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A note on Diophantine inequalities in function fields

Kathryn Wilson

Department of Mathematics, Kansas State University
138 Cardwell Hall, 1228 N. 17th St, Manhattan, KS 66506, USA
e-mail: kathry18@ksu.edu

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Abstract: We will discuss how the Bentkus–Götze–Freeman variant of the Davenport–Heilbronn circle method can be used to study $\mathbb{F}_q[t]$ solutions to inequalities of the form

$$\mathrm{ord}(\lambda_1 p_1^k + \cdots + \lambda_s p_s^k - \gamma) < \tau,$$

where constants $\lambda_1, \ldots, \lambda_s \in \mathbb{F}_q((1/t))$ satisfy certain conditions. This result is a generalization of the work done by Spencer in [11] to count the number of solutions to inequalities of the form

$$\mathrm{ord}(\lambda_1 p_1^k + \cdots + \lambda_s p_s^k) < \tau.$$

Keywords: Diophantine inequalities, Davenport–Heilbronn method, Hardy–Littlewood circle method, Function fields.
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1 Introduction and statement of the result

The Davenport–Heilbronn method was developed to study the number of integral solutions to Diophantine inequalities of the form

$$|\lambda_1 x_1^k + \cdots + \lambda_s x_s^k| < \tau, \tag{1}$$

where k and s are positive integers with $k > 1$, τ is a fixed positive real number, and $\lambda_1, \ldots, \lambda_s$ are non-zero real numbers not all in rational ratio. The number of solutions to (1) such that $\mathbf{x} \in [-P, P]^s \cap \mathbb{Z}^s$ is denoted by $N_0(P, \boldsymbol{\lambda})$. The condition that the λ_i $(1 \leq i \leq s)$ do not all have the same sign is added in the case that k is even, to guarantee the existence of a real solution. The method was constructed by Davenport and Heilbronn in [3]. In the aforementioned paper, they showed that for $s > 2^k$ and $(P_n)_{n=1}^{\infty}$ a sequence increasing to infinity that depends on $\boldsymbol{\lambda}$,

$$N_0(P_n, \boldsymbol{\lambda}) \gg P_n^{s-k}.$$

In 2001, Hsu in [7] used the Davenport–Heilbronn method in function fields to study the number of solutions of

$$\operatorname{ord}(\lambda_1 p_1^k + \cdots + \lambda_s p_s^k) < \tau,$$

where each p_i is a monic irreducible polynomial in $\mathbb{F}_q[t]$, the λ_i satisfy appropriate conditions, $k < \operatorname{char}(\mathbb{F}_q)$, and

$$s \geq \begin{cases} 2^k + 1, & \text{when } 2 \leq k \leq 11, \\ 2[2k^2 \log\, k + k^2 \log\, \log\, k + 2k^2 - 2k] + 1, & \text{when } k \geq 11. \end{cases}$$

This broadened in scope the previous work Hsu had done on counting solutions in the linear case with three monic irreducible polynomials (see [6]). In 2008, Spencer (see [11]) continued the study of such methods to count solutions of Diophantine inequalities in the function field setting by developing the Bentkus–Götze–Freeman variant of the Davenport–Heilbronn circle method for function fields.

The Bentkus–Götze–Freeman variant of the Davenport–Heilbronn method was developed over 50 years after the paper in which Davenport and Heilbronn first presented their method. The variant established an asymptotic lower bound and asymptotic formula for $N_0(P, \boldsymbol{\lambda})$, for all sufficiently large values of P (see [1, 4, 5, 13]). For

$$s \geq k(\log\, k + \log\, \log\, k + O(1)),$$

an asymptotic lower bound for $N_0(P, \boldsymbol{\lambda})$ was established, and for

$$s \geq k^2(\log\, k + \log\, \log\, k + 2 + o(1)),$$

an asymptotic formula for $N_0(P, \boldsymbol{\lambda})$ was established. Spencer used the Bentkus–Götze–Freeman variant of the Davenport–Heilbronn circle method for function fields to provide an asymptotic lower bound for all sufficiently large positive numbers P on the number of $\mathbb{F}_q[t]$ solutions to

$$\operatorname{ord}(\lambda_1 x_1^k + \cdots + \lambda_s x_s^k) < \tau$$

in $\mathbb{F}_q((1/t))$, for a real number τ and for s sufficiently large in terms of k and q. In this paper, we follow much of the same methodology while considering the inequality

$$\operatorname{ord}(\lambda_1 x_1^k + \cdots + \lambda_s x_s^k - \gamma) < \tau,$$

where γ is an element of $\mathbb{F}_q((1/t))$.

