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The complex-type Pell p-numbers

Yeşim Akuzüm[1], Hüseyin Aydın[2] and Ömür Deveci[3]

[1] Department of Mathematics, Faculty of Science and Letters,
Kafkas University 36100, Türkiye
e-mail: yesim_036@hotmail.com

[2] Department of Mathematics and Science Education, Education Faculty,
Biruni University, Istanbul, Türkiye
e-mail: haydin@biruni.edu.tr

[3] Department of Mathematics, Faculty of Science and Letters,
Kafkas University 36100, Türkiye
e-mails: odeveci36@hotmail.com

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Abstract: In this paper, we define the complex-type Pell p-numbers and give the generating matrix of these defined numbers. Then, we produce the combinatorial representation, the generating function, the exponential representation and the sums of the complex-type Pell p-numbers. Also, we derive the determinantal and the permanental representations of the complex-type Pell p-numbers by using certain matrices which are obtained from the generating matrix of these numbers. Finally, we obtain the Binet formula for the complex-type Pell p-number.
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1 Introduction

The generalized Pell (p, i)-numbers are defined [12] by the following equation for any given $p\,(p = 1, 2, 3, \ldots)\,, n > p + 1$ and $0 \le i \le p$

$$P_p^{(i)}(n) = 2P_p^{(i)}(n-1) + P_p^{(i)}(n-p-1)$$

with initial conditions $P_p^{(i)}(1) = \cdots = P_p^{(i)}(i) = 0$ and $P_p^{(i)}(i+1) = P_p^{(i)}(i+2) = \cdots = P_p^{(i)}(p+1) = 1$.

The complex Fibonacci sequence $\{F_n^*\}$ is defined [8] by the following equation: for $n \geq 0$

$$F_n^* = F_n + iF_{n+1},$$

where $\sqrt{-1} = i$ and F_n is the n-th Fibonacci number (cf. [1,9]).

Suppose the $(n+k)$-th term of a sequence is defined recursively as a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are constants.

Kalman [10] showed that number sequences can be derived by a matrix representation. He derived closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Also, he proved that

$$(A_k)^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

There have been many studies on this paper in the literature: see, for example, [6, 7, 14–20]. Some linear recurrence sequences are defined and their various properties are given using the matrix methods by many authors in [5, 11–13]. Also, in [5] and [4], the authors defined new sequences using the quaternions and complex numbers and then they gave miscellaneous properties. In this paper, we define the complex-type Pell p-numbers and give the generating matrix of these defined numbers. Then, we determine the relationships between the complex-type Pell p-numbers and the generalized Pell (p,i)-numbers. Also, we give number-theoretic properties of the complex-type Pell p-numbers such as the generating function, the exponential representation, the combinatorial representation, the sums, permanental and determinantal representations, and the Binet formula.

2 The complex-type Pell p-numbers

Define the complex-type Pell p-numbers as shown:

$$P_p^*(n+p+1) = 2i^{p+1} \cdot P_p^*(n+p) + i \cdot P_p^*(n)$$

for any given $p\,(p = 2, 3, \ldots)$ and $n \geq 1$, with the initial conditions $P_p^*(1) = \cdots = P_p^*(p) = 0$, $P_p^*(p+1) = 1$ and $\sqrt{-1} = i$.

From the definition of generalized the complex-type Pell p-numbers, we can write the following matrix relation:

$$\begin{bmatrix} P_p^*(n+p+1) \\ P_p^*(n+p) \\ \vdots \\ P_p^*(n+2) \\ P_p^*(n+1) \end{bmatrix} = K_p \cdot \begin{bmatrix} P_p^*(n+p) \\ P_p^*(n+p-1) \\ \vdots \\ P_p^*(n+1) \\ P_p^*(n) \end{bmatrix}$$

where K_p is a $(p+1)$-square companion matrix as following:

$$K_p = \begin{bmatrix} 2i^{p+1} & 0 & \cdots & 0 & i \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

The matrix K_p is called to be the complex-type Pell p-matrix.

