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Monophonic domination polynomial of the path graph

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Abstract: Let $MD(G, i)$ be the family of monophonic dominating sets of a graph G with cardinality i and let $\mathrm{md}(G, i) = |MD(G, i)|$. Then the monophonic domination polynomial $MD(G,x)$ of G is defined as $MD(G,x) = \sum_{i=\gamma_m(G)}^{p} \mathrm{md}(G,i)x^i$, where $\gamma_m(G)$ is the monophonic domination number of G. In this paper we have determined the family of monophonic dominating sets of the path graph P_n with cardinality i. Also, the monophonic domination polynomial of the path graph is calculated and some properties of the coefficient $\mathrm{md}(P_n, i)$ is discussed.
Keywords: Monophonic set, Monophonic dominating set, Monophonic domination polynomial, Path graph.
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1 Introduction

The graph G considered in this paper is finite, simple, undirected and connected with vertex set $V(G)$ and edge set $E(G)$, respectively. The order and size of G are denoted by n and m, respectively. For basic graph theoretic definitions we refer to [6,7]. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest $u - v$ path in G. The *neighborhood* of a vertex v denoted by $N(v)$ is the set of all vertices adjacent to v. For any subset S of $V(G)$, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with the vertex set S. A vertex v is said to be an *extreme vertex* if the subgraph $\langle N(v) \rangle$ is complete. $F \subseteq V(G)$ is said to be a dominating set of G if every vertex in $V(G) - F$ is adjacent to at least one vertex in F. The least order of the dominating sets of G is said to be the domination number of G and is denoted by $\gamma(G)$. The monophonic number of a graph was studied by Pelayo et al. in [8, 11]. Any chordless path connecting the vertices u and v is called a $u - v$ m-path. The *monophonic closure* of a subset S of $V(G)$ is given by $J_G[S] = \bigcup_{u,v \in S} J_G[u, v]$, where $J_G[u, v]$ is the set containing u and v and all vertices lying on some $u - v$ m-path. If $J_G[S] = V(G)$, then S is said to be a *monophonic set* in G. A monophonic set in G of least order is called a minimum monophonic set of G. The order of the minimum monophonic set of G is called the *monophonic number* of G and is denoted by $m(G)$. Monophonic domination number of a graph is studied in [9, 12]. $M \subseteq V(G)$ is said to be a monophonic dominating set if it is both monophonic and dominating. The minimum cardinality of a monophonic dominating set of G is the monophonic domination number and is denoted by $\gamma_m(G)$. The domination polynomial of a graph was introduced by Arocha and Llano in [4] and was further studied by Saeid Alikhani et al. in [1–3]. The monophonic polynomial of a graph was introduced and studied in [10]. The monophonic domination polynomial of a graph was introduced and studied by P. Arul Paul Sudhahar et al. in [5].

The notation $\lceil x \rceil$ represents the smallest integer greater than or equal to x and $|F|$ denotes the cardinality of F. Let us define the vertices of P_n by $V(P_n) = [n]$, where $[n] = \{1, 2, \ldots, n\}$.

In the second section, the family of monophonic dominating sets of the path P_n is constructed. The third section determines the number of monophonic dominating sets of the path graph P_n of cardinality i by using the recurrence relation connecting $\mathrm{md}(P_{n-1}, i - 1)$, $\mathrm{md}(P_{n-2}, i - 1)$ and $\mathrm{md}(P_{n-3}, i - 1)$. Also, the monophonic domination polynomial of the path P_n is determined by using the above recurrence relation.

2 Monophonic dominating set of the path P_n

Let $MD(P_n, i)$ denote the collection of all monophonic dominating sets of P_n with cardinality i and $\mathrm{md}(P_n, i) = |MD(P_n, i)|$.

Lemma 2.1. *For the path graph* P_n, $\gamma_m(P_n) = 2 + \left\lceil \frac{n-4}{3} \right\rceil$.

Lemma 2.2. $MD(P_n, l) = \varnothing$ *if and only if* $l > n$ *or* $l < 2 + \left\lceil \frac{n-4}{3} \right\rceil$.

Lemma 2.3. *If* $Y \in MD(P_{n-1}, i - 1)$ *or* $MD(P_{n-2}, i - 1)$ *or* $MD(P_{n-3}, i - 1)$, *then* $Y \cup \{n\} \in MD(P_n, i)$.