First, we establish some basic notation. Let p be the characteristic of $\mathbb{F}_q$. Let $\mathbb{A} = \mathbb{F}_q[t]$ denote the ring of polynomials over $\mathbb{F}_q$, let $\mathbb{K} = \mathbb{F}_q(t)$, and let $\mathbb{K}_\infty = \mathbb{F}_q((1/t))$ be the completion of $\mathbb{K}$ at the infinite place. If $\alpha \in \mathbb{K}_\infty \backslash \{0\}$, it can be expressed as

$$\alpha = \sum_{-\infty < i \le n} a_i t^i,$$

where $n \in \mathbb{Z}$, each $a_i \in \mathbb{F}_q$, and $a_n \neq 0$. In this case, we define $\operatorname{ord} \alpha = n$ and $\operatorname{lead} \alpha = a_n$, and if α is a polynomial, then $\operatorname{ord} \alpha = \deg \alpha$. We set $\operatorname{ord} 0 = -\infty$ and let $\operatorname{res} \alpha$ be the coefficient of t^{-1}.

There is a non-Archimedean valuation on $\mathbb{K}_\infty$ defined by $|\alpha| = \langle \alpha \rangle = q^{\operatorname{ord} \alpha}$. If u is a real number, let $\hat{u} = q^u$. We now rewrite

$$\operatorname{ord}(\lambda_1 x_1^k + \cdots + \lambda_s x_s^k - \gamma) < \tau$$

as

$$\langle \lambda_1 x_1^k + \cdots + \lambda_s x_s^k - \gamma \rangle < \hat{\tau}.$$

Here $k > 1$ and s are positive integers with $p \nmid k$. Let $\operatorname{Log} x = \max\{1, \log x\}$ for any positive real number x. Define $\psi(k) = \psi_q(k)$ by $\psi(k) = a_0 + a_1 + \cdots + a_n$, where k has base p expansion $k = a_0 + a_1 p + \cdots + a_n p^n$ with $0 \le a_i \le p-1$. We denote $B_q(k)$ by

$$B_q(k) = \begin{cases} 1, & \text{when } k \le 2^{\psi - 2}, \\ (1 - 2^{-\psi(k)})^{-1}, & \text{when } k > 2^{\psi - 2}. \end{cases}$$

Let

$$s_{q,k} = B_q(k) k (\operatorname{Log} k + \operatorname{Log}\operatorname{Log} k + 2 + B_q(k) \operatorname{Log}\operatorname{Log} k / \operatorname{Log} k).$$

We now state our generalization of [11, Theorem 1.1].

Theorem 1.1. *There exists a positive absolute constant C with the following property. Suppose that k and s are natural numbers with $k > 1$,*

$$s \ge s_{q,k} + Ck\sqrt{\operatorname{Log}\operatorname{Log} k}/\operatorname{Log} k,$$

and $\operatorname{char}(\mathbb{F}_q) \nmid k$. Let τ be some fixed integer, let γ be an element in $\mathbb{K}_\infty$, and let $\lambda_1, \ldots, \lambda_s$ be fixed non-zero elements of $\mathbb{K}_\infty$, not all in $\mathbb{F}_q(t)$ ratio. Suppose also that the equation

$$\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0 \tag{2}$$

has a non-trivial solution $\mathbf{z}$ in $\mathbb{K}_\infty^s$. Then, for all sufficiently large positive real numbers P, the number of $\mathbb{F}_q[t]$-solutions $N(P, \boldsymbol{\lambda})$ of

$$\langle \lambda_1 x_1^k + \cdots + \lambda_s x_s^k - \gamma \rangle < \hat{\tau}$$

with $\langle x_i \rangle \le \hat{P}$ for $1 \le i \le s$ satisfies

$$N(P, \boldsymbol{\lambda}) \gg \hat{P}^{s-k}.$$

The implicit constant may depend on s, k, q, $\boldsymbol{\lambda}$, γ, and τ.

