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Revisiting some r-Fibonacci sequences and Hessenberg matrices

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Abstract: The relationship between different generalizations of Fibonacci numbers and matrices is common in the literature. However, the basic relation of such sequences with Hessenberg matrices is often not properly explored. In this work we revisit some classic results and present some applications in recent contexts.

Keywords: Hessenberg matrices, r-Fibonacci sequence, Recurrence relations, Determinant, Matrices.
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1 Introduction

Given the numbers $b_1, \ldots, b_r$, let us consider the sequence (a_n) defined by the homogeneous recurrence relation

$$a_n = p_{n,n-1}\, a_{n-1} + \cdots + p_{n,n-r}\, a_{n-r}\,, \tag{1}$$

for $n > r$, with initial conditions

$$a_1 = b_1\,, \ldots\,, a_r = b_r\,. \tag{2}$$

It is well-known that (a_n) can be given explicitly as the determinant of a Hessenberg matrix, namely,

$$a_n = \det \begin{pmatrix} b_1 & b_2 & \cdots & b_r & & & & \\ -1 & 0 & \cdots & 0 & p_{r+1,1} & & & \\ & -1 & \ddots & \vdots & \vdots & \ddots & & \\ & & \ddots & 0 & \vdots & & \ddots & \\ & & & -1 & p_{r+1,r} & \cdots & \cdots & p_{n,n-r} \\ & & & & -1 & \ddots & & \vdots \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & -1 & p_{n,n-1} \end{pmatrix}. \tag{3}$$

For details and general applications the reader is referred to [16, 19, 28, 29]. A general result can be found for example in [27, Theorem 4.20]. Furthermore, if we replace the -1s of the subdiagonal of the Hessenberg matrix defined in (3) by 1s, then a_n is the permanent of such matrix (cf. also [11]).

Using classical results on Hessenberg matrices it is possible to find in many instances explicit formulas or the generating function of (3). The next result is known as *Trudi's formula* [21, Ch. VII] and provides an explicit formula for the determinant of a Toeplitz-Hessenberg matrix.

Theorem 1.1. *Given a positive integer n, we have*

$$\det \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ -1 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & a_2 \\ & & -1 & a_1 \end{pmatrix} = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} a_1^{t_1}\cdots a_n^{t_n},$$

where

$$\binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$$

is the multinomial coefficient.

In 2013, Merca [19] gave an elegant proof of this equality.

Nonetheless, we can also obtain the generating function of these sequences. Namely, in [12] we may find the following result.

Theorem 1.2. *The generating function of the sequence* (d_n) *defined by the determinants*

$$d_n = \det \begin{pmatrix} b_0 & b_1 & \cdots & b_k & & & \\ -1 & c_1 & c_2 & \cdots & c_\ell & & \\ & -1 & c_1 & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & c_\ell \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & -1 & c_1 & c_2 \\ & & & & & -1 & c_1 \end{pmatrix}_{(n+1)\times(n+1)}$$

is

$$\frac{b_0 + b_1 z + \cdots + b_k z^k}{1 - c_1 z - \cdots - c_\ell z^\ell} = \sum_{n=0}^{\infty} d_n z^n .$$

The reader is also invited to read [14] for an earlier look and historical context. We observe that we are assuming that $k < \ell$.

The purpose of this note is to bring together a few recent results on generalizations of the Fibonacci sequence into a common ground using Hessenberg matrices and the above results. With this, we will be able to simplify several proofs and obtain new characterizations.

2 Generalized (k, r)-Pell numbers

Given a number k, the k-Pell sequence $(P_n^{(k)})$ is defined by the recurrence relation

$$P_n^{(k)} = 2P_{n-1}^{(k)} + kP_{n-2}^{(k)}, \quad \text{for } n \geqslant 2,$$

with initial conditions $P_0^{(k)} = 0$ and $P_1^{(k)} = 1$. An elegant explicit formula for the n-th k-Pell number is

$$P_n^{(k)} = (-i\sqrt{k})^{n-1} U_{n-1}\left(\frac{i}{\sqrt{k}}\right),$$

where i stands for the imaginary unit and $(U_n(x))$ is the sequence of the Chebyshev polynomials of the second kind (cf. [3]). In [17], for a fixed integer $r > 2$, it was defined the generalized (k, r)-Pell numbers $(P_n^{(k,r)})$ given by the recurrence relation

$$P_n^{(k,r)} = kP_{n-2}^{(k,r)} + 2P_{n-r}^{(k,r)}, \quad \text{for } n \geqslant r,$$

with initial conditions $P_n^{(k,r)} = 1$, for $n = 0, 1, \ldots, r-1$ (cf. [2]). We note that, contrary to what the name might suggest, we do not obtain from them either the Pell or the k-Pell numbers.