Proof. First let us assume that $Y \in MD(P_{n-1}, i-1)$. The vertices labelled 1 and $n-1$ will belong to Y. It can be clearly seen that $Y \cup \{n\}$ is a monophonic dominating set of P_n. Now, let us assume that $Y \in MD(P_{n-2}, i-1)$. Vertices labelled 1 and $n-2$ will belong to Y. Clearly Y is a dominating set of the path P_{n-1}. $Y \cup \{n\}$ forms the monophonic dominating set of P_n. Now, let us consider the case when $Y \in MD(P_{n-3}, i-1)$. Y contains the vertices labelled 1 and $n-3$. The vertex labelled $n-2$ can be dominated by the vertex $n-3$ in P_n and the vertex labelled $n-1$ can be dominated by the vertex n. Hence $Y \cup \{n\}$ forms the dominating set of P_n. Since 1 and n belongs to $Y \cup \{n\}$, it forms the monophonic set of P_n. Hence $Y \cup \{n\} \in MD(P_n, i)$. □

Theorem 2.1.

(i) If $MD(P_{n-1}, i-1) = MD(P_{n-3}, i-1) = \varnothing$*, then* $MD(P_{n-2}, i-1) = \varnothing$.

(ii) If $MD(P_{n-1}, i-1) \neq \varnothing$ *and* $MD(P_{n-3}, i-1) \neq \varnothing$*, then* $MD(P_{n-2}, i-1) \neq \varnothing$.

(iii) If $MD(P_{n-1}, i-1) = MD(P_{n-3}, i-1) = MD(P_{n-2}, i-1) = \varnothing$*, then* $MD(P_n, i) = \varnothing$.

Proof.

(i) $MD(P_{n-1}, i-1) = MD(P_{n-3}, i-1) = \varnothing$
$\Rightarrow i-1 > n-1$ or $i-1 < 2 + \left\lceil \frac{n-7}{3} \right\rceil$. Hence $i-1 > n-2$ or $i-1 < 2 + \left\lceil \frac{n-6}{3} \right\rceil$. Therefore $MD(P_{n-2}, i-1) = \varnothing$.

(ii) $MD(P_{n-1}, i-1) \neq \varnothing$ and $MD(P_{n-3}, i-1) \neq \varnothing$
$\Rightarrow 2 + \left\lceil \frac{n-5}{3} \right\rceil \leq i-1 \leq n-1$ and $2 + \left\lceil \frac{n-7}{3} \right\rceil \leq i-1 \leq n-3$. Hence $2 + \left\lceil \frac{n-6}{3} \right\rceil \leq i-1 \leq n-2$, which implies that $MD(P_{n-2}, i-1) \neq \varnothing$.

(iii) $MD(P_{n-1}, i-1) = MD(P_{n-3}, i-1) = MD(P_{n-2}, i-1) = \varnothing$
$\Rightarrow i-1 > n-1$ or $i-1 < 2 + \left\lceil \frac{n-7}{3} \right\rceil$. If $i-1 > n-1$, then $MD(P_n, i) = \varnothing$. If $i-1 < 2 + \left\lceil \frac{n-7}{3} \right\rceil$ then $i < 2 + \left\lceil \frac{n-4}{3} \right\rceil$. Thus in both cases $MD(P_n, i) = \varnothing$. □

Now, we are on the way to find the recurrence relation between $MD(P_n, i)$ and the monophonic dominating sets $MD(P_{n-1}, i-1)$, $MD(P_{n-2}, i-1)$ and $MD(P_{n-3}, i-1)$. First, let us find the nature of $MD(P_n, i)$ to be empty or not; depending on whether the family of monophonic dominating sets $MD(P_{n-1}, i-1)$, $MD(P_{n-2}, i-1)$ and $MD(P_{n-3}, i-1)$ is empty or not. Hence, we get eight combinations for the family of monophonic dominating sets $MD(P_{n-1}, i-1)$, $MD(P_{n-2}, i-1)$ and $MD(P_{n-3}, i-1)$ to be empty or not. The combination $MD(P_{n-1}, i-1) = MD(P_{n-2}, i-1) = MD(P_{n-3}, i-1) = \varnothing$ is not considered, as it implies $MD(P_n, i) = \varnothing$. Since $MD(P_{n-3}, i-1) \neq \varnothing$, $MD(P_{n-1}, i-1) \neq \varnothing$ implies $MD(P_{n-2}, i-1) \neq \varnothing$ (by Theorem 2.1(ii)), the family of monophonic dominating sets that come under this category is same as that of case (v) in Theorem 2.2. Hence we have neglected the above case in Theorem 2.2. Also the combination $MD(P_{n-1}, i-1) = MD(P_{n-3}, i-1) = \varnothing$ implies $MD(P_n, i) = \varnothing$ [by Theorem 2.1(i)]. Hence it is not considered in Theorem 2.2.