Stemming from results of Chevalley and Weil (see [2] and [12], respectively) and discussion in [11], the following conditions on s, k, and q provide cases where a non-trivial solution to (2) exists in $\mathbb{K}_\infty^s$.

Proposition 1.1. *Suppose that* $\mathrm{char}(\mathbb{F}_q) \nmid k$ *and let* $\lambda_1, \ldots, \lambda_s$ *be non-zero elements of* $\mathbb{K}_\infty$. *The equation* $\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$ *has a non-trivial solution* $\mathbf{z} \in \mathbb{K}_\infty^s$ *whenever one of the following three conditions are met:*

(1) $s \geq k^2 + 1$,

(2) $q > k^4$ *and* $s \geq 2k + 1$,

(3) $(k, q-1) = 1$ *and* $s \geq k + 1$.

2 Proof of the theorem

2.1 The Davenport–Heilbron method for function fields

To produce the results found in Theorem 1.1, we utilize the Davenport–Heilbron method for function fields. Let $e_q : \mathbb{F}_q \to \mathbb{C}^\times$ be defined by $e_q(a) = e^{2\pi i \, \mathrm{tr}(a)/p}$ where $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map. We define $e : \mathbb{K}_\infty \to \mathbb{C}^\times$ by $e(\alpha) = e_q(\mathrm{res}\, \alpha)$. Let $\mathbb{T}$ be a compact additive subgroup of $\mathbb{K}_\infty$ defined as $\mathbb{T} = \{\alpha : \mathrm{ord}\, \alpha < 0\}$. We normalize a Haar measure on $\mathbb{K}_\infty$ so that

$$\int_{\mathbb{T}} d\alpha = 1.$$

First, like in [7], define the function $\chi_\tau : \mathbb{K}_\infty \to \mathbb{R}$ by

$$\chi_\tau(\alpha) = \begin{cases} \hat{\tau}, & \text{when } \langle \alpha \rangle < \hat{\tau}^{-1}, \\ 0, & \text{when } \langle \alpha \rangle \geq \hat{\tau}^{-1}. \end{cases}$$

Then as in Lemma 2.2 of [7], we construct an indicator function:

$$\int_{\mathbb{K}_\infty} e(\alpha\beta)\chi_\tau(\alpha) d\alpha = \begin{cases} 1, & \text{when } \langle \beta \rangle < \hat{\tau}, \\ 0, & \text{when } \langle \beta \rangle \geq \hat{\tau}. \end{cases}$$

We define the set of *R-smooth polynomials* $\mathcal{A}(P, R)$ as

$$\mathcal{A}(P, R) = \{x \in \mathbb{A} : \langle x \rangle \leq \hat{P}; \omega \text{ irreducible and } \omega \mid x \Rightarrow \langle \omega \rangle \leq \hat{R}\},$$

where $R \leq P$ for some real numbers P and R. Whenever R is used in the remainder of the paper, we take $R = \eta P$, and when R occurs in a statement, we are asserting that there exists a positive number η_0 such that the statement holds for all $0 < \eta \leq \eta_0$. Denote

$$F(\alpha) = F(\alpha; P) = \sum_{\langle x \rangle \leq \hat{P}} e(\alpha x^k)$$

and

$$f(\alpha) = f(\alpha; P, R) = \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^k).$$

Set $F_i(\alpha) = F(\lambda_i\alpha)$ for $1 \le i \le s$ and $f_j(\alpha) = f(\lambda_j\alpha)$ for $3 \le j \le s$. We now have that the integral

$$\int_{\mathbb{K}_\infty} F_1(\alpha)F_2(\alpha)f_3(\alpha)\cdots f_s(\alpha)e(-\alpha\gamma)\chi_\tau(\alpha)d\alpha$$

counts the number of solutions $\mathbf{x} \in \mathbb{A}^s$ of

$$\langle \lambda_1 x_1^k + \cdots + \lambda_s x_s^k - \gamma \rangle < \hat{\tau},$$

where $\langle x_i \rangle \le \hat{P}$ for $i = 1, 2$ and $x_j \in \mathcal{A}(P, R)$ for $3 \le j \le s$.

