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An analytical formula for Bell numbers

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Abstract: We present an analytic formula for Bell numbers through counting the number of uniform structures on a finite set.
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1 Introduction

The Bell number ϵ_n is defined as the number of partitions of a set of n elements. Eric Temple Bell [1] presented an explicit formula for the Bell numbers as follows:

$$\epsilon_n = \sum_{s=1}^{n} \frac{1}{(s-1)!} \left[\sum_{r=0}^{s-1} (-1)^r \binom{s-1}{r} (s-r)^{n-1} \right], \; n \geq 1. \tag{1}$$

André Weil [2] introduced the uniform structure to study the uniform continuity in the context of topological spaces. We first recall the definition of the uniform structure on a set X.

Definition 1.1 ([3]). *Let X be a non-empty set. A collection $\mathfrak{U}$ of subsets of $X \times X$ is said to be a uniform structure or diagonal uniformity on X if*

($U1$) $\Delta \subseteq E$, for all $E \in \mathfrak{U}$, where $\Delta = \{(x,x) : x \in X\}$.

($U2$) $E \cap F \in \mathfrak{U}$ whenever $E, F \in \mathfrak{U}$.

($U3$) If $E \in \mathfrak{U}$ and $E \subseteq F \subseteq X \times X$, then $F \in \mathfrak{U}$.

($U4$) $E^{-1} \in \mathfrak{U}$ whenever $E \in \mathfrak{U}$, where $E^{-1} = \{(x,y) : (y,x) \in E\}$.

($U5$) For each $E \in \mathfrak{U}$, there exists $F \in \mathfrak{U}$ such that $F \circ F \subseteq E$, where $F \circ F = \{(x,y) : (x,z),(z,y) \in F$, for some $z \in X\}$.

Throughout this paper, we fix X as a nonempty finite set. We first introduce some notations and phrases.

- For any set S, $|S|$ denotes the cardinality of S.
- Let $\mathcal{P}_n$ be the class of all partitions of the positive integer n. Every element of $\mathcal{P}_n$ can be considered as a function $P : \{1,2,\ldots,n\} \to \{0,1,2,\ldots,n\}$ which satisfies the condition $\sum_{i=1}^{n} P(i)i = n$. If P is the function representing a partition of n, then $P(i)$ counts the number of occurrences of i in the partition of n. In particular, if

$$n = \underbrace{n_1 + n_1 + \cdots + n_1}_{q_1\text{-times}} + \underbrace{n_2 + n_2 + \cdots + n_2}_{q_2\text{-times}} + \cdots + \underbrace{n_t + n_t + \cdots + n_t}_{q_t\text{-times}}$$

  is a partition of n, then $P(n_k) = q_k$, for $k = 1,2,\ldots,t$ and $P(i) = 0$ for $i \neq n_k$ for any k.
- By a partition of X, we mean a collection $\mathcal{Q} = \{S_1, S_2, \ldots, S_\ell\}$ of nonempty subsets of X such that $\bigcup_{i=1}^{\ell} S_i = X$ and $S_i \cap S_j = \varnothing$ for $i \neq j$.
- For a given $P \in \mathcal{P}_n$, let

$$\{n_1, n_2, \ldots, n_t\} = \{i : P(i) \neq 0\} \text{ and } P(n_k) = q_k, \forall k = 1,2,\ldots,t. \quad (2)$$

  Let $\ell = \sum_{i=1}^{n} P(i)$ and $\mathscr{E}_P$ denote the class of all partitions $\{S_1, S_2, \ldots, S_\ell\}$ of the set X such that

$$|S_i| = \begin{cases} n_1, & \text{if } 1 \leq i \leq q_1 \\ n_2, & \text{if } q_1 + 1 \leq i \leq q_1 + q_2 \\ \vdots & \\ n_t, & \text{if } \sum_{k=1}^{t-1} q_k + 1 \leq i \leq \sum_{k=1}^{t} q_k. \end{cases}$$

  Note that $\ell = \sum_{i=1}^{n} P(i) = \sum_{k=1}^{t} P(n_k) = \sum_{k=1}^{t} q_k$.
- For given positive integers n and r such that $r \leq n$, $\binom{n}{r}$ denotes the number of ways of choosing r objects from n objects.

In this paper, we count the number of uniform structures on a set of n elements and find the cardinality of each uniform structure on a finite set. Finally, we present yet another formula for the Bell number ϵ_n, which is written in terms of the partitions of n.

