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On (k,p)-Fibonacci numbers and matrices

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Abstract: In this paper, some relations between the powers of any matrices X satisfying the equation $X^k - pX^{k-1} - (p-1)X - I = \mathbf{0}$ and (k,p)-Fibonacci numbers are established with $k \geq 2$. First, a result is obtained to find the powers of the matrices satisfying the condition above via (k,p)-Fibonacci numbers. Then, new properties related to (k,p)-Fibonacci numbers are given. Moreover, some relations between the sequence $\{F_{3,s}(n)\}$ and the generalized Fibonacci sequence $\{U_n(p,q)\}$ are also examined.
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1 Introduction and Preliminaries

The second-order homogeneous linear recurrence sequence $\{W_n(a,b;p,q)\}$, or briefly $\{W_n\}$ defined by $W_n = pW_{n-1} + qW_{n-2}$ for all integers $n \geq 2$, with the initial conditions $W_0 = a$

and $W_1 = b$ is called as Horadam sequence, where a, b, p, and q are arbitrary real numbers. The characteristic equation of the sequence $\{W_n\}$ is $x^2 - px - q = 0$. The relations between the roots of this equation and the terms of the sequence $\{W_n\}$ have been studied in literature (see, e.g., [1,4,5,8,15]).

The sequence $\{U_n(p,q)\}$, or briefly $\{U_n\}$, defined by the recurrence relation $U_n = pU_{n-1} + qU_{n-2}$ for all integers $n \geq 2$, with the initial conditions $U_0 = 0$ and $U_1 = 1$ is called generalized Fibonacci sequence, where p and q are nonzero real numbers. The sequence $\{U_n\}$ is a well-known special case of the sequence $\{W_n\}$. Generalized Fibonacci numbers for negative subscript are defined as $U_{-n} = \frac{-U_n}{(-q)^n}$. The classical Fibonacci sequence $\{F_n\}$ is a special case of the sequence $\{U_n\}$ for $p = q = 1$ (see, e.g., [6,12,13]).

In recent years, some special number sequences have been introduced. One of them is the (k,p)-Fibonacci sequence. The sequence $\{F_{k,p}(n)\}$ defined by the recurrence relation

$$F_{k,p}(n) = pF_{k,p}(n-1) + (p-1)F_{k,p}(n-k+1) + F_{k,p}(n-k) \tag{1}$$

for all integers $n \geq k$ with the initial conditions $F_{k,p}(0) = 0$ and $F_{k,p}(n) = p^{n-1}$ for $1 \leq n \leq k-1$ is called (k,p)-Fibonacci sequence, where $k \geq 2$, $n \geq 0$ are integers and $p \geq 1$ is a rational number. The sequence $\{F_{k,p}(n)\}$ contains some important sequences for some special values of k and p. For example, the Fibonacci sequence $\{F_n\}$ and the Pell sequence $\{P_n\}$ are obtained from $\{F_{k,p}(n)\}$ for $k = 2, p = 1$, and $k = 2, p = \frac{3}{2}$, respectively (see, e.g., [2,11]).

The characteristic polynomial of $\{F_{k,p}(n)\}$ was given as

$$f_{k,p}(x) = x^k - px^{k-1} - (p-1)x - 1$$

in [14]. For detailed information about (k,p)-Fibonacci sequences, see, e.g, [2,3,11,14].

There are important relations between some special number sequences and matrices. For example, the relation $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for all integers n, where $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The Q-matrix is a well-known matrix and termed as the Fibonacci Q-matrix [7,9]. It is a well-known fact that the characteristic equation of the Q-matrix is $x^2 - x - 1 = 0$. Since every square matrix satisfies its own characteristic equation, it is clear that $Q^2 - Q - I = \mathbf{0}$. Therefore, it is natural to question whether there are other matrices X that satisfy the equation $X^2 - X - I = \mathbf{0}$ and whose powers are related to Fibonacci numbers. In response to this question, a relation between the powers of square matrices X satisfying the condition $X^2 - X - I = \mathbf{0}$ and the Fibonacci sequence $\{F_n\}$ was shown in [7]. Inspired by the study just mentioned, a relation between the powers of the matrices X satisfying the condition $X^2 - pX - qI = \mathbf{0}$ with the generalized Fibonacci sequence $\{U_n(p,q)\}$ was established in [13]. Then, in Theorem 2.1 of [10], the authors obtained some relations between the positive powers of 3×3 matrices with eigenvalues