We say that a positive number $u > 2k - 2$ is *accessible to the exponent* k when there exists a positive number δ for which

$$\int_{\mathfrak{n}} |F(\alpha; P)^2 f(\alpha; P, R)^u| d\alpha \ll \hat{P}^{u+2-k-\delta},$$

where $\mathfrak{n}$ denotes the set of $\alpha \in \mathbb{T}$ such that for a and g in $\mathbb{A}$, when $\langle g\alpha - a \rangle < \hat{P}^{1-k}$ and $g \neq 0$, then $\langle g \rangle > \hat{P}$. By Theorem 9.4, Corollary 13.3, and Lemma 14.1 of [10], Theorem 1.1 is the consequence of the following theorem.

Theorem 2.1. *Suppose that k and s are natural numbers with $k > 1$ and $\mathrm{char}(\mathbb{F}_q) \nmid k$. Assume that $u > 2k - 2$ is accessible to the exponent k and that $s \ge u + 5$. Let τ be some fixed integer, let γ be a non-zero element in $\mathbb{K}_\infty$, and let $\lambda_1, \ldots, \lambda_s$ be fixed non-zero elements of $\mathbb{K}_\infty$, not all in $\mathbb{F}_q(t)$ ratio. Suppose also that the equation $\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$ has a non-trivial solution $\mathbf{z}$ in $\mathbb{K}_\infty^s$. Then, for all sufficiently large positive real numbers P, the number of $\mathbb{F}_q[t]$-solutions $N(P, \boldsymbol{\lambda}, \gamma)$ of*

$$\langle \lambda_1 x_1^k + \cdots + \lambda_s x_s^k - \gamma \rangle < \hat{\tau}$$

with $\langle x_i \rangle \le P$ for $1 \le i \le s$ satisfies

$$N(P, \boldsymbol{\lambda}, \gamma) \gg \hat{P}^{s-k}.$$

The implicit constant may depend on s, k, q, $\boldsymbol{\lambda}$, γ, and τ.

Following the Davenport–Heilbronn method, we split the region $\mathbb{K}_\infty$ into three arcs. First, set $S_1(P) = (\mathrm{Log}\,\hat{P})^{1/8}$. Define the major arc by

$$\mathfrak{M} = \{\alpha \in \mathbb{K}_\infty : \langle \alpha \rangle < S_1(P)\hat{P}^{-k}\},$$

the minor arc by

$$\mathfrak{m} = \{\alpha \in \mathbb{K}_\infty : S_1(P)\hat{P}^{-k} \le \langle \alpha \rangle < \hat{\tau}^{-1}\},$$

and the trivial arc by

$$\mathfrak{t} = \{\alpha \in \mathbb{K}_\infty : \langle \alpha \rangle \ge \hat{\tau}^{-1}\}.$$

The trivial arcs provide no contribution to the bound, as

$$\int_{\langle \alpha \rangle \ge \hat{\tau}^{-1}} F_1(\alpha)F_2(\alpha)f_3(\alpha)\cdots f_s(\alpha)e(-\alpha\gamma)\chi_\tau(\alpha)d\alpha = 0.$$

Thus, for Theorem 2.1 to hold we want

$$\int_{\mathfrak{m}} F_1(\alpha)F_2(\alpha)f_3(\alpha)\cdots f_s(\alpha)e(-\alpha\gamma)\chi_\tau(\alpha)d\alpha = o(\hat{P}^{s-k})$$

and

$$\int_{\mathfrak{M}} F_1(\alpha)F_2(\alpha)f_3(\alpha)\cdots f_s(\alpha)e(-\alpha\gamma)\chi_\tau(\alpha)d\alpha \gg \hat{P}^{s-k}$$

for sufficiently large values of P.

2.2 The minor arc

To establish the desired bound on the minor arcs, we need an appropriate Weyl-type estimate and a suitable mean value estimate for $f(\alpha)$. We will closely follow the analysis provided in Sections 3 and 4 of [11], with a slight difference in the proof of one of the lemmas.

