda^n + Z\lambda^n\psi^n\right) \\ &= X\lambda^{2n} + Y\psi^{2n} - Z + (X+Y-Z)\left(\lambda^n+\psi^n+\lambda^n\psi^n\right) - \left(X\psi^n + Y\lambda^n - Z\lambda^n\psi^n\right) \\ &= BE_{2n} + \frac{1}{2}\left(BC_n^2 - BC_{2n}\right) - (-1)^n BE_{-n} \end{aligned}$$

This completes the proof. □

In the following theorems, we associate the terms of the Bronze Leonardo BE_n, Bronze Leonardo–Lucas BC_n and Modified Bronze Leonardo BM_n sequences with matrices. In addition, we obtain the Binet formulas of these sequences with the help of the matrices.

Theorem 3.5. *Let $n \in \mathbb{N}$. The following equations are true:*

(i). *For the Bronze Leonardo sequence,*

1. $\begin{bmatrix} BE_{n+2} \\ BE_{n+1} \\ BE_n \end{bmatrix} = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$,

2. $\begin{bmatrix} BE_n \\ BE_{n+1} \\ BE_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -2 & -1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$,

3. $\begin{bmatrix} BE_{n+2} \\ BE_{n+1} \\ BE_n \end{bmatrix} = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} BE_{n+1} \\ BE_n \\ BE_{n-1} \end{bmatrix}$.

(ii). *For the Bronze Leonardo–Lucas sequence,*

1. $\begin{bmatrix} BC_{n+2} \\ BC_{n+1} \\ BC_n \end{bmatrix} = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 12 \\ 4 \\ 3 \end{bmatrix},$ 2. $\begin{bmatrix} BC_n \\ BC_{n+1} \\ BC_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -2 & -1 \end{bmatrix}^n \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix},$

3. $\begin{bmatrix} BC_{n+2} \\ BC_{n+1} \\ BC_n \end{bmatrix} = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} BC_{n+1} \\ BC_n \\ BC_{n-1} \end{bmatrix}.$

(iii). *For the Modified Bronze Leonardo–Lucas sequence,*

1. $\begin{bmatrix} BM_{n+2} \\ BM_{n+1} \\ BM_n \end{bmatrix} = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix},$ 2. $\begin{bmatrix} BM_n \\ BM_{n+1} \\ BM_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -2 & -1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix},$

3. $\begin{bmatrix} BM_{n+2} \\ BM_{n+1} \\ BM_n \end{bmatrix} = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} BM_{n+1} \\ BM_n \\ BM_{n-1} \end{bmatrix}.$

Proof. (i). Let us show the proof by induction over n. For $n = 1$, the equality is true. For $n - 1$, assume the equality is true. We obtain

$$\begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} BE_{n+1} \\ BE_n \\ BE_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} BE_{n+2} \\ BE_{n+1} \\ BE_n \end{bmatrix}.$$

From the last equation, for n, it can be seen that the equality is true.
The other items (ii) and (iii) may be proven similarly. □

Theorem 3.6. (Simpson Formulas) *Let* $n \in \mathbb{N}$. *The following equations are satisfied:*

(i) $\det \begin{bmatrix} BE_{n+2} & BE_{n+1} & BE_n \\ BE_{n+1} & BE_n & BE_{n-1} \\ BE_n & BE_{n-1} & BE_{n-2} \end{bmatrix} = (-1)^{n+1} 16,$ (ii) $\det \begin{bmatrix} BC_{n+2} & BC_{n+1} & BC_n \\ BC_{n+1} & BC_n & BC_{n-1} \\ BC_n & BC_{n-1} & BC_{n-2} \end{bmatrix} = (-1)^n 117,$

(ii) $\det \begin{bmatrix} BM_{n+2} & BM_{n+1} & BM_n \\ BM_{n+1} & BM_n & BM_{n-1} \\ BM_n & BM_{n-1} & BM_{n-2} \end{bmatrix} = (-1)^n.$

Proof. (i) Let show the proof by induction over n. For $n = 1$, the equality is true. For $n – 1$, assume the equality is true. We obtain

$$\det\begin{bmatrix} BE_{n+2} & BE_{n+1} & BE_n \\ BE_{n+1} & BE_n & BE_{n-1} \\ BE_n & BE_{n-1} & BE_{n-2} \end{bmatrix} = \det\begin{bmatrix} 4BE_{n+1}-2BE_n-BE_{n-1} & BE_{n+1} & BE_n \\ 4BE_n-2BE_{n-1}-BE_{n-2} & BE_n & BE_{n-1} \\ 4BE_{n-1}-2BE_{n-2}-BE_{n-3} & BE_{n-1} & BE_{n-2} \end{bmatrix}$$