Theorem 2.2. *If $MD(P_n, i) \neq \varnothing$, then*

(i) $MD(P_{n-1}, i-1) = \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ *and* $MD(P_{n-3}, i-1) \neq \varnothing$ *if and only if* $n = 3k$ *and* $i = k+1$ *for some* $k \in \mathbb{N}$.

(ii) $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ *and* $MD(P_{n-3}, i-1) = \varnothing$ *if and only if* $i = n-1$.

(iii) $MD(P_{n-1}, i-1) = \varnothing$, $MD(P_{n-2}, i-1) = \varnothing$ *and* $MD(P_{n-3}, i-1) \neq \varnothing$ *if and only if* $n = 3k+1$ *and* $i = k+1$ *for some* $k \in \mathbb{N}$.

(iv) $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) = \varnothing$ *and* $MD(P_{n-3}, i-1) = \varnothing$ *if and only if* $i = n$.

(v) $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ *and* $MD(P_{n-3}, i-1) \neq \varnothing$ *if and only if* $3 + \left\lceil \frac{n-5}{3} \right\rceil \leq i \leq n-2$.

Proof.

(i) Since $MD(P_{n-1}, i-1) = \varnothing$, by Lemma 2.2 $i-1 > n-1$ or $i-1 < 2 + \left\lceil \frac{n-5}{3} \right\rceil$. If $i-1 > n-1$, then $i > n$ and hence by Lemma 2.2, $MD(P_n, i) = \varnothing$. Therefore $i-1 < 2 + \left\lceil \frac{n-5}{3} \right\rceil$. Also, Since $MD(P_{n-2}, i-1) \neq \varnothing$ we have $i-1 \geq 2 + \left\lceil \frac{n-6}{3} \right\rceil$. Hence $2 + \left\lceil \frac{n-6}{3} \right\rceil \leq i-1 < 2 + \left\lceil \frac{n-5}{3} \right\rceil$. Therefore $n = 3k$ and $i = k+1$ for some $k \in \mathbb{N}$.
Conversely, let $n = 3k$ and $i = k+1$ for some $k \in \mathbb{N}$. It follows that $MD(P_{n-1}, i-1) = \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ and $MD(P_{n-3}, i-1) \neq \varnothing$.

(ii) Since $MD(P_{n-1}, i-1) \neq \varnothing$ and $MD(P_{n-2}, i-1) \neq \varnothing$, by Lemma 2.2 we have $2 + \left\lceil \frac{n-5}{3} \right\rceil \leq i-1 \leq n-2$. Since $MD(P_{n-3}, i-1) = \varnothing$, by Lemma 2.2 we have $i-1 > n-3$ or $i-1 < 2 + \left\lceil \frac{n-7}{3} \right\rceil$. If $i-1 < 2 + \left\lceil \frac{n-7}{3} \right\rceil$, then $MD(P_{n-2}, i-1) = \varnothing$, which is a contradiction. Hence $i-1 > n-3$. Thus $i = n-1$ or $i = n$. If $i = n$, then $MD(P_{n-2}, i-1) = \varnothing$. Hence $i = n-1$.
Conversely, let us assume $i = n-1$. By Lemma 2.2 we have $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ and $MD(P_{n-3}, i-1) = \varnothing$.

(iii) Since $MD(P_{n-1}, i-1) = MD(P_{n-2}, i-1) = \varnothing$, we have $i-1 > n-1$ or $i-1 < 2 + \left\lceil \frac{n-6}{3} \right\rceil$. If $i-1 > n-1$, then $MD(P_n, i)$ is empty. Hence $i-1 < 2 + \left\lceil \frac{n-6}{3} \right\rceil$. Also, $MD(P_{n-3}, i-1) \neq \varnothing$ by Lemma 2.2, $2 + \left\lceil \frac{n-7}{3} \right\rceil \leq i-1 \leq n-3$. Thus $2 + \left\lceil \frac{n-7}{3} \right\rceil \leq i-1 < \left\lceil \frac{n-6}{3} \right\rceil$. Therefore, $n = 3k+1$ and $i = k+1$ for some $k \in \mathbb{N}$.
Conversely, let $n = 3k+1$, $i = k+1$ for some $k \in \mathbb{N}$. Then by Lemma 2.2 $MD(P_{n-1}, i-1) = MD(P_{n-2}, i-1) = \varnothing$ and $MD(P_{n-3}, i-1) \neq \varnothing$.