2 Characterization theorem

Theorem 2.1. *Let X be a finite set and $\mathfrak{U}$ be a collection of subsets of $X \times X$. Then $\mathfrak{U}$ is a uniform structure on X if and only if*

$$\mathfrak{U} = \{F \subseteq X \times X : F \supseteq A\} \text{ where } A = \bigcup_{i=1}^{k} (S_i \times S_i)$$

for some partition $\mathcal{Q} = \{S_1, S_2, \ldots, S_k\}$ of X.

Proof. Assume that $\mathfrak{U}$ is a uniform structure and let $A = \bigcap_{F \in \mathfrak{U}} F$. Since X is finite, using the axiom $(U2)$, we get $A \in \mathfrak{U}$ and clearly $F \supseteq A$, for all $F \in \mathfrak{U}$. We observe that A is an equivalence relation on X. Indeed, the axioms $(U1)$, $(U4)$, and $(U5)$ of the uniform structure imply

$$\Delta \subseteq A, \; A^{-1} = A, \text{ and } A \circ A \subseteq A,$$

respectively. The equivalence relation A gives a partition of X, say $\{S_1, S_2, \ldots, S_k\}$. We observe that $A = \bigcup_{i=1}^{k} (S_i \times S_i)$ because $(x, y) \in A$ if and only if $x, y \in S_{i_0}$ for some $i_0 \in \{1, 2, \ldots k\}$ if and only if $(x, y) \in S_{i_0} \times S_{i_0}$ if and only if $(x, y) \in \bigcup_{i=1}^{k} (S_i \times S_i)$. Clearly, by the definition of A, we get $\mathfrak{U} = \{F \subseteq X \times X : F \supseteq A\}$.

Conversely, let $\{S_1, S_2, \ldots, S_k\}$ be a partition of X, $A = \bigcup_{i=1}^{k} (S_i \times S_i)$ and $\mathfrak{U} = \{F \subseteq X \times X : F \supseteq A\}$. We shall show that $\mathfrak{U}$ is a uniform structure.

$(U1)$ Given $x \in X$, there exists $i_x \in \{1, 2, \ldots, k\}$ such that $x \in S_{i_x}$. Therefore, $(x, x) \in S_{i_x} \times S_{i_x} \subseteq A \subseteq E$, for all $x \in X$ and for all $E \in \mathfrak{U}$. Hence $\Delta \subseteq E$, for all $E \in \mathfrak{U}$.

$(U2)$ If $E, F \in \mathfrak{U}$, then $E \supseteq A$ and $F \supseteq A$ and hence $E \cap F \supseteq A$. Thus $E \cap F \in \mathfrak{U}$.

$(U3)$ If $E \in \mathfrak{U}$ and $F \supseteq E$, then $F \supseteq E \supseteq A$ and hence $F \in \mathfrak{U}$.

$(U4)$ Since $(S_i \times S_i)^{-1} = (S_i \times S_i)$, we have $A^{-1} = A$. Therefore, for a given $E \in \mathfrak{U}$, we have $A \subseteq E$ and hence $A = A^{-1} \subseteq E^{-1}$. Thus $E^{-1} \in \mathfrak{U}$.

$(U5)$ We first show that $A \circ A \subseteq A$. If $(x, z) \in A \circ A$, then there exists $y \in X$ such that $(x, y), (y, z) \in A$. Then, there exist $i, j \in \{1, 2, \ldots, k\}$ such that $(x, y) \in S_i \times S_i$ and $(y, z) \in S_j \times S_j$. Therefore, $y \in S_i \cap S_j$. Since $\{S_1, S_2, \ldots, S_k\}$ is a partition of X, we have $i = j$ and hence $x, z \in S_i$, which implies that $(x, z) \in S_i \times S_i \subseteq A$. Thus, we get $A \circ A \subseteq A$. Therefore, for each $E \in \mathfrak{U}$, we see that $A \circ A \subseteq A \subseteq E$.

Thus $\mathfrak{U}$ is a uniform structure. □

Hereafter, for a given partition $\mathcal{Q}$ of X, $\mathfrak{U}_{\mathcal{Q}}$ denotes the unique uniform structure associated with $\mathcal{Q}$ as in the above theorem. That is, $\mathcal{Q}$ and $\mathfrak{U}_{\mathcal{Q}}$ satisfy the following condition:

$$\bigcup_{S \in \mathcal{Q}} (S \times S) = \bigcap_{F \in \mathfrak{U}_{\mathcal{Q}}} F.$$

We say that a uniform structure $\mathfrak{U}$ on X is associated with $P \in \mathcal{P}_n$ if $\mathfrak{U} = \mathfrak{U}_{\mathcal{Q}}$, for some $\mathcal{Q} \in \mathscr{E}_P$.