The machinery provided by Lemma 4.1 of [11] will still be used to establish a mean value estimate for $f(\alpha)$, without any changes. However, our discussion of the Weyl-type estimate will slightly diverge from what is given by Section 3 of [11]. In [11], the proof of Lemma 3.1 cites a result from a preprint that never appeared in the literature. We will provide a proof analogous to Lemma 3.1 of [11] using a result that does currently exist in the literature.

We will first provide definitions for terms that are used in the statement of the result we will cite. Let $\mathcal{K}$ be a finite set of positive integers. We define the *shadow* of $\mathcal{K}$ to be

$$S(\mathcal{K}) = \left\{ j \in \mathbb{Z}^+ : p \nmid \binom{r}{j} \text{ for some } r \in \mathcal{K} \right\},$$

where we adopt the convention that if $j > r$, $\binom{r}{j} = 0$. We can then construct

$$\mathcal{K}^* = \{k \in \mathcal{K} : p \nmid k \text{ and } p^v k \notin S(\mathcal{K}) \text{ for any } v \in \mathbb{Z}^+\}.$$

We are now able to state the result we will use in proving Lemma 3.1, which is Theorem 3.1 from [9].

Theorem 2.2. *Fix q and a finite set $\mathcal{K} \subset \mathbb{Z}^+$. There exist positive constants ξ and C, depending only on $\mathcal{K}$ and q, such that the following holds. Let $\epsilon > 0$ and let M be sufficiently large in terms of $\mathcal{K}$, ϵ, and q. Suppose that $h(x) = \sum_{r\in\mathcal{K}\cup\{0\}} \alpha_r x^r$ is a polynomial with coefficients in $\mathbb{K}_\infty$ satisfying the bound*

$$\left| \sum_{\langle x\rangle<\hat{M}} e(h(x)) \right| \geq q^{M-\sigma}, \tag{3}$$

for some positive number σ with $\sigma \leq \xi M$. Then for each maximal $k \in \mathcal{K}^*$, there exist $a \in \mathbb{F}_q[t]$ and monic $g \in \mathbb{F}_q[t]$ having the property that

$$\mathrm{ord}(g\alpha_k - a) < -kM + \epsilon M + C\sigma \ \textit{ and } \ \mathrm{ord}\, g \leq \epsilon M + C\sigma.$$

For the purpose of this paper, we set $\mathcal{K} = \{k\}$, $h(x) = \alpha x^k$, $M = P + 1$, $\epsilon = 1/4$, and $\sigma = \min\{P/(4C), \xi P\}$. Furthermore, we have that $\{k\} \subseteq S(\mathcal{K}) \subseteq \{1, \ldots, k\}$ and $\mathcal{K}^* = \{k\}$, since $p \nmid k$ and $p^v k > k$ for all $v \in \mathbb{Z}^+$. Thus, if (3) holds and P is sufficiently large in terms of k and q, it follows that there exists $a \in \mathbb{F}_q[t]$ and monic $g \in \mathbb{F}_q[t]$ satisfying

$$\operatorname{ord}(g\alpha - a) < -k(P+1) + \epsilon(P+1) + C\sigma < (3/4 - k)P$$

and

$$\operatorname{ord} g \leq \epsilon(P+1) + C\sigma \leq 3P/4.$$

Let $\mathfrak{n}_*$ denote the set of $\alpha \in \mathbb{T}$ such that, for a and monic g in $\mathbb{A}$ satisfying $\operatorname{ord}(g\alpha - a) < (3/4 - k)P$, we have $\operatorname{ord} g > 3P/4$. Since k is maximal in $\mathcal{K}^*$, by the contrapositive of Theorem 2.2, there exists a small positive constant $\nu = \nu(q, k)$ such that

$$\sup_{\alpha \in \mathfrak{n}_*} |F(\alpha; P)| \ll \hat{P}^{1-\nu}.$$

We adapt the proof of Lemma 3.1 of [11] using the set $\mathfrak{n}_*$ defined above.