$$\alpha = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad \beta = \frac{p - \sqrt{p^2 + 4q}}{2},$$

and r, and generalized Fibonacci sequence $\{U_n(p,q)\}$, where r is any real number with $r^2 - rp - q \neq 0$. Moreover, it was pointed out that if $r \neq 0$, then the mentioned theorem is valid not only for positive powers of the matrices but also for all integer powers. So, we have the following theorem:

Theorem 1.1. *If* $A = \begin{pmatrix} a & b & 0 \\ \frac{ap-a^2+q}{b} & p-a & 0 \\ \frac{br-ab-ap+a^2-q}{b} & r+a-b-p & r \end{pmatrix}$, *then*

$$A^n = \begin{pmatrix} aU_n + qU_{n-1} & bU_n & 0 \\ \frac{ap-a^2+q}{b}U_n & (p-a)U_n + qU_{n-1} & 0 \\ \frac{-ab-ap+a^2-q}{b}U_n - qU_{n-1} + r^n & (a-b-p)U_n - qU_{n-1} + r^n & r^n \end{pmatrix}$$

for all $n \in \mathbb{Z}$*, where* a *is any real number and* b, p, q, r *are any nonzero real numbers with* $p^2 + 4q > 0$ *and* $r^2 - rp - q \neq 0$.

Just like Fibonacci sequences, it is possible to talk about some relations between (k, p)-Fibonacci sequences and matrices. In [2], the author introduced the (k, p)-Fibonacci matrix Q_k as follows: $Q_k = [q_{ij}]_{k \times k}$, where for a fixed $1 \leq i \leq k$, an entry q_{i1} is equal to coefficient of $F_{k,p}(n-i)$ in the equality (1), and for $j \geq 2$, $q_{ij} = \begin{cases} 1, \text{if} j = i+1 \\ 0, \text{otherwise} \end{cases}$.

For $k = 2, 3, 4$, the matrices

$$Q_2 = \begin{pmatrix} 2p-1 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} p & 1 & 0 \\ p-1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} p & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p-1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

are obtained.

In general, for $k > 2$, we have

$$Q_k = \begin{pmatrix} p & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{k \times k},$$

which is called a (k, p)- Fibonacci matrix. In [2], the author also gave the matrix

$$A_k = \begin{pmatrix} F_{k,p}(2k-2) & F_{k,p}(2k-3) & \cdots & F_{k,p}(k-1) \\ F_{k,p}(2k-3) & F_{k,p}(2k-4) & \cdots & F_{k,p}(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,p}(k-1) & F_{k,p}(k-2) & \cdots & F_{k,p}(0) \end{pmatrix}_{k \times k},$$

which is the matrix of initial conditions of (k, p)-Fibonacci sequences. And then, the author showed that

$$A_k Q_k^n = \begin{pmatrix} F_{k,p}(n+2k-2) & F_{k,p}(n+2k-3) & \cdots & F_{k,p}(n+k-1) \\ F_{k,p}(n+2k-3) & F_{k,p}(n+2k-4) & \cdots & F_{k,p}(n+k-2) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,p}(n+k-1) & F_{k,p}(n+k-2) & \cdots & F_{k,p}(n) \end{pmatrix}_{k \times k}$$

for all the integers $n \geq 1$, where $k \geq 2$ is any integer and $p \geq 1$ is a rational number.

In this study, we establish some relations between the powers of any square matrices X satisfying the condition $X^k - pX^{k-1} - (p-1)X - I = \mathbf{0}$ and the sequence $\{F_{k,p}(n)\}$. Then some relations between $\{F_{3,s}(n)\}$ and $\{U_n(p,q)\}$ are obtained.

2 Results

In this section, we deal with the problem of finding the powers of the matrices X satisfying the condition $X^k - pX^{k-1} - (p-1)X - I = \mathbf{0}$ via (k,p)-Fibonacci numbers. To do so, we examine the main result separately for $k = 2$ and $k \geq 3$. Now, we give the result for $k = 2$.

