Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2024, Volume 30, Number 4, 803–810
DOI: 10.7546/nntdm.2024.30.4.803-810

Irrationality and transcendence of infinite series of rational terms

Lazhar Dhaouadi

Carthage University, Preparatory Institute for Engineering Studies of Bizerte
7021 Zarzouna, Tunisia
e-mail: lazhardhaouadi@yahoo.fr

Received: 31 May 2024 **Revised:** 13 November 2024
Accepted: 26 November 2024 **Online First:** 27 November 2024

Abstract: The aim of this paper is to prove two results concerning the irrationality and transcendence of a certain class of infinite series which consist of rational numbers and converge very quickly.
Keywords: Infinite series, Irrational number, Transcendental number.
2020 Mathematics Subject Classification: 11J72, 11J81.

1 Introduction

There are a number of sufficient conditions known within the literature for an infinite series of positive rationals to converge to an irrational or transcendal number. These conditions, which are quite varied in form, share one common feature, namely, they all require rapid convergence. In 1844, Joseph Liouville give the first example of transcendal number as a sum of rapidly convergente series with rational coefficient

$$x = \sum_{n=1}^{\infty} \frac{a_n}{b^{n!}}.$$

for any integer $b \geq 2$ and any sequence of integers $(a_1, a_2, \ldots)$ such that $0 \leq a_k < b$ and $a_k \neq 0$

for infinitely many k. In 1932, Cantor gave a criterion for the irrationality of a real number x given by the infinite series

$$x = \sum_{n=1}^{\infty} \frac{a_n}{b_1 \cdots b_n},$$

and later, Oppenheim [8] extend hits method to prove other criterion for irrationality. Also, Hančl and Tijdeman [4] study other conditions under which the Cantor series is irrational. Erdős [2] proved that if the sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers converges quickly to infinity, then the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

is an irrational number.

Mahler in [5] introduced the main method of proving the transcendence of sums of infinite series with rational coefficient. This method has been extended several times and Nishioka's book [6] contains a survey of these results. Another technique makes use of the Roth's theorem [9]. One can also use Nyblom's theorem which can be found in [7]. Other methods are given by Sándor [10] and Badea [1].

These paper deals with a criteria concerning the irrationality and transcendence of the sums of infinite series. The terms of these series consist of positive rational numbers which converge rapidly to zero.

The main result of the first section is Theorem 2.2. It deals with a criterion for the irrationality of sums of infinite series of rational numbers which depends on the speed and character of the convergence. Our main result is a consequence of the Badea theorem [1]. The aim of the second section is to prove the Theorem 3.2 on transendence of infinite series of positive rational terms, which is a consequence of [3, Theorem 2.1]. In the next section, we prove a general theorem which emphasizes the relationship between the irrationality and the transcendence of convergent infinite series of positive rational numbers and their speed of convergence.

2 The irrationality cases

Definition 2.1. We introduce the following sets:

- $\mathcal{L}$ as the set of rational positive numbers $r = \{r_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \sup \frac{r_{n+1}}{r_n} \leq 1$.
- $\mathcal{J}$ as the set of sequences of positive integers $v = \{v_n\}_{n=1}^{\infty}$ such that $\left\{\frac{v_{n+1}}{v_n}\right\}_{n=1}^{\infty}$ is strictly increasing.

Definition 2.2. Let $v, u \in \mathcal{J}$. The sequence u precedes the sequence v if there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} \geq v_{n_0}$ and for every $n \geq n_0$ we have

$$\frac{u_{n+1}}{u_n} \geq \frac{v_{n+1}}{v_n},$$

and we write $u \geq v$.

Definition 2.3. Let $r = \{r_n\}_{n=1}^{\infty} \in \mathcal{L}$ and $v = \{v_n\}_{n=1}^{\infty} \in \mathcal{J}$. We introduce the function $v * r$ as follows

$$v * r(x) = \sum_{n=1}^{\infty} r_n x^{v_n}.$$

Then $v * r$ is well defined on $(0, 1)$. The function $v * r$ is called irrational if $v * r(x)$ is irrational for any rational number $x \in (0, 1)$ and we write $r \in \mathrm{Ir}(v)$.

In 1987 in [1] Badea proved the following theorem.

Theorem 2.1. [1] *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that for every large n,*

$$a_{n+1} > \frac{b_{n+1}}{b_n} a_n^2 - \frac{b_{n+1}}{b_n} a_n + 1. \tag{1}$$

Then the sum $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is an irrational number.