(iv) Since $MD(P_{n-1}, i-1) \neq \varnothing$ by Lemma 2.2, $2 + \left\lceil \frac{n-5}{3} \right\rceil \leq i-1 \leq n-1$. Since $MD(P_{n-2}, i-1) = \varnothing$ and $MD(P_{n-3}, i-1) = \varnothing$ by Lemma 2.2, $i-1 > n-3$ or $i-1 < 2 + \left\lceil \frac{n-6}{3} \right\rceil$. If $i-1 \geq 2 + \left\lceil \frac{n-5}{3} \right\rceil$, then $MD(P_{n-2}, i-1) \neq \varnothing$. Hence $i-1 \leq n-1$ and $i-1 > n-3$. This implies that $i = n-1, n$. When $i = n-1$, $MD(P_{n-2}, i-1) \neq \varnothing$, which is a contradiction. Hence $i = n$.
Conversely, Assume that $i = n$. By Lemma 2.2, $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) = MD(P_{n-3}, i-1) = \varnothing$.

(v) Since $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ and $MD(P_{n-3}, i-1) \neq \varnothing$, by Lemma 2.2, $2 + \left\lceil \frac{n-5}{3} \right\rceil \leq i-1 \leq n-1$, $2 + \left\lceil \frac{n-6}{3} \right\rceil \leq i-1 \leq n-2$ and $2 + \left\lceil \frac{n-7}{3} \right\rceil \leq i-1 \leq n-3$. Thus $2 + \left\lceil \frac{n-5}{3} \right\rceil \leq i-1 \leq n-3$. Therefore, $3 + \left\lceil \frac{n-5}{3} \right\rceil \leq i \leq n-2$.
Conversely, assume that $3 + \left\lceil \frac{n-5}{3} \right\rceil \leq i \leq n-2$. Then by Lemma 2.2, $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ and $MD(P_{n-3}, i-1) \neq \varnothing$. □

Theorem 2.3. *For every path P_n, $n \geq 5$,*

(i) If $MD(P_{n-1}, i-1) = \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ and $MD(P_{n-3}, i-1) \neq \varnothing$, then $MD(P_n, i) = [A \cup \{3k\}/A \in MD(P_{3k-3}, k)] \cup [B \cup \{3k\}/B \in MD(P_{3k-2}, k)]$.

(ii) If $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ and $MD(P_{n-3}, i-1) = \varnothing$, then $MD(P_n, i) = \{[n] - \{x\}/x \in [n] - \{1, n\}\}$.

(iii) If $MD(P_{n-1}, i-1) = \varnothing$, $MD(P_{n-2}, i-1) = \varnothing$ and $MD(P_{n-3}, i-1) \neq \varnothing$, then $MD(P_n, i) = \{1, 4, 7, \ldots, 3k-2, 3k+1\}$.

(iv) If $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) = \varnothing$ and $MD(P_{n-3}, i-1) = \varnothing$, then $MD(P_n, i) = \{[n]\}$.

(v) If $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ and $MD(P_{n-3}, i-1) \neq \varnothing$, then $MD(P_n, i) = \{A_1 \cup \{n\}/A_1 \in MD(P_{n-1}, i-1)\} \cup \{A_2 \cup \{n\}/A_2 \in MD(P_{n-2}, i-1)\} \cup \{A_3 \cup \{n\}/A_3 \in MD(P_{n-3}, i-1)\}$.

Proof.

(i) Let $MD(P_{n-1}, i-1) = \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ and $MD(P_{n-3}, i-1) \neq \varnothing$. By Theorem 2.2(i), we have $n = 3k$ and $i = k+1$ for some $k \in \mathbb{N}$. Let $A = \{1, 4, 7, \ldots, 3k-5, 3k-3\} \in MD(P_{3k-3}, k)$. Clearly, $A \cup \{3k\} \in MD(P_{3k}, k+1)$. Similarly, if $B \in MD(P_{3k-2}, k)$ then $B \cup \{3k\} \in MD(P_{3k}, k+1)$. Hence

$$[A \cup \{3k\}/A \in MD(P_{3k-3}, k)] \cup [B \cup \{3k\}/B \in MD(P_{3k-2}, k)] \subseteq MD(P_{3k}, k+1).$$

Now, let $Y \in MD(P_{3k}, k+1)$. Then the vertices labelled 1 and $3k$ must belong to $MD(P_{3k}, k+1)$. If the vertex $3k-3$ belongs to Y, then $Y = \{A \cup \{3k\}/A \in MD(P_{3k-3}, k)\}$.

Similarly, if the vertex $3k-2$ belongs to Y, then $Y = \{B \cup \{3k\}/B \in MD(P_{3k-2}, k)\}$. Hence $MD(P_{3k}, k+1) \subseteq [A \cup \{3k\}/A \in MD(P_{3k-3}, k)] \cup [B \cup \{3k\}/B \in MD(P_{3k-2}, k)]$.