Theorem 2.1. *If X is a square matrix with $X^2 = (2p-1)X + I$, then*

$$X^m = F_{2,p}(m)X + F_{2,p}(m-1)I \tag{2}$$

for all integers $m \geq 2$.

Proof. First of all, if we use the relation (1) for $k = 2$, then we get

$$F_{2,p}(m) = (2p-1)F_{2,p}(m-1) + F_{2,p}(m-2). \tag{3}$$

Now, we will use mathematical induction to prove the theorem. First, we will show that the equality (2) is true for $m = 2$. Then, assuming that the assertion is true for some integers $m \geq 2$, we will show that it is also true for $m+1$. It is clear that $F_{2,p}(1) = 1$ and $F_{2,p}(2) = 2p - 1$. So, we get

$$X^2 = (2p-1)X + I = F_{2,p}(2)X + F_{2,p}(1)I,$$

as desired. Now, assume that the relation (2) is true for some integers $m \geq 2$. We have to show that the relation (2) is true for $m+1$, too. It is obvious that

$$X^{m+1} = F_{2,p}(m)X^2 + F_{2,p}(m-1)X \tag{4}$$

since $X^m = F_{2,p}(m)X + F_{2,p}(m-1)I$.

If we use the equalities $X^2 = (2p-1)X + I$ and (3) in (4), then

$$X^{m+1} = F_{2,p}(m+1)X + F_{2,p}(m)I$$

is obtained and the proof is completed. □

Now, we give the general result for the integers $k \geq 3$.

Theorem 2.2. *If X is a square matrix satisfying $X^k - pX^{k-1} - (p-1)X - I = \mathbf{0}$ for an integer $k \geq 3$, then*

$$\begin{aligned} X^m &= F_{k,p}(m-k+2)X^{k-1} + F_{k,p}(m-k+1)I \\ &\quad + \sum_{i=1}^{k-2}[(p-1)F_{k,p}(m-k+2-i) + F_{k,p}(m-k+1-i)]X^i \end{aligned}$$

for all integers $m \geq k$, provided that $F_{k,p}(a) = 0$ for all negative integers a.

Proof. We will use mathematical induction on m to prove the theorem. First, we will show that the assertion is true for $m = k$. It is clear that

$$\sum_{i=1}^{k-2}[(p-1)F_{k,p}(2-i) + F_{k,p}(1-i)]X^i = (p-1)X, \tag{5}$$

because $F_{k,p}(0) = F_{k,p}(-1) = F_{k,p}(-2) = \cdots = 0$. Taking into account the equalities (5) and $m = k$, we get

$$\begin{aligned} X^m &= pX^{k-1} + I + (p-1)X \\ &= F_{k,p}(2)X^{k-1} + F_{k,p}(1)I + \sum_{i=1}^{k-2}[(p-1)F_{k,p}(2-i) + F_{k,p}(1-i)]X^i \\ &= F_{k,p}(m-k+2)X^{k-1} + F_{k,p}(m-k+1)I \\ &\quad + \sum_{i=1}^{k-2}[(p-1)F_{k,p}(m-k+2-i) + F_{k,p}(m-k+1-i)]X^i. \end{aligned}$$

So, the assertion is true for $m = k$. Now assume that the assertion is true for some integers $m \geq k$. We will show that it is true for $m+1$, too. Considering the equality $X^k = pX^{k-1} + (p-1)X + I$ together with the induction hypothesis, after simplification, we get

$$\begin{aligned} X^{m+1} &= pF_{k,p}(m-k+2)X^{k-1} + [(p-1)F_{k,p}(m-k+2) + F_{k,p}(m-k+1)]X \\ &\quad + F_{k,p}(m-k+2)I + \sum_{i=1}^{k-2}[(p-1)F_{k,p}(m-k+2-i) + F_{k,p}(m-k+1-i)]X^{i+1}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} X^{m+1} &= pF_{k,p}(m-k+2)X^{k-1} + F_{k,p}(m-k+2)I \\ &\quad + \sum_{i=0}^{k-2}[(p-1)F_{k,p}(m-k+2-i) + F_{k,p}(m-k+1-i)]X^{i+1}. \end{aligned} \tag{6}$$