Remark 2.1. *The condition (1) is satisfied if for every large n we have*

$$1 > \frac{b_{n+1}}{b_n} \frac{a_n^2}{a_{n+1}} \quad \textit{and} \quad \frac{b_{n+1}}{b_n} a_n > 1. \tag{2}$$

The main result of this section is the following theorem.

Theorem 2.2. *Let $r \in \mathcal{L}$. Then there exists $v = \{v_n\}_{n=1}^{\infty} \in \mathcal{J}$ such that $r \in \mathrm{Ir(v)}$. On the other hand, for any sequence of positive integers $u = \{u_n\}_{n=1}^{\infty} \in \mathcal{J}$ such that $v \le u$, we have $r \in \mathrm{Ir(u)}$.*

Proof. Let

$$0 < x = \frac{p}{q} < 1 \quad \text{and } r_n = \frac{b_n}{a_n}.$$

Our infinite series is

$$\sum_{n=1}^{\infty} r_n x^{v_n} = \sum_{n=1}^{\infty} \frac{b_n p^{v_n}}{a_n q^{v_n}}.$$

Then (2) becomes

$$1 > \frac{b_{n+1}}{b_n} \frac{a_n^2}{a_{n+1}} \frac{p^{v_{n+1}-v_n}}{q^{v_{n+1}-2v_n}} \text{ and } \frac{b_{n+1}}{b_n} a_n p^{v_{n+1}-v_n} q^{v_n} > 1. \tag{3}$$

Sufficient conditions for (3) to be satisfied are

$$\lim_{n\to\infty} \log \left[\frac{b_{n+1}}{b_n} \frac{a_n^2}{a_{n+1}} \frac{p^{v_{n+1}-v_n}}{q^{v_{n+1}-2v_n}} \right] = -\infty \text{ and } \lim_{n\to\infty} \log \left[\frac{b_{n+1}}{b_n} a_n p^{v_{n+1}-v_n} q^{v_n} \right] = +\infty.$$

Now by a simple calculation we obtain

$$\begin{aligned}
&\log\left[\frac{b_{n+1}}{b_n}\frac{a_n^2}{a_{n+1}}\frac{p^{v_{n+1}-v_n}}{q^{v_{n+1}-2v_n}}\right] \\
= \;& \log\left(\frac{b_{n+1}}{b_n}\frac{a_n^2}{a_{n+1}}\right) + (v_{n+1}-v_n)\log p - (v_{n+1}-2v_n)\log q \\
= \;& v_n\left[\frac{1}{v_n}\log\left(\frac{b_{n+1}}{b_n}\frac{a_n^2}{a_{n+1}}\right) + \left(\frac{v_{n+1}}{v_n}-1\right)\log p - \left(\frac{v_{n+1}}{v_n}-2\right)\log q\right] \\
= \;& v_n\left[\frac{1}{v_n}\log\left(\frac{r_{n+1}}{r_n}\right) + \frac{1}{v_n}\log(a_n) + \left(\frac{v_{n+1}}{v_n}-1\right)\log p - \left(\frac{v_{n+1}}{v_n}-2\right)\log q\right].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\log\left[\frac{b_{n+1}}{b_n}a_n p^{v_{n+1}-v_n}q^{v_n}\right] &= \log\left(\frac{b_{n+1}}{b_n}a_n\right) + (v_{n+1}-v_n)\log p + v_n\log q \\
&= \log\left(\frac{r_{n+1}}{r_n}\right) + \log(a_{n+1}) + (v_{n+1}-v_n)\log p + v_n\log q.
\end{aligned}$$

It suffices to take $v = \{v_n\}_{n=1}^{\infty}$ satisfying the following conditions

$$\lim_{n\to\infty}\frac{1}{v_n}\log(a_n) = 0 \;\text{ and }\; v_n = o(v_{n+1}). \tag{4}$$

On the other hand, if we take another increasing sequence of positive integers $u = \{u_n\}_{n=1}^{\infty}$ such that $u \geq v$, then we obtain for every $n \geq n_0$

$$u_n \geq v_n \quad \Rightarrow \quad \lim_{n\to\infty}\frac{1}{u_n}\log(a_n) = 0.$$

Also we have

$$\frac{u_{n+1}}{u_n} \geq \frac{v_{n+1}}{v_n} \quad \Rightarrow \quad \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = +\infty.$$

So u satisfies (4) and we obtain the result $r \in \mathrm{Ir}(u)$. □

3 The transcendence cases

Definition 3.1. Let $r = \{r_n\}_{n=1}^{\infty} \in \mathcal{L}$ and $v = \{v_n\}_{n=1}^{\infty} \in \mathcal{J}$. The function $v * r$ is called transcendental if $v * r(x)$ is transcendental number for any rational number $x \in (0,1)$ and we write $r \in \mathrm{Tr}(v)$.