By an inductive argument for $n \geq p$, it is easy to see that the n-th powers of the matrix K_p is

$$(K_p)^n = \begin{bmatrix} P_p^*(n+p+1) & iP_p^*(n+1) & iP_p^*(n+2) & \cdots & iP_p^*(n+p) \\ P_p^*(n+p) & iP_p^*(n) & iP_p^*(n+1) & \cdots & iP_p^*(n+p-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_p^*(n+2) & iP_p^*(n-p+2) & iP_p^*(n-p+3) & \cdots & iP_p^*(n+1) \\ P_p^*(n+1) & iP_p^*(n-p+1) & iP_p^*(n-p+2) & \cdots & iP_p^*(n) \end{bmatrix}. \quad (1)$$

Using the $(K_p)^n$ matrix, we derive the following relationships between the complex-type Pell p-numbers and the generalized Pell (p,i)-numbers for $n \geq p$ such that every even p integer:

$$(K_p)^n = \begin{bmatrix} \left(i^{p+1}\right)^n P_p^{(i)}(n+p+1) & \left(i^{p+1}\right)^{n+1} P_p^{(i)}(n+1) & \left(i^{p+1}\right)^{n+2} P_p^{(i)}(n+2) & \cdots & \left(i^{p+1}\right)^{n+p} P_p^{(i)}(n+p) \\ \left(i^{p+1}\right)^{n-1} P_p^{(i)}(n+p) & \left(i^{p+1}\right)^{n} P_p^{(i)}(n) & \left(i^{p+1}\right)^{n+1} P_p^{(i)}(n+1) & \cdots & \left(i^{p+1}\right)^{n+p-1} P_p^{(i)}(n+p-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(i^{p+1}\right)^{n-p+1} P_p^{(i)}(n+2) & \left(i^{p+1}\right)^{n-p+2} P_p^{(i)}(n-p+2) & \left(i^{p+1}\right)^{n-p+3} P_p^{(i)}(n-p+3) & \cdots & \left(i^{p+1}\right)^{n+1} P_p^{(i)}(n+1) \\ \left(i^{p+1}\right)^{n-p} P_p^{(i)}(n+1) & \left(i^{p+1}\right)^{n-p+1} P_p^{(i)}(n-p+1) & \left(i^{p+1}\right)^{n-p+2} P_p^{(i)}(n-p+2) & \cdots & \left(i^{p+1}\right)^{n} P_p^{(i)}(n) \end{bmatrix},$$

where the generalized Pell (p,i)-numbers are considered for the case $i = p$.

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2.1. (Chen and Louck [3]) *The (t,j)-th entry $k_{t,j}^{(n)}(k_1, k_2, \ldots, k_v)$ in the matrix $K^n(k_1, k_2, \ldots, k_v)$ is given by the following formula:*

$$k_{t,j}^{(n)}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v} \quad (2)$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$,

$$\binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} = \frac{(t_1 + \cdots + t_v)!}{t_1! \cdots t_v!}$$

is a multinomial coefficient, and the coefficients in (2) *are defined to be* 1 *if* $n = i - j$.

Then we have the following Corollary for the complex-type Pell p-numbers.

Corollary 2.1. *Let* $P_p^*(n)$ *be the complex-type Pell* p*-number. Then*

$$\begin{aligned} P_p^*(n) &= \frac{1}{i} \sum_{(t_1, t_2 \ldots, t_k)} \frac{t_2 + t_3 + \cdots + t_{p+1}}{t_1 + t_2 + \cdots + t_{p+1}} \times \binom{t_1 + \cdots + t_{p+1}}{t_1, \ldots, t_{p+1}} \left(2i^{p+1}\right)^{t_1} (i)^{t_{p+1}} \\ &= \frac{1}{i} \sum_{(t_1, t_2 \ldots, t_k)} \frac{t_3 + t_4 + \cdots + t_{p+1}}{t_1 + t_2 + \cdots + t_{p+1}} \times \binom{t_1 + \cdots + t_{p+1}}{t_1, \ldots, t_{p+1}} \left(2i^{p+1}\right)^{t_1} (i)^{t_{p+1}} \\ &= \cdots \\ &= \frac{1}{i} \sum_{(t_1, t_2 \ldots, t_k)} \frac{t_{p+1}}{t_1 + t_2 + \cdots + t_{p+1}} \times \binom{t_1 + \cdots + t_{p+1}}{t_1, \ldots, t_{p+1}} \left(2i^{p+1}\right)^{t_1} (i)^{t_{p+1}} \end{aligned}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+1)\, t_{p+1} = n$.