Corollary 2.1. *The number of uniform structures on a set of n elements is the Bell number ϵ_n.*

Proof. Let X be a set with n elements. The map $\mathcal{Q} \mapsto \mathfrak{U}_{\mathcal{Q}}$ is a bijection between the class of all partitions on X and the class of all uniform structures on X. Hence the number of uniform structures on X is the number of partitions of X which is the Bell number ϵ_n. □

Since $\mathfrak{U} = \{F \subseteq X \times X : F \supseteq A\} = \{A \cup B : B \subseteq (X \times X) \setminus A\}$, we obtain the following corollary.

Corollary 2.2. *The cardinality of a uniform structure on a finite set X is $2^{|X|^2 - |A|}$, where $A \subseteq X \times X$ is such that $\mathfrak{U} = \{F \subseteq X \times X : F \supseteq A\}$.*

Corollary 2.3. *The cardinality of any uniform structure $\mathfrak{U}$ is 4^j for some $j \in \mathbb{N}$.*

Proof. Let $\mathfrak{U}$ be a uniform structure on X. Then, $\mathfrak{U} = \mathfrak{U}_{\mathcal{Q}}$, for some partition $\mathcal{Q}$ of X. If $\mathcal{Q} = \{S_1, S_2, \ldots, S_\ell\}$ and $A = \bigcup_{i=1}^{k} (S_i \times S_i)$, then $\mathfrak{U} = \{F \subset X \times X : F \supseteq A\}$. Therefore,

$$|X| = \sum_{i=1}^{k} |S_i|, \ |A| = \sum_{i=1}^{k} |S_i|^2, \ \text{and } |\mathfrak{U}| = 2^{|X|^2 - |A|}.$$

Now we show that $|X|^2 - |A|$ is even. If $|S_i|$ is even for each $i = 1, 2, \ldots, k$, then $|X|$ is even and $|A|$ is even. Thus, $|X|^2 - |A|$ is even. Suppose that there is some S_i such that $|S_i|$ is odd. Without loss of generality, we assume that

$$|S_i| \text{ is odd for } 1 \leq i \leq \ell \text{ and } |S_i| \text{ is even for } \ell + 1 \leq i \leq k.$$

If $|X|$ is even, then ℓ is even and hence $|A|$ is even. Thus, $|X|^2 - |A|$ is even. If $|X|$ is odd, then ℓ is odd and hence $|A|$ is odd. Thus, $|X|^2 - |A|$ is even. Therefore, $|X|^2 - |A| = 2j$, for some $j \in \mathbb{N}$. So, $|\mathfrak{U}| = 2^{2j} = 4^j$. □

3 Cardinality of uniform structures

We first find the number of partitions of a set X with n elements, associated with a given partition P of n. We also find the number of elements in each uniform structure on X, which are associated with P.

Theorem 3.1. *For a given $P \in \mathcal{P}_n$, let t, n_k's, and q_k's be as in* (2)*, and $m = n^2 - \sum_{k=1}^{t} q_k {n_k}^2$. Then,*

(a) $$|\mathscr{E}_P| = \prod_{k=1}^{t} \left(\frac{\prod_{j=0}^{q_k - 1} \binom{n - \left(\sum_{i=1}^{k-1} q_i n_i\right) - j n_k}{n_k}}{q_k!} \right).$$

(b) $|\mathfrak{U}_{\mathcal{Q}}| = 2^m$, *for all $\mathcal{Q} \in \mathscr{E}_P$.*

Proof. If t, n_k's, and q_k's are depending on P as mentioned in (2), we can write

$$n = \underbrace{n_1 + n_1 + \cdots + n_1}_{q_1\text{-times}} + \underbrace{n_2 + n_2 + \cdots + n_2}_{q_2\text{-times}} + \cdots + \underbrace{n_t + n_t + \cdots + n_t}_{q_t\text{-times}}.$$

where $n_i \neq n_j$ for $i \neq j$. To choose a partition of X in $\mathscr{E}_P$, we first choose a set S_1 with $|S_1| = n_1$ from the n elements of X, then choose a set S_2 with $|S_2| = n_1$ from the remaining $n-n_1$ elements of X, then choose a set S_3 with $|S_3| = n_1$ from the remaining $n - 2n_1$ elements of X, and so on. We repeat this procedure for q_1 times. In the q_1-st time, we choose a set S_{q_1} with $|S_{q_1}| = n_1$ from the remaining $n - (q_1 - 1)n_1$ elements of X. Thus there are

$$\binom{n}{n_1}\binom{n-n_1}{n_1}\binom{n-2n_1}{n_1}\cdots\binom{n-(q_1-1)n_1}{n_1}$$

such selections. Since each permutation on $\{S_1, S_2, \ldots, S_{q_1}\}$ will contribute a repetition in the counting, there are

$$\frac{\binom{n}{n_1}\binom{n-n_1}{n_1}\binom{n-2n_1}{n_1}\cdots\binom{n-(q_1-1)n_1}{n_1}}{q_1!} = \frac{\prod_{j=0}^{q_1-1}\binom{n-jn_1}{n_1}}{q_1!}$$

selections of the pairwise disjoint subsets $S_1, S_2, \ldots, S_{q_1}$ of X with $|S_i| = n_1$, for all $1 \leq i \leq q_1$.