On the other hand, there are two different cases by the definition of the (k,p)-Fibonacci sequence. If $m \geq 2k-3$, then it is seen that

$$F_{k,p}(m-k+3) = pF_{k,p}(m-k+2) + (p-1)F_{k,p}(m-2k+4) + F_{k,p}(m-2k+3). \tag{7}$$

If (7) is used in (6), then

$$\begin{aligned} X^{m+1} &= F_{k,p}(m-k+3)X^{k-1} + [-(p-1)F_{k,p}(m-2k+4) - F_{k,p}(m-2k+3)]X^{k-1} \\ &\quad + F_{k,p}(m-k+2)I + \sum_{i=0}^{k-2}[(p-1)F_{k,p}(m-k+2-i) + F_{k,p}(m-k+1-i)]X^{i+1} \end{aligned}$$

is obtained. Also, since

$$
\begin{aligned}
&\sum_{i=0}^{k-2}[(p-1)F_{k,p}(m-k+2-i)+F_{k,p}(m-k+1-i)]X^{i+1} \\
&-[(p-1)F_{k,p}(m-2k+4)+F_{k,p}(m-2k+3)]X^{k-1} \\
=&\sum_{i=0}^{k-3}[(p-1)F_{k,p}(m-k+2-i)+F_{k,p}(m-k+1-i)]X^{i+1} \\
=&\sum_{i=1}^{k-2}[(p-1)F_{k,p}(m-k+3-i)+F_{k,p}(m-k+2-i)]X^{i},
\end{aligned}
$$

we get

$$
\begin{aligned}
X^{m+1}=&F_{k,p}(m-k+3)X^{k-1}+F_{k,p}(m-k+2)I \\
&+\sum_{i=1}^{k-2}[(p-1)F_{k,p}(m-k+3-i)+F_{k,p}(m-k+2-i)]X^{i}.
\end{aligned}
$$

So, the assertion is true for $m+1$ in case $m \geq 2k-3$.

If $m < 2k-3$, then it is clear that

$$F_{k,p}(m-k+3)=pF_{k,p}(m-k+2)$$

by the definition of (k,p)-Fibonacci sequence. If we use the last equality in (6), then we get

$$
\begin{aligned}
X^{m+1}=&F_{k,p}(m-k+3)X^{k-1}+F_{k,p}(m-k+2)I \\
&+\sum_{i=0}^{k-2}[(p-1)F_{k,p}(m-k+2-i)+F_{k,p}(m-k+1-i)]X^{i+1}.
\end{aligned}
$$

Also, we have $F_{k,p}(m-2k+4)=F_{k,p}(m-2k+3)=0$ because $m < 2k-3$. So,

$$
\begin{aligned}
&\sum_{i=0}^{k-2}[(p-1)F_{k,p}(m-k+2-i)+F_{k,p}(m-k+1-i)]X^{i+1} \\
=&\sum_{i=1}^{k-1}[(p-1)F_{k,p}(m-k+3-i)+F_{k,p}(m-k+2-i)]X^{i} \\
=&\sum_{i=1}^{k-2}[(p-1)F_{k,p}(m-k+3-i)+F_{k,p}(m-k+2-i)]X^{i},
\end{aligned}
$$

that is

$$
\begin{aligned}
X^{m+1}=&F_{k,p}(m-k+3)X^{k-1}+F_{k,p}(m-k+2)I \\
&+\sum_{i=1}^{k-2}[(p-1)F_{k,p}(m-k+3-i)+F_{k,p}(m-k+2-i)]X^{i}
\end{aligned}
$$

is obtained in the case when $m < 2k-3$. Thus, the proof is completed. □

Corollary 2.2.1. *If* $\lambda \in \{\lambda_1, \lambda_2\}$ *with* $\lambda_1 = \frac{2p-1+\sqrt{(2p-1)^2+4}}{2}$ *and* $\lambda_2 = \frac{2p-1-\sqrt{(2p-1)^2+4}}{2}$*, where* $p \geq 1$ *is a rational number, then* $\lambda^m = F_{2,p}(m)\lambda + F_{2,p}(m-1)$ *for all the integers* $m \geq 2$*.*

Proof. Notice that λ is a root of the equation $x^2 - (2p-1)x - 1 = 0$. So, the matrix $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ satisfies the equation $A^2 = (2p-1)A + I$.