$$= \det\begin{bmatrix} 4BE_{n+1} & BE_{n+1} & BE_n \\ 4BE_n & BE_n & BE_{n-1} \\ 4BE_{n-1} & BE_{n-1} & BE_{n-2} \end{bmatrix} + \det\begin{bmatrix} -2BE_n & BE_{n+1} & BE_n \\ -2BE_{n-1} & BE_n & BE_{n-1} \\ -2BE_{n-2} & BE_{n-1} & BE_{n-2} \end{bmatrix}$$

$$+ \det\begin{bmatrix} -BE_{n-1} & BE_{n+1} & BE_n \\ -BE_{n-2} & BE_n & BE_{n-1} \\ -BE_{n-3} & BE_{n-1} & BE_{n-2} \end{bmatrix}$$

$$= (-1)\det\begin{bmatrix} BE_{n+1} & BE_n & BE_{n-1} \\ BE_n & BE_{n-1} & BE_{n-2} \\ BE_{n-1} & BE_{n-2} & BE_{n-3} \end{bmatrix}$$

$$= (-1)^{n+1}16.$$

From the last equation, for n, it can be seen that the equality is true.
The other items (ii) and (iii) may be proven similarly. ☐

Theorem 3.7. *Let* $n \in \mathbb{N}$. *The following equation is satisfied:*

$$\begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} BM_{n+1} & -2BM_n - BM_{n-1} & -BM_n \\ BM_n & -2BM_{n-1} - BM_{n-2} & -BM_{n-1} \\ BM_{n-1} & -2BM_{n-2} - BM_{n-3} & -BM_{n-2} \end{bmatrix}.$$

Proof. Let show the proof by induction over n. For $n = 1$, the equality is true. For $n – 1$, assume the equality is true. We obtain

$$\begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} BM_n & -2BM_{n-1} - BM_{n-2} & -BM_{n-1} \\ BM_{n-1} & -2BM_{n-2} - BM_{n-3} & -BM_{n-2} \\ BM_{n-2} & -2BM_{n-3} - BM_{n-4} & -BM_{n-3} \end{bmatrix} \begin{bmatrix} 4 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} BM_{n+1} & -2BM_n - BM_{n-1} & -BM_n \\ BM_n & -2BM_{n-1} - BM_{n-2} & -BM_{n-1} \\ BM_{n-1} & -2BM_{n-2} - BM_{n-3} & -BM_{n-2} \end{bmatrix}.$$

From the last equation, for n, it can be seen that the equality is true. ☐

Theorem 3.8. *For BE_n, BC_n and BM_n sequences, the Binet formulas can be obtained with the help of matrices.*

Proof. The following relation is used for proof (see for details Corollary 3.1 in [13]):

$$t_n = \frac{1}{\det(\Lambda)} \sum_{j=1}^{i} r_{m+1-j} \det(\Lambda_j).$$

Thus, for BE_n

$$BE_n = \frac{1}{\det(\Lambda)} \sum_{j=1}^{i} BE_{m+1-j} \det(\Lambda_j).$$

Let $m = i = 3$,

$$\Lambda = \begin{bmatrix} \lambda^2 & \lambda & 1 \\ \psi^2 & \psi & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} \lambda^{n-1} & \lambda & 1 \\ \psi^{n-1} & \psi & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} \lambda^2 & \lambda^{n-1} & 1 \\ \psi^2 & \psi^{n-1} & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ and } \Lambda_3 = \begin{bmatrix} \lambda^2 & \lambda & \lambda^{n-1} \\ \psi^2 & \psi & \psi^{n-1} \\ 1 & 1 & 1 \end{bmatrix}.$$

So, we obtain

$$\begin{aligned} BE_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^{3} BE_{4-j} \det(\Lambda_j) \\ &= \frac{1}{\det(\Lambda)} \left(BE_3 \det(\Lambda_1) + BE_2 \det(\Lambda_2) + BE_1 \det(\Lambda_3) \right) \\ &= \frac{\lambda^2 + \lambda - 1}{(\lambda - \psi)(\lambda - 1)} \lambda^{n-1} + \frac{\psi^2 + \psi - 1}{(\psi - \lambda)(\psi - 1)} \psi^{n-1} - \frac{1}{(\lambda - 1)(\psi - 1)} \end{aligned}$$

Similarly, the Binet formulas of the BC_n and BM_n, sequences are found. □

4 Conclusion

In this paper, we defined the Bronze Leonardo and Bronze Leonardo Lucas sequences. Then, we obtained the many features of these sequences. Also, we found the relationships between the terms of these sequences. In addition, we gave the summation formulas and generating functions of these sequences. Moreover, we obtained the Binet formulas in three different ways. The first is in the known classical way, the second is with the help of the sequence's generating functions, and the third is with the help of the matrices. Finally, we associated these sequences with the matrices. If this study is examined, such features can be found in other sequences such as Bronze Pell, and Bronze Pell-Lucas sequences.

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