Lemma 2.1. *There is a positive constant c, depending at most on k and q, with the following property. Suppose that P is a real number, sufficiently large in terms of k and q. Suppose that δ is a positive number with $\hat{P}^{-\nu/2} < \delta \leq 1$. Then, whenever $|F(\alpha)| \geq \delta \hat{P}$, there exist a and g in $\mathbb{A}$ such that $(a, g) = 1$, $1 \leq \langle g \rangle \leq c\delta^{-k}$ and $\langle g\alpha - a \rangle \leq \delta^{-k} \hat{P}^{-k}$.*

Proof. Suppose there exists an $\alpha \in \mathbb{K}_\infty$ such that $|F(\alpha)| \geq \delta \hat{P}$, where δ is a positive number with $\hat{P}^{-\nu/2} < \delta \leq 1$. It follows from Lemma 3 of [8], that for all $\alpha \in \mathbb{T}$, there exist unique a and g in $\mathbb{A}$, where $(a, g) = 1$, g is monic, $\langle a \rangle < \langle g \rangle \leq \hat{P}^{k-3/4}$ and $\langle g\alpha - a \rangle < \hat{P}^{-k+3/4}$.

Suppose that $\langle g \rangle > \hat{P}^{3/4}$. Thus, $\alpha \in \mathfrak{n}_*$, and $|F(\alpha)| \ll \hat{P}^{1-\nu}$. Taking P sufficiently large,

$$|F(\alpha)| < \frac{1}{2}\hat{P}^{1-\nu/2} \leq \frac{1}{2}\delta \hat{P}.$$

This contradicts our assumption on the size of $|F(\alpha)|$; thus we assume that $\langle g \rangle \leq \hat{P}^{3/4}$.

By Lemma 4.1 of [10], there exists a positive constant c such that

$$F(\alpha) \leq c^{1/k} \hat{P} (\langle g \rangle + \hat{P}^k \langle g\alpha - a \rangle)^{-1/k}.$$

By assumption, we have that $|F(\alpha)| \geq \delta \hat{P}$, thus

$$\langle g \rangle + \hat{P}^k \langle g\alpha - a \rangle \leq c\delta^{-k}.$$

The result follows. □

The following are Lemma 3.2 and Lemma 3.3 of [11], respectively.

Lemma 2.2. *Suppose that S is a fixed real number with $0 < S < \hat{\tau}^{-1}$. Then one has*

$$\lim_{P \to \infty} \sup_{S \leq \langle \alpha \rangle < \hat{\tau}^{-1}} |F_1(\alpha) F_2(\alpha)| = 0.$$

Lemma 2.3. *Suppose that $S(P)$ is a function on $(0,\infty)$ that increases monotonically to infinity and satisfies $1 \le S(P) \le \hat{P}$. Then there exists a function $T(P)$ on $(0,\infty)$ depending only on λ_1, λ_2, k, q, τ, and $S(P)$, that increases monotonically to infinity, satisfies $1 \le T(P) \le S(P)$ and satisfies the property that*

$$\sup_{S(P)\hat{P}^{-k} \le \langle\alpha\rangle < \hat{\tau}^{-1}} |F_1(\alpha)F_2(\alpha)| \ll \hat{P}^2 T(P)^{-\nu/(2k)}.$$

We have now established a Weyl-type estimate that can provide our desired bound on the minor arc.

Lemma 2.4. *Suppose that $u > 2k-2$ is accessible to the exponent k and that $s \ge u+5$. One has*

$$\int_{\mathfrak{m}} F_1(\alpha)F_2(\alpha)f_3(\alpha)\cdots f_s(\alpha)e(-\alpha\gamma)\chi_\tau(\alpha)d\alpha = o(\hat{P}^{s-k}).$$