Proof. If we take $v = p + 1$, $i = j = \lambda$ such that $2 \leq \lambda \leq p + 1$ in Theorem 2.1, then the proof is immediately seen from (1). □

The generating function of the complex-type Pell p-numbers is given by:

$$g_p^{(i)}(x) = \frac{x^{p+1}}{1 - 2i^{p+1}x - ix^{p+1}}.$$

Theorem 2.2. *The complex-type Pell* p*-numbers have the following exponential representation:*

$$g_p^{(i)}(x) = x^{p+1} \exp\left(\sum \frac{x^k}{k} \left(2i^{p+1} + ix^p\right)^k\right).$$

Proof. It is clear that

$$\ln \frac{g_p^{(i)}(x)}{x^{p+1}} = -\ln\left(1 - 2i^{p+1}x - ix^{p+1}\right).$$

By the function $\ln x$ we obtain the relation

$$\begin{aligned} -\ln\left(1 - 2i^{p+1}x - ix^{p+1}\right) &= -\left[-x\left(2i^{p+1} + ix^p\right) - \frac{1}{2}x^2\left(2i^{p+1} + ix^p\right)^2 - \cdots \right. \\ &\quad \left. -\frac{1}{n}x^n\left(2i^{p+1} + ix^p\right)^n - \cdots\right]. \end{aligned}$$

A simple calculation shows that

$$\ln \frac{g_p^{(i)}(x)}{x^{p+1}} = \exp\left(\sum \frac{x^k}{k} \left(2i^{p+1} + ix^p\right)^k\right).$$

Thus, this completes the proof. □

Let

$$S_n = \sum_{j=1}^{n} P_p^* (j)$$

and suppose that P_p is the $(p+2) \times (p+2)$ matrix such that

$$P_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & K_p & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

Then it can be shown by induction that

$$(P_p)^n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ S_{n+p} & & & \\ S_{n+p-1} & & (K_p)^n & \\ \vdots & & & \\ S_n & & & \end{bmatrix}.$$

Definition 2.1. *A $u \times v$ real matrix $M = [m_{i,j}]$ is called a contractible matrix in the k-th column (respectively, row) if the k-th column (respectively, row) contains exactly two non-zero entries.*

Brualdi and Gibson [2] show that $\operatorname{per}(M) = \operatorname{per}(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Let $p \geq 2$ be a positive integer and let $R_u^{(p,i)} = \left[r_{t,j}^{(p,i,u)} \right]$ be the $u \times u$ super-diagonal matrix, defined by

$$\begin{matrix} (p+1)\text{-th} \\ \downarrow \end{matrix}$$

$$R_u^{(p,i)} = \begin{bmatrix} 2i^{p+1} & 0 & 0 & \cdots & 0 & i & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2i^{p+1} & 0 & 0 & \cdots & 0 & i & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2i^{p+1} & 0 & 0 & \cdots & 0 & i & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2i^{p+1} & 0 & 0 & \cdots & 0 & i & 0 \\ 0 & 0 & \cdots & 0 & 1 & 2i^{p+1} & 0 & 0 & \cdots & 0 & i \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2i^{p+1} & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2i^{p+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2i^{p+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2i^{p+1} \end{bmatrix},$$

for $u > p + 1$.

Then we have the following theorem.

Theorem 2.3. *For $u > p+1$ and $p \geq 2$,*

$$\mathrm{per}R_u^{(p,i)} = P_p^*\left(u+p+1\right).$$

Proof. For the proof, we apply the inductive method on u. Let the equation hold for $u > p+1$. Now we prove that the equation is satisfied for $u+1$. Then expanding the $\mathrm{per}R_u^{(p,i)}$ with the Laplace expansion relative to the first row, so we get

$$\mathrm{per}R_{u+1}^{(p,i)} = 2i^{p+1} \cdot \mathrm{per}R_u^{(p,i)} + i \cdot \mathrm{per}R_{u-p}^{(p,i)}.$$

Since $\mathrm{per}R_u^{(p,i)} = P_p^*\left(u+p+1\right)$ and $\mathrm{per}R_{u-p}^{(p,i)} = P_p^*\left(u+1\right)$, from definition of the complex-type Pell p-numbers $P_p^*\left(n\right)$, the following equality is achieved:

$$\mathrm{per}R_{u+1}^{(p,i)} = P_p^*\left(u+p+2\right).$$

Thus, the proof is complete. □

Let $p \geq 2$ and let $V_u^{(p,i)} = \left[v_{t,j}^{(p,i,u)}\right]$ be the $u \times u$ matrix, defined by

$$v_{t,j}^{(p,i,u)} = \begin{cases} 2i^{p+1}, & \text{if } t = k \text{ and } j = k \text{ for } 1 \leq k \leq u-p-1, \\ i, & \text{if } t = k \text{ and } j = k+p \text{ for } 1 \leq k \leq u-p, \\ 1, & (\text{if } t = k+1 \text{ and } j = k \text{ for } 1 \leq k \leq u-p-2) \\ & \quad \text{and } (\text{if } t = k \text{ and } j = k \text{ for } u-p \leq k \leq u), \\ 0, & \text{otherwise.} \end{cases}$$

for $u > p+1$.