In [3, Theorem 2.1] J. Hančl and P. Rucki have proven the following theorem.

Theorem 3.1. [3] *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two sequences of positive integers such that*

$$\lim_{n\to\infty}\sup\frac{a_{n+1}}{(a_1\cdots a_n)}\frac{1}{b_{n+1}} = \infty, \tag{5}$$

and

$$\lim_{n\to\infty}\inf\frac{a_{n+1}}{a_n}\frac{b_n}{b_{n+1}} > 1. \tag{6}$$

Then the number

$$\xi = \sum_{n=1}^{\infty} \frac{b_n}{a_n} < \infty$$

is transcendental.

The main result of these section is the following:

Theorem 3.2. *Let $r \in \mathcal{L}$. Then there exists $v = \{v_n\}_{n=1}^{\infty} \in \mathcal{J}$ such that $r \in \mathrm{Tr}(v)$. On the other hand, for any sequence $u = \{u_n\}_{n=1}^{\infty} \in \mathcal{J}$ such that $v \leq u$ we have $r \in \mathrm{Tr}(u)$.*

Proof. Let

$$0 < x = \frac{p}{q} < 1 \quad \text{and } r_n = \frac{b_n}{a_n}.$$

Our infinite series is

$$\sum_{n=1}^{\infty} r_n x^{v_n} = \sum_{n=1}^{\infty} \frac{b_n p^{v_n}}{a_n q^{v_n}}.$$

Then (5) becomes

$$\lim_{n\to\infty} \sup \frac{a_{n+1}}{(a_1 \cdots a_n)^{2+\delta}} \frac{1}{b_{n+1}} q^{v_{n+1}-(v_1+\cdots+v_n)} p^{v_{n+1}} = +\infty \tag{7}$$

and (6) takes the following form

$$\lim_{n\to\infty} \inf \frac{a_{n+1}}{a_n} \frac{b_n}{b_{n+1}} \left(\frac{q}{p}\right)^{v_{n+1}-v_n} > 1. \tag{8}$$

Now by a simple calculation we obtain

$$\lim_{n\to\infty} v_{n+1} \left[\frac{1}{v_{n+1}} \log\left(\frac{a_{n+1}}{(a_1\cdots a_n)} \frac{1}{b_{n+1}}\right) + \left(1 - \frac{1}{v_{n+1}}(v_1 + \cdots + v_n)\right) \log q - \log p\right] = +\infty.$$

A sufficient conditions for (7) to be satisfied is

$$\lim_{n\to\infty} \frac{1}{v_{n+1}} \log\left(\frac{a_{n+1}}{(a_1\cdots a_n)} \frac{1}{b_{n+1}}\right) = 0 \text{ and } (v_1 + \cdots + v_n) = o(v_{n+1}). \tag{9}$$

The condition (8) is a simple consequence of the fact that

$$\lim_{n\to\infty} \sup \frac{r_{n+1}}{r_n} \leq 1.$$

On the other hand, if we take another increasing sequence of positive integers $u = \{u_n\}_{n=1}^{\infty}$ such that $u \geq v$, then we obtain for every $n \geq n_0$

$$u_n \geq v_n \Rightarrow \lim_{n\to\infty} \frac{1}{v_{n+1}} \log\left(\frac{a_{n+1}}{(a_1\cdots a_n)} \frac{1}{b_{n+1}}\right) = 0.$$

Without loss of generality, we take $n_0 = 1$, then

$$\begin{aligned} u_1 + \cdots + u_n &= \frac{u_1}{v_1} v_1 + \cdots + \frac{u_n}{v_n} v_n \\ &\leq \frac{u_{n+1}}{v_{n+1}} (v_1 + \cdots + v_n) \\ &= u_{n+1} o(1) \\ &= o(u_{n+1}). \end{aligned}$$