(ii) $MD(P_{n-1}, i–1) \neq \varnothing$, $MD(P_{n-2}, i–1) \neq \varnothing$ and $MD(P_{n-3}, i–1) = \varnothing$. By Theorem 2.2(ii), we have $i = n-1$. Thus, $MD(P_n, i) = MD(P_n, n-1) = \{[n] - \{x\}/x \in [n] - \{1, n\}\}$.

(iii) $MD(P_{n-1}, i–1) = \varnothing$, $MD(P_{n-2}, i–1) = \varnothing$ and $MD(P_{n-3}, i–1) \neq \varnothing$. By Theorem 2.2(iii), we have $n=3k+1$ and $i=k+1$ for some $k \in \mathbb{N}$. Therefore, $MD(P_n, i) = MD(P_{3k+1}, k+1)$ $= \{1, 4, 7, \ldots, 3k-2, 3k+1\}$.

(iv) $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) = \varnothing$ and $MD(P_{n-3}, i-1) = \varnothing$, by Theorem 2.2(iv) we have $i = n$. Hence $MD(P_n, i) = MD(P_n, n) = \{[n]\}$.

(v) $MD(P_{n-1}, i-1) \neq \varnothing$, $MD(P_{n-2}, i-1) \neq \varnothing$ and $MD(P_{n-3}, i-1) \neq \varnothing$. By Theorem 2.2(v), we have $3 + \left\lceil \frac{n-5}{3} \right\rceil \leq i \leq n-2$. Let $A_1 \in MD(P_{n-1}, i-1)$. Then the vertex labelled 1 and $n-1$ will belongs to $MD(P_{n-1}, i-1)$. Thus, $A_1 \cup \{n\} \in MD(P_n, i)$. Let $A_2 \in MD(P_{n-2}, i-1)$, Then by Lemma 2.3, we have $A_2 \cup \{n\} \in MD(P_n, i)$. Let $A_3 \in MD(P_{n-3}, i-1)$, Then by Lemma 2.3, we have $A_3 \cup \{n\} \in MD(P_n, i)$. Hence

$$\{A_1 \cup \{n\}/A_1 \in MD(P_{n-1}, i-1)\} \cup \{A_2 \cup \{n\}/A_2 \in MD(P_{n-2}, i-1)\}$$
$$\cup \{A_3 \cup \{n\}/A_3 \in MD(P_{n-3}, i-1)\} \subseteq MD(P_n, i).$$

Let $Y \in MD(P_n, i)$. The vertices labelled 1 and n will belong to $MD(P_n, i)$. The vertices labelled $n-3$ or $n-2$ or $n-1$ will belong to $MD(P_n, i)$. If $n-3 \in Y$, then $Y = A_1 \cup \{n\}$ for some $A_1 \in MD(P_{n-3}, i-1)$. If $n-2 \in Y$, then $Y = A_2 \cup \{n\}$ for some $A_2 \in MD(P_{n-2}, i-1)$. If $n-1 \in Y$, then $Y = A_3 \cup \{n\}$ for some $A_3 \in MD(P_{n-1}, i-1)$. Thus

$$\begin{aligned} MD(P_n, i) \subseteq &\{A_1 \cup \{n\}/A_1 \in MD(P_{n-1}, i-1)\} \\ &\cup \{A_2 \cup \{n\}/A_2 \in MD(P_{n-2}, i-1)\} \\ &\cup \{A_3 \cup \{n\}/A_3 \in MD(P_{n-3}, i-1)\}. \end{aligned}$$ □

The monophonic dominating sets of size i, $2+\left\lceil \frac{n-4}{3} \right\rceil \leq i \leq n$ is determined using Theorem 2.3 in Example 2.1 for the path graph P_8.

Example 2.1. *Consider the path P_8 with $V(P_8) = [8]$. By using Theorem 2.3, we have constructed $MD(P_8, i)$ for $4 \leq i \leq 8$.*
Since $MD(P_5, 3) = \{\{1, 3, 5\}, \{1, 2, 5\}, \{1, 4, 5\}\}$, $MD(P_6, 3) = \{\{1, 6, 3\}, \{1, 6, 4\}\}$ and $MD(P_7, 3) = \{1, 7, 4\}$ by Theorem 2.3 we have,

$$\begin{aligned} MD(P_8, 4) &= \{A_1 \cup \{8\}/A_1 \in MD(P_7, 3)\} \cup \{A_2 \cup \{8\}/A_2 \in MD(P_6, 3)\} \\ &\quad \cup \{A_3 \cup \{8\}/A_3 \in MD(P_5, 3)\} \\ &= \{\{1, 3, 5, 8\}, \{1, 2, 5, 8\}, \{1, 4, 5, 8\}, \{1, 6, 3, 8\}, \{1, 6, 4, 8\}, \{1, 7, 4, 8\}\}. \end{aligned}$$