Now we define the $u \times u$ matrix $L_u^{(p,i)} = \left[l_{t,j}^{(p,i,u)}\right]$ as follows:

$$L_u^{(p,i)} = \begin{bmatrix} 1 & \cdots & \overset{\overset{(u-p-1)\text{th}}{\downarrow}}{1} & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & V_{u-1}^{(p,i)} & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix}.$$

Then we can give the following Theorem by using the permanental representations.

Theorem 2.4. (i). *For $u > p+1$,*

$$\mathrm{per}V_u^{(p,i)} = P_p^*\left(u\right).$$

(ii). *For $u > p+2$,*

$$\mathrm{per}L_u^{(p,i)} = \sum_{n=1}^{u-1} P_p^*\left(n\right).$$

Proof. $(i.)$ Let the equation hold for $u > p+1$, then we show that the equation holds for $u+1$. If we expand the $\mathrm{per}V_u^{(p,i)}$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\begin{aligned}\text{per}V_{u+1}^{(p,i)} &= 2i^{p+1}\cdot \text{per}V_{u}^{(p,i)} + i\cdot \text{per}V_{u-p}^{(p,i)} \\ &= 2i^{p+1}\cdot P_{p}^{*}(u) + i\cdot P_{p}^{*}(u-p).\end{aligned}$$

So, we have the conclusion.

(ii). Since we expand the $\text{per}L_{u}^{(p,i)}$ with the Laplace expansion relative to the first row, we reach

$$\text{per}L_{u}^{(p,i)} = \text{per}L_{u-1}^{(p,i)} + \text{per}V_{u-1}^{(p,i)}.$$

The inductive argument and by the result of part (i) in Theorem 2.4, the result has been reached. □

A matrix M is called convertible if there is an $n\times n$ $(1,-1)$-matrix K such that $\text{per}M = \det(M\circ K)$, where $M\circ K$ denotes the Hadamard product of M and K. We will now address the determinantal representations for the complex-type Pell p-numbers. Let $u > p+2$ and let J be the $u\times u$ matrix, defined by

$$J = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 2.2. *For $u > p+2$,*

$$\begin{aligned}\det\left(R_{u}^{(p,i)}\circ J\right) &= P_{p}^{*}(u+p+1), \\ \det\left(V_{u}^{(p,i)}\circ J\right) &= P_{p}^{*}(u)\end{aligned}$$

and

$$\det\left(L_{u}^{(p,i)}\circ J\right) = \sum_{n=1}^{u-1} P_{p}^{*}(n).$$

Proof. Since $\text{per}R_{u}^{(p,i)} = \det\left(R_{u}^{(p,i)}\circ J\right)$, $\text{per}V_{u}^{(p,i)} = \det\left(V_{u}^{(p,i)}\circ J\right)$ and $\text{per}L_{u}^{(p,i)} = \det\left(L_{u}^{(p,i)}\circ J\right)$ for $u > p+2$, by Theorem 2.3 and Theorem 2.4, we have the conclusion. □

We now derive a generalized Binet formula for the complex-type Pell p-numbers.

From companion matrices, it is known that the chacteristic equation of the complex-type Pell p-matrix is $x^{p+1} - 2i^{p+1}\cdot x^{p} - i = 0$, which is also the characteristic equation of the complex-type Pell p-numbers.