Hence, from Theorem 2.1, $A^m = F_{2,p}(m)A + F_{2,p}(m-1)I$, and therefore the desired result is obtained. □

Corollary 2.2.2. *If* λ *is a root of the equation* $x^k - px^{k-1} - (p-1)x - 1 = 0$*, then*

$$\begin{aligned} \lambda^m &= F_{k,p}(m-k+2)\lambda^{k-1} + F_{k,p}(m-k+1) \\ &\quad + \sum_{i=1}^{k-2} [(p-1)F_{k,p}(m-k+2-i) + F_{k,p}(m-k+1-i)]\lambda^i \end{aligned}$$

for all integers $m \geq k$*, where* $k \geq 3$ *is an integer,* $p \geq 1$ *is a rational number, and* $F_{k,p}(a) = 0$ *for all negative integers* a*.*

Proof. Let

$$B = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix}_{k \times k}.$$

If we use Theorem 2.2 for matrix B, the desired result is obtained. □

Corollary 2.2.3. *If*

$$Q_k = \begin{pmatrix} p & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{k \times k},$$

then

$$\begin{aligned} Q_k^m &= F_{k,p}(m-k+2)Q_k^{k-1} + F_{k,p}(m-k+1)I \\ &\quad + \sum_{i=1}^{k-2} [(p-1)F_{k,p}(m-k+2-i) + F_{k,p}(m-k+1-i)]Q_k^i \end{aligned}$$

for all integers $m \geq k$*, where* $k \geq 3$ *is an integer,* $p \geq 1$ *is a rational number, and* $F_{k,p}(a) = 0$ *for all negative integers* a*.*

Proof. It is easily seen that the characteristic polynomial of the matrix Q_k is

$$\det(Q_k - xI) = (-1)^k[x^k - px^{k-1} - (p-1)x - 1].$$

We can write $Q_k^k - pQ_k^{k-1} - (p-1)Q_k - I = \mathbf{0}$ by the Cayley–Hamilton theorem. So, the matrix Q_k satisfies the condition in Theorem 2.2 and the desired result is obtained. □

In Corollary 2.2.3, it has been obtained a relation between the powers of Q_k and the (k,p)-Fibonacci numbers. Notice that the powers of Q_k for $m \geq k$ are a linear combination of the matrices $Q_k^{k-1}, Q_k^{k-2}, \ldots, Q_k$, and I.

Now, we give a special result for $k = 3$.

Corollary 2.2.4. *If*

$$Q_3 = \begin{pmatrix} p & 1 & 0 \\ p-1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with $p \geq 1$ being a rational number, then

$$Q_3^m = \begin{pmatrix} F_{3,p}(m+1) & F_{3,p}(m) & F_{3,p}(m-1) \\ (p-1)F_{3,p}(m) + F_{3,p}(m-1) & \begin{matrix}(p-1)F_{3,p}(m-1)\\ +F_{3,p}(m-2)\end{matrix} & \begin{matrix}(p-1)F_{3,p}(m-2)\\ +F_{3,p}(m-3)\end{matrix} \\ F_{3,p}(m) & F_{3,p}(m-1) & F_{3,p}(m-2) \end{pmatrix}$$

for all integers $m \geq 3$.