Since $MD(P_5, 4) = \{\{1, 5, 2, 3\}, \{1, 5, 2, 4\}, \{1, 5, 3, 4\}\}$*,*
$MD(P_6, 4) = \{\{1, 6, 2, 4\}, \{1, 6, 2, 3\}, \{1, 6, 2, 5\}, \{1, 6, 3, 4\}, \{1, 6, 3, 5\}, \{1, 6, 4, 5\}\}$ *and*
$MD(P_7, 4) = \{\{1, 7, 4, 2\}, \{1, 7, 4, 3\}, \{1, 7, 4, 5\}, \{1, 7, 4, 6\}\}$ *by Theorem 2.3 we have*

$$\begin{aligned} MD(P_8, 5) = &\{A_1 \cup \{8\}/A_1 \in MD(P_7, 4)\} \cup \{A_2 \cup \{8\}/A_2 \in MD(P_6, 4)\} \\ &\cup \{A_3 \cup \{8\}/A_3 \in MD(P_5, 4)\}. \\ = &\{\{1, 5, 2, 3, 8\}, \{1, 5, 2, 4, 8\}, \{1, 5, 3, 4, 8\}, \{1, 6, 2, 4, 8\}, \{1, 6, 2, 3, 8\}, \\ &\{1, 6, 2, 5, 8\}, \{1, 6, 3, 4, 8\}, \{1, 6, 3, 5, 8\}\{1, 6, 4, 5, 8\}, \{1, 7, 4, 2, 8\}, \\ &\{1, 7, 4, 3, 8\}, \{1, 7, 4, 5, 8\}, \{1, 7, 4, 6, 8\}\}. \end{aligned}$$

Since $MD(P_5, 5) = \{1, 2, 3, 4, 5\}$*,*
$MD(P_6, 5) = \{\{1, 2, 3, 4, 6\}, \{1, 6, 5, 2, 3\}, \{1, 6, 5, 2, 4\}, \{1, 6, 5, 3, 4\}\}$ *and*
$MD(P_7, 5) = \{\{1, 7, 2, 3, 4\}, \{1, 7, 5, 2, 3\}, \{1, 7, 5, 2, 4\}, \{1, 7, 5, 3, 4\}, \{1, 7, 6, 2, 3\}, \{1, 7, 6, 2, 4\}, \{1, 7, 6, 2, 5\}, \{1, 7, 6, 3, 4\}, \{1, 7, 6, 3, 5\}\}$ *by Theorem 2.3 we have,*
$MD(P_8, 6) = \{\{1, 2, 3, 4, 5, 8\}, \{1, 2, 3, 4, 6, 8\}, \{1, 6, 5, 2, 3, 8\}, \{1, 6, 5, 2, 4, 8\}, \{1, 6, 5, 3, 4, 8\}, \{1, 7, 2, 3, 4, 8\}, \{1, 7, 5, 2, 3, 8\}, \{1, 7, 5, 2, 4, 8\}, \{1, 7, 5, 3, 4, 8\}, \{1, 7, 6, 2, 3, 8\}, \{1, 7, 6, 2, 4, 8\}, \{1, 7, 6, 2, 5, 8\}, \{1, 7, 6, 3, 4, 8\}, \{1, 7, 6, 3, 5, 8\}\}$.
Since $MD(P_5, 6) = \varnothing$*,* $MD(P_6, 6) = \{1, 2, 3, 4, 5, 6\}$ *and* $MD(P_7, 6) = \{\{1, 2, 3, 4, 5, 7\}, \{1, 7, 2, 3, 4, 6\}, \{1, 7, 2, 3, 5, 6\}, \{1, 7, 2, 4, 5, 6\}, \{1, 7, 3, 4, 5, 6\}\}$ *by Theorem 2.3 we have*

$$\begin{aligned} MD(P_8, 7) &= \{[8] - \{x\}/x \in [8] - \{1, 8\}\} \\ &= \{\{1, 2, 3, 4, 5, 6, 8\}, \{1, 2, 3, 4, 5, 7, 8\}, \{1, 7, 2, 3, 4, 6, 8\}, \\ &\quad \{1, 7, 2, 3, 5, 6, 8\}, \{1, 7, 2, 4, 5, 6, 8\}, \{1, 7, 3, 4, 5, 6, 8\}\} \end{aligned}$$

Since $MD(P_5, 7) = \varnothing$*,* $MD(P_6, 7) = \varnothing$ *and* $MD(P_7, 7) = \{1, 2, 3, 4, 5, 6, 7\}$ *by Theorem 2.3 we have,* $MD(P_8, 8) = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

3 Monophonic domination polynomial of the path graph P_n

Let $MD(P_n, x) = \sum_{i=\gamma_m(P_n)}^{n} \mathrm{md}(P_n, i)x^i$ be the monophonic domination polynomial of the path graph P_n.