Hence, u satisfies (9) and we obtain the result $r \in \mathrm{Tr}(u)$. □

4 A general overview

Theorem 4.1. *Let A be a countable subset of $\mathcal{L}$. Then there exist $\alpha, \beta \in \mathcal{J}$ such that $A \subset \mathrm{Ir}(\alpha)$ and $A \subset \mathrm{Tr}(\beta)$.*

Proof. We only expose the proof of $\mathrm{Ir}(\alpha)$ because the case of $\mathrm{Tr}(\beta)$ is similar. Let $(r^1, r^2, \ldots)$ be an enumeration of the elements of A. From Theorem 2.2 we can assume that for every $k = 1, 2, \ldots,$

$$r^k \in \mathrm{Ir}(v^k),$$

for some $v^k = \left(v_1^k, v_2^k, \ldots\right) \in \mathcal{J}$. We shall prove that there exists $\alpha \in \mathcal{J}$ such that for every $k = 1, 2, \ldots,$

$$r^k \in \mathrm{Ir}(\alpha),$$

A sufficient condition is that for every large n and $k = 1, 2, \ldots$

$$v^k \leq \alpha \quad \Leftrightarrow \quad \frac{v_{n+1}^k}{v_n^k} \leq \frac{\alpha_{n+1}}{\alpha_n}.$$

For the construction of the sequence $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ we use the diagonalization method. The n-th sequence is in the n-th line.

$$\begin{array}{cccccc}
\frac{v_2^1}{v_1^1} & \frac{v_3^1}{v_2^1} & \frac{v_4^1}{v_3^1} & \frac{v_5^1}{v_4^1} & \frac{v_6^1}{v_5^1} & \cdots \\
\frac{v_2^2}{v_1^2} & \frac{v_3^2}{v_2^2} & \frac{v_4^2}{v_3^2} & \frac{v_5^2}{v_4^2} & \frac{v_6^2}{v_5^2} & \cdots \\
\frac{v_1^3}{v_1^3} & \frac{v_1^3}{v_1^3} & \frac{v_1^3}{v_1^3} & \frac{v_1^3}{v_1^3} & \frac{v_1^3}{v_1^3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}$$

For our first element we take $\alpha_1 = 1$. Now take the first number of sequence one, and multiply by α_1. That is our second element α_2. Now take the second number of sequence two, and the number from the previous step. Take the larger of both and multiply by α_2. That is our element α_3. Keep doing that and construct the sequence $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ of monotonically increasing integers. By assumption, this sequence belongs to $\mathcal{J}$ and precedes any sequence v^k, $\forall k = 1, 2, \ldots$ □

Proposition 4.1. *Let $v, w \in \mathcal{J}$ and p be a positive integer. Then we have*

$$\mathrm{Ir}(v) \subset \mathrm{Ir}(v+p) \quad \textit{and} \quad \mathrm{Tr}(v) \subset \mathrm{Tr}(v+p). \tag{10}$$

If $\frac{v_{n+1}}{v_n} / \frac{v_n}{v_{n-1}} > \frac{p+1}{p}$ for every large n, we obtain

$$\mathrm{Ir}(v) \subset \mathrm{Ir}(pv) \quad \textit{and} \quad \mathrm{Tr}(v) \subset \mathrm{Tr}(pv), \tag{11}$$

$$\mathrm{Ir}(v) \subset \mathrm{Ir}(vw) \quad \textit{and} \quad \mathrm{Tr}(v) \subset \mathrm{Tr}(vw). \tag{12}$$

Proof. We only expose the proof of $\mathrm{Ir}(v)$ because the case of $\mathrm{Tr}(v)$ is similar. Let $v \in \mathcal{J}$, then $v+p=\{v_n+p\}_{n=1}^{\infty} \in \mathcal{J}$. In fact,

$$\frac{v_{n+1}}{v_n} > \frac{v_n}{v_{n-1}} \quad \Rightarrow \quad v_{n+1}v_{n-1} > v_n^2,$$

and for every large n

$$\frac{v_{n+1}+v_{n-1}}{v_n} > \frac{v_{n+1}}{v_n} > 2.$$

This implies

$$v_{n+1}v_{n-1} + p\left(v_{n+1}+v_{n-1}\right) + p^2 > v_n^2 + 2pv_n + p^2,$$

which gives

$$\frac{v_{n+1}+p}{v_n+p} > \frac{v_n+p}{v_{n-1}+p} \quad \Rightarrow \quad v+p \in \mathcal{J}.$$

Now, let $r=\{r_n\}_{n=1}^{\infty} \in \mathrm{Ir}(v)$, then

$$v * r(x) = \sum_{n=0}^{\infty} r_n x^{v_n}$$

is an irrational number for any rational $x \in (0,1)$. Therefore,

$$(v+p) * r(x) = x^p \sum_{n=0}^{\infty} r_n x^{v_n}$$

is also an irrational number for any rational $x \in (0,1)$ and then $r \in \mathrm{Ir}(v+p)$.