Proof. If we use Corollary 2.2.3 for $k = 3$, then we get

$$Q_3^m = F_{3,p}(m-1)Q_3^2 + [(p-1)F_{3,p}(m-2) + F_{3,p}(m-3)]Q_3 + F_{3,p}(m-2)I. \qquad (8)$$

Also, it is clear that

$$F_{3,p}(m) = pF_{3,p}(m-1) + (p-1)F_{3,p}(m-2) + F_{3,p}(m-3) \qquad (9)$$

by definition of (k,p)-Fibonacci numbers. From (8) and (9), the desired result is obtained. □

Corollary 2.2.5. *For any integers $a, b \geq 3$ and a rational number $p \geq 1$, the relation $F_{3,p}(a+b) = F_{3,p}(a+1)F_{3,p}(b) + (p-1)F_{3,p}(a)F_{3,p}(b-1) + F_{3,p}(a)F_{3,p}(b-2) + F_{3,p}(a-1)F_{3,p}(b-1)$ holds.*

Proof. If we use Corollary 2.2.4, then we can calculate the matrices Q_3^a, Q_3^b and Q_3^{a+b}, easily. In view of the equality $Q_3^{a+b} = Q_3^a Q_3^b$, the desired result is obtained from the equality of the $(1,2)$-entries of the matrices. □

Some relations between $\{F_{k,p}(n)\}$ and other sequences for some special values of k can be examined. We will find some relations between the sequences $\{F_{3,s}(n)\}$ and $\{U_n(p,q)\}$.

Theorem 2.3. *Let p, q and r be nonzero real numbers satisfying the conditions $p = \frac{q^2+q+1}{q-1}$, $r = -\frac{1}{q}$, $p^2 + 4q > 0$ and $r^2 - rp - q \neq 0$. Let $s \geq 1$ be a rational number satisfying the condition $s = \frac{q^3+q^2+1}{q(q-1)}$. Then, there are following identities for the integers $m \geq 3$:*

(i) $qF_{3,s}(m-1) + F_{3,s}(m-2) = qU_{m-1}$;

(ii) $pF_{3,s}(m-1) + (s-1)F_{3,s}(m-2) + F_{3,s}(m-3) = U_m$.

Proof. Let

$$B = \begin{pmatrix} 0 & 1 & 0 \\ q & p & 0 \\ r-q & r-p-1 & r \end{pmatrix}$$

with $p = \dfrac{q^2+q+1}{q-1}$ and $r = -\dfrac{1}{q}$. The matrix B is a special case of the matrix A in Theorem 1.1 for $a = 0$ and $b = 1$. So, we get

$$B^m = \begin{pmatrix} qU_{m-1} & U_m & 0 \\ qU_m & U_{m+1} & 0 \\ -qU_m - qU_{m-1} + r^m & -U_m - U_{m+1} + r^m & r^m \end{pmatrix} \tag{10}$$

for all $m \in \mathbb{Z}$. Also, it is seen that

$$B^3 - (r+p)B^2 - (-rp+q)B + qrI = \mathbf{0}$$

or, equivalently,

$$B^3 - (\tfrac{q^3+q^2+1}{q(q-1)})B^2 - (\tfrac{q^3+q^2+1}{q(q-1)} - 1)B - I = \mathbf{0}.$$

So, we get

$$B^3 - sB^2 - (s-1)B - I = \mathbf{0}.$$

Thus, we can use Theorem 2.2 for $k = 3$. By doing so,

$$B^m = F_{3,s}(m-1)B^2 + [(s-1)F_{3,s}(m-2) + F_{3,s}(m-3)]B + F_{3,s}(m-2)I \tag{11}$$

is obtained for $m \geq 3$. The desired results are obtained from the equality of the two matrices by considering (10) together with (11). □

Example 2.4. *Consider the sequences* $\{F_{3,\frac{13}{2}}(m)\}$ *and* $\{U_m(7,2)\}$*. For* $q = 2$*, it is clear that* $p = 7 = \dfrac{q^2+q+1}{q-1}$ *and* $s = \dfrac{13}{2} = \dfrac{q^3+q^2+1}{q(q-1)}$*. If we use the item (i) of Theorem 2.3, then we get*

$$2F_{3,\frac{13}{2}}(m-1) + F_{3,\frac{13}{2}}(m-2) = 2U_{m-1}(7,2)$$

for $m \geq 3$.

The (k,p)-Fibonacci sequence is a sequence of numbers defined quite recently. Many of the problems dealt with in other sequences can be worked out for this sequence as well. For example, as in [10], the problem of obtaining new matrices related to (k,p)-Fibonacci numbers can also be addressed.

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