Theorem 3.1. *For the path* P_n*,*

1. $\mathrm{md}(P_n, i) = \mathrm{md}(P_{n-1}, i-1) + \mathrm{md}(P_{n-2}, i-1) + \mathrm{md}(P_{n-3}, i-1)$.

2. For every path P_n *with* $n \geq 5$*,* $MD(P_n, x) = x\{MD(P_{n-3}, x) + MD(P_{n-2}, x) + MD(P_{n-1}, x)\}$ *with* $MD(P_2, x) = x^2$*,* $MD(P_3, x) = x^2 + x^3$ *and* $MD(P_4, x) = x^2 + 2x^3 + x^4$.

Proof.

1. The result holds from Theorem 2.3.

2. Monophonic dominating set of P_{n-3}, P_{n-2} or P_{n-1} together with the vertex $\{n\}$ forms the monophonic dominating set of P_n. Hence the monophonic dominating polynomial of P_n will be generated by $x\{MD(P_{n-3}, x) + MD(P_{n-2}, x) + MD(P_{n-1}, x)\}$. □

By using the Theorem 3.1, we have calculated the values of $\mathrm{md}(P_n, j)$ for $2 \leq n \leq 16$ and it is shown in Table 1.

Table 1. $\mathrm{md}(P_n, j)$

$n \setminus j$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
2	1														
3	1	1													
4	1	2	1												
5	0	3	3	1											
6	0	2	6	4	1										
7	0	1	7	10	5	1									
8	0	0	6	16	15	6	1								
9	0	0	3	19	30	21	7	1							
10	0	0	1	16	45	50	28	8	1						
11	0	0	0	10	51	90	77	36	9	1					
12	0	0	0	4	45	126	161	112	45	10	1				
13	0	0	0	1	30	141	266	266	156	55	11	1			
14	0	0	0	0	15	126	357	504	414	210	66	12	1		
15	0	0	0	0	5	90	393	784	882	615	275	78	13	1	
16	0	0	0	0	1	50	357	1016	1554	1452	880	352	91	14	1

Theorem 3.2. *The following properties holds for the coefficient* $\mathrm{md}(P_n, j)$ *of* $MD(P_n, x)$

1. *For every* $n \geq 2$, $\mathrm{md}(P_n, n) = 1$.
2. *For every* $n \geq 3$, $\mathrm{md}(P_n, n-1) = n - 2$.
3. *For every* $n \geq 4$, $\mathrm{md}(P_n, n-2) = \frac{(n-2)(n-3)}{2}$.
4. *For every* $n \geq 5$, $\mathrm{md}(P_n, n-3) = \frac{n(n-4)(n-5)}{6}$.
5. $\mathrm{md}(P_{3n+1}, n+1) = 1$.
6. $\mathrm{md}(P_{3n}, n+1) = n$.
7. $\mathrm{md}(P_{3n-1}, n+1) = \frac{n(n+1)}{2}$.

Proof.

1. By Theorem 2.3 (iv), we have $MD(P_n, n) = \{1, 2, 3, \ldots, n\}$. Therefore, $\mathrm{md}(P_n, n) = 1$.
2. By Theorem 2.3 (ii), we get $MD(P_n, n-1) = \{[n] - \{x\}/x \in [n] - \{1, n\}\}$. Hence $\mathrm{md}(P_n, n-1) = n - 2$.

3. We prove this result by induction on n. $\mathrm{md}(P_4, 2) = 1 = \frac{(4-2)(4-3)}{2}$. Thus the result is true for $n = 4$. Assume that the result is true for all natural numbers less than n. Now, let us prove that the result is true for n.

$$\begin{aligned}\mathrm{md}(P_n, n-2) &= \mathrm{md}(P_{n-1}, n-3) + \mathrm{md}(P_{n-2}, n-3) + \mathrm{md}(P_{n-3}, n-3)\\ &= \frac{(n-3)(n-4)}{2} + n - 4 + 1\\ &= \frac{(n-2)(n-3)}{2}.\end{aligned}$$

4. We prove this by induction on n. Since $\mathrm{md}(P_5, 2) = 0$, the result holds for $n = 5$. Now let us assume that the result is true for all natural numbers less than n. Now, we prove the result for n.