Let $v \in \mathcal{J}$ then $pv=\{pv_n\}_{n=1}^{\infty} \in \mathcal{J}$. Therefore,

$$(pv) * r(x) = \sum_{n=0}^{\infty} r_n \left(x^p\right)^{v_n}$$

is also an irrational number for any rational $x \in (0,1)$ and then $r \in \mathrm{Ir}(pv)$. To prove (12) first we should verify that for any $v, w \in \mathcal{J}$,

$$vw \in \mathcal{J}.$$

In fact, we have

$$\frac{v_{n+1}}{v_n} > \frac{v_n}{v_{n-1}}, \frac{w_{n+1}}{w_n} > \frac{w_n}{w_{n-1}} \quad \Rightarrow \quad \frac{v_{n+1}w_{n+1}}{v_n w_n} > \frac{v_n w_n}{v_{n-1}w_{n-1}} \quad \Rightarrow \quad vw \in \mathcal{J}.$$

Now, let $r=\{r_n\}_{n=1}^{\infty} \in \mathrm{Ir}(v)$, then

$$(vw) * r(x) = \sum_{n=0}^{\infty} r_n \left(x^{w_n}\right)^{v_n}$$

is an irrational number for any rational $x \in (0,1)$ and $r \in \mathrm{Ir}(vw)$. Note that $vw \geq v$. □

Remark 4.1. *Many questions about* $\mathrm{Ir}(v)$ *and* $\mathrm{Tr}(v)$ *remain unsolved:*

- *For example, are they countable sets?*
- *Given a subset* A *of* $\mathcal{L}$*, can we affirm that there exists* $v \in \mathcal{J}$ *such that* $\mathrm{Ir}(v) = A$ *or* $\mathrm{Tr}(v) = A$ *?*
- *If we have* $\mathrm{Ir}(v) \subset \mathrm{Ir}(u)$ *or* $\mathrm{Tr}(v) \subset \mathrm{Tr}(u)$*, this does not necessarily imply that* $v \leq u$*. In fact, a counterexample is given in (10) for* $u = v + p \ngeq v$*.*
- *Motivated by the results proven in (11), (12), Theorem 3.2 and Theorem 2.2, we formulate the following conjecture.*

Conjecture 4.1. *Let* $v, u \in \mathcal{J}$ *such that* $v \leq u$*. Then*

$$\mathrm{Ir}(v) \subset \mathrm{Ir}(u) \quad \textit{and} \quad \mathrm{Tr}(v) \subset \mathrm{Tr}(u).$$

References

[1] Badea, C. (1987). The irrationality of certain infinite series. *Glasgow Mathematical Journal*, 29, 221–228.

[2] Erdős, P. (1975). Some problems and results on the irrationality of the sum of infinite series. *Journal of Mathematical Sciences*, 10, 1–7.

[3] Hančl, J., & Rucki, P. (2005). The transcendence of certain infinite series. *Rocky Mountain Journal of Mathematics*, 35(2), 531–537.

[4] Hančl, J., & Tijdeman, R. (2004). On the irrationality of Cantor series. *Journal für die reine und angewandte Mathematik*, 571, 145–158.

[5] Mahler, K. (1976). *Lectures on Transcendental Numbers*. Series “Lecture Notes in Mathematics”, Vol. 546, Springer.

[6] Nishioka, K. (1996). *Mahler Functions and Transcendence*. Series “Lecture Notes in Mathematics”, Vol. 631, Springer, New York.

[7] Nyblom, M. A. (2000). A theorem on transcendence of infinite series. *Rocky Mountain Journal of Mathematics*, 30(3), 1111–1120.

[8] Oppenheim, A. (1954). Criteria for irrationality of certain classes of numbers. *The American Mathematical Monthly*, 61, 235–241.

[9] Roth, K. F. (1955). Rational approximations to algebraic numbers. *Mathematica*, 2(1), 1–20.

[10] Sándor, J. (1984). Some classes of irrational numbers. *Studia Universitatis Babeș-Bolyai Mathematica*, 29, 3–12.