Proof. Let $v = \lfloor s/2 \rfloor - 2$. By the triangle inequality,

$$\int_{\mathfrak{m}} F_1(\alpha)F_2(\alpha)f_3(\alpha)\cdots f_s(\alpha)e(-\alpha\gamma)\chi_\tau(\alpha)d\alpha \ll \int_{\mathfrak{m}} |F_1(\alpha)F_2(\alpha)f_3(\alpha)\cdots f_s(\alpha)|\chi_\tau(\alpha)d\alpha,$$

so the result follows from the proof of Lemma 4.2 of [11]. □

2.3 The major arc

First, define

$$\mathcal{F}(\alpha) = F_1(\alpha)F_2(\alpha)f_3(\alpha)\cdots f_s(\alpha)$$

and

$$\mathcal{G}(\alpha) = F_1(\alpha)\cdots F_s(\alpha).$$

For the major arcs, we want to compare

$$\hat{\tau}\int_{\mathfrak{M}} \mathcal{F}(\alpha)e(-\alpha\gamma)d\alpha$$

to the singular integral

$$J_{s,k} = \int_{\langle\alpha\rangle < \hat{P}^{s-k}} \mathcal{G}(\alpha)e(-\alpha\gamma)d\alpha.$$

The following lemma uses the Dickman function, $\rho(u)$, which is the continuous function on the real numbers that is uniquely defined by the differential equation $u\rho'(u) = -\rho(u-1)$, with the initial condition that $\rho(u) = 1$ for $u \in [0,1]$.

Lemma 2.5. *Suppose that $s \ge k+1$. One has*

$$\int_{\mathfrak{M}} \mathcal{F}(\alpha)e(-\alpha\gamma)d\alpha - \rho(P/R)^{s-2}J_{s,k} \ll \hat{P}^{s-k}(\operatorname{Log}\hat{P})^{-1/(8k)}.$$

Proof. Set $P \geq 1$ large enough so that $2P/\log(2P) < R < P - \log(P)$. From [10] and [11], we have that

$$\mathcal{F}(\alpha)e(-\alpha\gamma) - \rho(P/R)\mathcal{G}(\alpha)e(-\alpha\gamma) \ll \hat{P}(\operatorname{Log}\hat{P})^{-3/8}$$

for $\alpha \in \mathfrak{M}$. It follows by the triangle inequality that

$$\int_{\mathfrak{M}} \mathcal{F}(\alpha)e(-\alpha\gamma)d\alpha - \rho(P/R)^{s-2}\int_{\mathfrak{M}} \mathcal{G}(\alpha)e(-\alpha\gamma)d\alpha \ll \hat{P}^{s-k}(\operatorname{Log}\hat{P})^{-1/4}.$$

Furthermore, Lemma 4.1 in [10] provides the bound

$$F_i(\alpha) \ll \hat{P}(1+\hat{P}^k\langle\alpha\rangle)^{-1/k}.$$

Thus,

$$\begin{aligned}\int_{\mathfrak{M}} \mathcal{G}(\alpha)e(-\alpha\gamma)d\alpha - J_{s,k} &\ll \int_{\mathfrak{T}} \Big|\mathcal{G}(\alpha)\Big|\Big|e(-\alpha\gamma)\Big|d\alpha \\ &\ll \hat{P}^s\int_{\mathfrak{T}}(1+\hat{P}^k\langle\alpha\rangle)^{-s/k}d\alpha,\end{aligned}$$

where $\mathfrak{T} = \{\alpha \in \mathbb{K}_\infty : S_1(P)\hat{P}^{-k} \leq \langle\alpha\rangle\}$. The remaining details follow from the proof of Lemma 5.1 in [11]. □

Lemma 2.6. *Let $s \geq k+1$, and suppose that the equation $\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$ has a non-trivial solution* $\mathbf{z}$ *in $\mathbb{K}_\infty^s$. For sufficiently large values of P, one has $J_{s,k} \gg \hat{P}^{s-k}$.*

Proof. By Lemma 1 of [8], we have

$$J_{s,k} = \int_{\langle\alpha\rangle<\hat{P}^{1-k}} \mathcal{G}(\alpha)e(\alpha\gamma)d\alpha = \hat{P}^{1-k}W, \tag{4}$$

where W denotes the number of s-tuples $(x_1,\ldots,x_s) \in \mathbb{A}^s$ with

$$\langle\lambda_1 x_1^k + \cdots + \lambda_s x_s^k - \gamma\rangle < \hat{P}^{k-1} \tag{5}$$