$$\begin{aligned}\mathrm{md}(P_n, n-3) &= \mathrm{md}(P_{n-1}, n-4) + \mathrm{md}(P_{n-2}, n-4) + \mathrm{md}(P_{n-3}, n-4)\\ &= \frac{(n-1)(n-5)(n-6)}{6} + \frac{(n-4)(n-5)}{2} + n - 5\\ &= \frac{n(n-4)(n-5)}{6}.\end{aligned}$$

5. By Theorem 2.3 (iii) we have $MD(P_{3k+1}, k+1) = \{1, 4, 7, \ldots, 3k-2, 3k+1\}$. Thus $\mathrm{md}(P_{3k+1}, k+1) = 1$.

6. Proof by induction on n. Since $\mathrm{md}(P_3, 2) = 1$, the result is true for $n = 1$. Assume that the result is true for all positive integers less than n. Now, we have to prove that the result holds for n. By Theorem 2.3 (i), we have

$$\begin{aligned}\mathrm{md}(P_{3n}, n+1) &= \mathrm{md}(P_{3n-2}, n) + \mathrm{md}(P_{3n-3}, n)\\ &= \mathrm{md}(P_{3(n-1)+1}, (n-1)+1) + \mathrm{md}(P_{3(n-1)}, (n-1)+1)\\ &= n.\end{aligned}$$

7. Proof by induction on n. Since $\mathrm{md}(P_2, 2) = 1$, the result is true for $n = 1$. Assume that the result is true for all positive integers less than n. Now, we have to prove that the result is true for n.

$$\begin{aligned}\mathrm{md}(P_{3n-1}, n+1) &= \mathrm{md}(P_{3n-2}, n) + \mathrm{md}(P_{3n-3}, n) + \mathrm{md}(P_{3n-4}, n)\\ &= \mathrm{md}(P_{3(n-1)+1}, (n-1)+1) + \mathrm{md}(P_{3(n-1)}, (n-1)+1)\\ &+ \mathrm{md}(P_{3(n-1)-1}, (n-1)+1)\\ &= \frac{n(n+1)}{2}.\end{aligned}$$ □

4 Conclusion

In this paper, we have determined the characterisation of monophonic dominating sets of the path graph P_n and have found the recurrence relation between the monophonic dominating sets of the path graph P_n, P_{n-1}, P_{n-2} and P_{n-3}. By using this we have determined the monophonic domination polynomial of the path graph P_n. In the near future, we can use this polynomial in various applications.

References

[1] Alikhani, S., & Peng, Y. H. (2009). *Introduction to domination polynomial of a graph*. Preprint. arXiv:0905.2251.

[2] Alikhani, S., & Peng, Y. H. (2009). Dominating sets and domination polynomials of paths. *International Journal of Mathematics and Mathematical Sciences*, 2009, Article ID 542040.

[3] Alikhani, S., & Peng, Y. H. (2009). *Dominating sets and domination polynomials of cycles*. Preprint. arXiv:0905.3268.

[4] Arocha, J., & Llano, B. (2000) Mean value for the matching and dominating polynomial. *Discussiones Mathematicae Graph Theory*, 20(1), 57–69.

[5] Arul Paul Sudhahar, P., & Jebi, W. (2024). On the monophonic and monophonic domination polynomial of a graph. *TWMS Journal of Applied and Engineering Mathematics*, 14(1), 197–205.

[6] Buckley, F., & Harary, F. (1990). *Distance in Graphs*. Addison Wesley, Redwood City.

[7] Harary, F. (1969). *Graph Theory*. Addison Wesley.

[8] Hernando, C., Jiang, T., Mora, M., Pelayo, I. M., & Seara, C. (2005). On the Steiner, geodetic and hull numbers of graphs. *Discrete Mathematics*, 293, 139–154.

[9] John, J., Arul Paul Sudhahar, P., & Stalin, D. (2019). On the (M, D) number of a graph. *Proyecciones (Antofagasta)*, 38(2), 255–266.

[10] Merlin Sugirtha, F. (2020). *Monophonic distance related parameters in graphs*. PhD thesis, Manonmaniam Sundaranar University, Tirunelveli.

[11] Pelayo, I. M. (2004). Comment on “The Steiner number of a graph” by G. Chartrand and P. Zhang [Discrete Mathematics 242 (2002) 41–54]. *Discrete Mathematics*, 280(1–3), 259–263.

[12] Sadiquali, A., & Arul Paul Sudhahar, P. (2017). Monophonic domination in special graph structures and related properties. *International Journal of Mathematical Analysis*, 11(22), 1089–1102.