Lemma 2.1. *The equation $x^{p+1} - 2i^{p+1}\cdot x^{p} - i = 0$ does not have multiple roots for $p \geq 2$.*

Proof. Let $f(x) = x^{p+1} - 2i^{p+1}\cdot x^{p} - i$. It is clear that $f(0) \neq 0$ and $f(1) \neq 0$. Let δ be a multiple root of $f(x)$, then $\delta \neq 0$ and $\delta \neq 1$. Since δ is a multiple root

$$f(\delta) = \delta^{p+1} - 2i^{p+1}\cdot \delta^{p} - i = 0$$

and

$$\begin{aligned} f'(\delta) &= (p+1)\,\delta^{p} - \left(2i^{p+1}\cdot p\right)\cdot \delta^{p-1} = 0 \\ &= \delta^{p-1}\left((p+1)\,\delta - \left(2i^{p+1}\cdot p\right)\right) = 0. \end{aligned}$$

Thus, we obtain $\delta = \frac{2i^{p+1}\cdot p}{p+1}$. For $p \geq 2$, $f(\delta) \neq 0$, which is a contradiction and with this contradiction the conclusion is reached. □

Let $q_1, q_2, \ldots, q_{p+1}$ be the eigenvalues of the matrix K_p. Then by Lemma 2.1, it is known that $q_1, q_2, \ldots, q_{p+1}$ are distinct. Let be a $(p+1)\times(p+1)$ Vandermonde matrix W^p as follows:

$$W^p = \begin{bmatrix} (q_1)^p & (q_2)^p & \cdots & (q_{p+1})^p \\ (q_1)^{p-1} & (q_2)^{p-1} & \cdots & (q_{p+1})^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ q_1 & q_2 & \cdots & q_{p+1} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Assume that

$$W_t^p = \begin{bmatrix} (q_1)^{n+p+1-t} \\ (q_2)^{n+p+1-t} \\ \vdots \\ (q_{p+1})^{n+p+1-t} \end{bmatrix}$$

and $W^p(t,j)$ is a $(p+1)\times(p+1)$ matrix obtained from the W^p by replacing the j-th column of W^p by W_t^p.

Theorem 2.5. *Let $(K_p)^n = \left[k_{t,j}^{p,n}\right]$, then*

$$k_{t,j}^{p,n} = \frac{\det W^p(t,j)}{\det W^p}$$

for $n \geq p$ and $p \geq 2$.

Proof. The matrix K_p is diagonalizable because the eigenvalues of the matrix K_p are distinct. Let $Q_p = \operatorname{diag}(q_1, q_2, \ldots, q_{p+1})$, then we easily see that $K_p W^p = W^p Q_p$. Since the matrix W^p is invertible, we may write $(W^p)^{-1} K_p W^p = Q_p$. Then the matrix K_p is similar to Q_p; so, we obtain $(K_p)^n W^p = W^p (Q_p)^n$. Thus, we have the following linear system of equations:

$$\begin{cases} k_{t,1}^{p,n}(q_1)^{k-1} + k_{t,2}^{p,n}(q_1)^{k-2} + \cdots + k_{t,p+1}^{p,n} = (q_1)^{n+p+1-t} \\ k_{t,1}^{p,n}(q_2)^{k-1} + k_{t,2}^{p,n}(q_2)^{k-2} + \cdots + k_{t,p+1}^{p,n} = (q_2)^{n+p+1-t} \\ \vdots \\ k_{t,1}^{p,n}(q_{p+1})^{k-1} + k_{t,2}^{p,n}(q_{p+1})^{k-2} + \cdots + k_{t,p+1}^{p,n} = (q_{p+1})^{n+p+1-t} \end{cases}.$$

Then for each $t, j = 1, 2, \ldots, p+1$, it is obtained $k_{t,j}^{p,n}$ as follows

$$k_{t,j}^{p,n} = \frac{\det W^p(t,j)}{\det W^p}.$$

This completes the proof. □

Corollary 2.3. *Suppose that $P_p^*(n)$ is the n-th element of complex-type Pell p-number $n \geq p$ such that $p \geq 2$, then*

$$\begin{aligned} P_p^*(n) &= \frac{\det W^p(2,2)}{i \cdot \det W^p} \\ &= \frac{\det W^p(3,3)}{i \cdot \det W^p} \\ &= \cdots \\ &= \frac{\det W^p(p+1,p+1)}{i \cdot \det W^p}. \end{aligned}$$

3 Conclusion

This paper focuses on defining and analyzing the complex-type Pell p-numbers, which extend the concept of traditional Pell numbers into the complex domain. We introduce the generating matrix for these numbers and derive various mathematical representations, including combinatorial, exponential, and determinantal. We also present the Binet formula in these complex numbers and explore their applications in matrix theory, contributing to the broader understanding of linear recurrence relations and their complex extensions.

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