Similarly, for each $k \in \{2, \ldots, t\}$, if $\mu_k = \sum_{i=1}^{k-1} q_i$ and $\sigma_k = \left|\bigcup_{i=1}^{\mu_k} S_i\right| = \sum_{i=1}^{k-1} q_i n_i$, then there are

$$\frac{\binom{n-\sigma_k}{n_k}\binom{n-\sigma_k-n_k}{n_k}\binom{n-\sigma_k-2n_k}{n_k}\cdots\binom{n-\sigma_k-(q_k-1)n_k}{n_k}}{q_k!} = \frac{\prod_{j=0}^{q_k-1}\binom{n-\sigma_k-jn_k}{n_k}}{q_k!}$$

selections of pairwise disjoint subsets $S_{\mu_k+1}, S_{\mu_k+2}, \ldots, S_{\mu_k+q_k}$ of $X \setminus \bigcup_{i=1}^{\mu_k} S_i$ with $|S_i| = n_k$, for all $\mu_k + 1 \leq i \leq \mu_k + q_k$. At the end of this process, we get that

$$\prod_{k=1}^{t}\left(\frac{\prod_{j=0}^{q_k-1}\binom{n-\left(\sum_{i=1}^{k-1} q_i n_i\right)-jn_k}{n_k}}{q_k!}\right)$$

number of selections of pairwise disjoint subsets $S_1, S_2, \ldots, S_\ell$ of X such that

$$|S_i| = \begin{cases} n_1, & \text{if } 1 \leq i \leq q_1 \\ n_2, & \text{if } q_1 + 1 \leq i \leq q_1 + q_2 \\ \vdots & \\ n_t, & \text{if } \sum_{k=1}^{t-1} q_k + 1 \leq i \leq \sum_{k=1}^{t} q_k = \ell, \end{cases}. \tag{3}$$

This completes the proof of (a).

To prove (b), let $\mathcal{Q} = \{S_1, S_2, \ldots, S_\ell\} \in \mathscr{E}_P$ be arbitrary and let S_i's satisfy (3). If $A = \bigcup_{i=1}^{\ell}(S_i \times S_i)$, then $|A| = \sum_{i=1}^{\ell} |S_i|^2 = \sum_{k=1}^{t} q_k {n_k}^2$. Using Corollary 2.2, we get that $|\mathfrak{U}_{\mathcal{Q}}| = 2^{n^2-|A|} = 2^m$. $\square$

The following corollary is a particular case of the previous theorem.

Corollary 3.1. *If P is a partition of n into distinct parts $n_1, n_2, \dots, n_t$, then*

(a) $|\mathscr{E}_P| = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \binom{n}{n_1,n_2,\dots,n_k}$*, where* $\binom{n}{n_1,n_2,\dots,n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$.

(b) $|\mathfrak{U}_{\mathcal{Q}}| = 2^m,$ *for all* $\mathcal{Q} \in \mathscr{E}_P$ *where* $m = n^2 - \left(\binom{n}{n_1}^2 + \binom{n-n_1}{n_2}^2 + \binom{n-n_1-n_2}{n_3}^2 + \cdots + \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}^2\right).$

The corollary follows at once, since the given partition is same as $P \in \mathcal{P}_n$ such that $\{n_1, n_2, \dots, n_t\} = \{i : P(i) \neq 0\}$, $P(n_k) = 1,$ for all $k = 1, 2, \dots, t$.

From what we have developed so far, we get the following analytical formula for the Bell numbers.

Corollary 3.2. *The Bell number ϵ_n is given by*

$$\epsilon_n = \sum_{P \in \mathcal{P}_n} \left(\prod_{k=1}^{t} \left(\frac{\prod_{j=0}^{q_k-1} \binom{n - \left(\sum_{i=1}^{k-1} q_i n_i\right) - j n_k}{n_k}}{q_k!} \right) \right)$$

where t, n_k's, and q_k's are depending on P as mentioned in (2).

4 Conclusion

We find the number of uniform structures on a given set of n elements and we obtain a novel expression for the Bell numbers.

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