and $\langle x_i\rangle \leq \hat{P}$ for $1 \leq i \leq s$. As in [11], choose r such that $\langle\lambda_r z_r^k\rangle$ is maximal. Let $d = \operatorname{ord}\lambda_r$ and $w = \operatorname{lead}\lambda_r$. For $1 \leq i \leq s$, we define a_i by

$$a_i = \begin{cases}\operatorname{lead} z_i, & \text{when } \langle\lambda_i z_i^k\rangle = \langle\lambda_r z_r^k\rangle,\\ 0, & \text{otherwise,}\end{cases}$$

and m_i as

$$m_i = \left\lfloor \frac{d - \operatorname{ord}\lambda_i + k\cdot\operatorname{ord} z_r}{k}\right\rfloor.$$

Let $n = \lfloor P\rfloor - \max_{1\leq i\leq s} m_i - \max\left\{0, \left\lceil\frac{d}{k-1}\right\rceil\right\}$, and suppose that P is large enough so that $n + m_i > 0$ for $1 \leq i \leq s$ and $d + k(n+m_r) > \operatorname{ord}\gamma$. For $1 \leq i \leq s$, write $x_i \in \mathbb{A}$ as $x_i = a_i t^{n+m_i} + y_i$, where $y_i \in \mathbb{A}$ and $\operatorname{ord} y_i < n + m_i$. Let

$$x_r = a_r t^{n+m_r} + b_{n+m_r-1}t^{n+m_r-1} + \cdots + b_0,$$

where each $b_i \in \mathbb{F}_q$. We define $c_l \in \mathbb{F}_q$ via the relation

$$\lambda_1 x_1^k + \cdots + \lambda_s x_s^k - \gamma = \sum_{l=-\infty}^{\infty} c_l t^l.$$

Thus, (5) holds when $c_l = 0$ for all $l \geq (k-1)P$. We note that $c_l = 0$ for all $l > d + k(n + m_r)$ via the relation

$$\operatorname{ord} \lambda_i + k(n + m_i) \leq d + k(n + m_r).$$

Also, the coefficient $c_{d+k(n+m_r)} = 0$ by the definition for a_i, the construction of the x_i, and our hypothesis.

We consider what occurs when

$$d + (k-1)(n + m_r) \leq l < d + k(n + m_r).$$

From our construction of the x_i, we have

$$c_l = kwa_r^{k-1} b_{l-d-(k-1)(n+m_r)} + h_l,$$

where h_l is an element of $\mathbb{F}_q$ depending at most on $\boldsymbol{\lambda}$, $\mathbf{a}$, γ, b_i with $i > l - d - (k-1)(n + m_r)$, and y_j with $j \neq r$.

For $j \neq r$, let y_j be arbitrarily selected. Note that $kwa_r^{k-1} \neq 0$. We can choose b_{n+m_r-1} so that $c_{d+k(n+m_r)-1} = 0$, and similarly we can then select b_{n+m_r-2} so that $c_{d+k(n+m_r)-2} = 0$. Continuing in this manner, we can choose x_r such that $c_l = 0$ for all $l \geq d + (k-1)(n + m_r)$. From our construction of n,

$$\begin{aligned} d + (k-1)(n + m_r) &\leq (k-1)(\lfloor P \rfloor - \max_{1 \leq i \leq s} m_i + m_r) \\ &\leq (k-1)P. \end{aligned}$$

Then, due to the y_j being arbitrarily selected for $j \neq r$, $W \gg \hat{P}^{s-1}$ for P sufficiently large. In summary, by (4), $J_{s,k} \gg \hat{P}^{s-k}$. □

Combining the results of the previous lemmas gives us the following:

Lemma 2.7. *Let $s \geq k+1$, and suppose that the equation $\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$ has a non-trivial solution $\mathbf{z}$ in $\mathbb{K}_\infty^s$. For sufficiently large values of P, one has*

$$\int_{\mathfrak{M}} F_1(\alpha) F_2(\alpha) f_3(\alpha) \cdots f_s(\alpha) e(-\gamma\alpha) \chi_\tau(\alpha) d\alpha \gg \hat{P}^{s-k}.$$

Theorem 2.1 now follows by Lemma 2.4 and Lemma 2.7.

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