Taking into account (1)–(2), from (3), we have

$$
\begin{aligned}
P_{n-1}^{(k,r)} &= \det \begin{pmatrix}
1 & 1 & \cdots & 1 & & & & \\
-1 & 0 & \cdots & 0 & 2 & & & \\
 & -1 & \ddots & \vdots & & 2 & & \\
 & & \ddots & 0 & k & & \ddots & \\
 & & & -1 & 0 & k & & 2 \\
 & & & & -1 & \ddots & \ddots & \\
 & & & & & \ddots & \ddots & k \\
 & & & & & & -1 & 0
\end{pmatrix}_{n\times n} \qquad (4) \\
&= \det \begin{pmatrix}
1 & 1 & 1-k & \cdots & 1-k & & & \\
-1 & 0 & k & & & 2 & & \\
 & -1 & \ddots & \ddots & & & \ddots & \\
 & & \ddots & \ddots & \ddots & & & 2 \\
 & & & \ddots & 0 & k & & \\
 & & & & -1 & \ddots & \ddots & \\
 & & & & & \ddots & \ddots & k \\
 & & & & & & -1 & 0
\end{pmatrix}_{n\times n}, \qquad (5)
\end{aligned}
$$

where the first row of the matrix in (4) contains r consecutive 1s, while in the first row of the matrix in (5) the two 1s are followed by $r-2$ entries $(1-k)$.

Remark 2.1. *We notice that*

$$
P_6^{(k,3)} = \det \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & -1 & 0 & k & 2 & 0 & 0 \\
0 & 0 & -1 & 0 & k & 2 & 0 \\
0 & 0 & 0 & -1 & 0 & k & 2 \\
0 & 0 & 0 & 0 & -1 & 0 & k \\
0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix} = k^2 + 4k + 4\,.
$$

Therefore there is a typographical error in [17, p. 768].

In [17, Theorem 2.2], it is stated that for an even $r = 2t$ we always have

$$P_{2n}^{(k,2t)} = P_{2n+1}^{(k,2t)}\,.$$

However, if we eliminate the penultimate row and column of the Hessenberg matrix and then applying the same procedure to the obtained submatrices, we reach to the conclusion that

$$P_{2n}^{(k,2t)} = P_{2n+1}^{(k,2t)} = \tilde{P}_n^{(k,t)}\,,$$

where $(\tilde{P}_n^{(k,t)})$ is the sequence defined by the recurrence relation

$$\tilde{P}_n^{(k,t)} = k\tilde{P}_{n-1}^{(k,t)} + 2\tilde{P}_{n-t}^{(k,t)}\,, \quad \text{for } n \geqslant t,$$

with initial conditions $\tilde{P}_n^{(k,t)} = 1$, for $n = 0, 1, \ldots, t-1$. Ultimately, from [3, 26], for $t = 2$, we have the explicit formula

$$\tilde{P}_n^{(k,2)} = (-i\sqrt{2})^{n-1} \left(U_{n-1}\left(\frac{ik}{2\sqrt{2}}\right) + i\sqrt{2}U_{n-2}\left(\frac{ik}{2\sqrt{2}}\right) \right).$$

Regarding the generating function of $(P_n^{(k,r)})$, in [17] it is claimed the following:

$$\frac{1+z}{1-kz^2-2z^r}.$$

Nevertheless, this is indeed the generating function of the sequence $(R_n^{(k,r)})$ of determinants defined by

$$\begin{aligned} R_{n-1}^{(k,r)} &= \det \begin{pmatrix} 1 & 1 & & & & & & & \\ -1 & 0 & k & & 2 & & & & \\ & -1 & \ddots & \ddots & & 2 & & & \\ & & \ddots & 0 & k & & \ddots & & \\ & & & -1 & 0 & k & & 2 \\ & & & & -1 & \ddots & \ddots & \\ & & & & & \ddots & \ddots & k \\ & & & & & & -1 & 0 \end{pmatrix}_{n\times n} \\ &= \det \begin{pmatrix} 1 & 1 & k & \cdots & k & & & \\ -1 & 0 & \cdots & \cdots & 0 & 2 & & \\ & -1 & \ddots & & \vdots & & \ddots & \\ & & \ddots & \ddots & \vdots & & & 2 \\ & & & \ddots & 0 & k & & \\ & & & & -1 & 0 & \ddots & \\ & & & & & \ddots & \ddots & k \\ & & & & & & -1 & 0 \end{pmatrix}_{n\times n}. \end{aligned}$$

This means that $(R_n^{(k,r)})$ is defined by the same recurrence relation as $(P_n^{(k,r)})$ but now with initial conditions $R_0^{(k,r)} = R_1^{(k,r)} = 1$ and $R_n^{(k,r)} = k$, for $n = 2, \ldots, r-1$. For example,

$$R_6^{(k,3)} = \det \begin{pmatrix} 1 & 1 & k & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & k & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & k & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & k & 2 \\ 0 & 0 & 0 & 0 & -1 & 0 & k \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} = k^3 + 4k + 4.$$

In fact, from Theorem 1.2 and (5), we readily obtain the correct generating function.

Proposition 2.1. *The generating function of the sequence* $(P_n^{(k,r)})$ *is*

$$\frac{1+z-(k-1)z^2-\cdots-(k-1)z^{r-1}}{1-kz^2-2z^r}\,.$$

Finally, for the generalized (k,r)-Pell-Lucas numbers defined in [17], we observe that they differ from the (k,r)-Pell numbers by the initial conditions where the 1 is replaced by 2. So, in the Hessenberg matrix of (4), we replace in the first row of 1s by 2s. This means that the (k,r)-Pell-Lucas sequence is $(2P_n^{(k,r)})$.

3 Generalized bivariate Fibonacci sequence

In this section, for a given integer $r>1$, we consider the sequence (u_n) defined by the recurrence relation

$$u_n = x\,u_{n-1} + y\,u_{n-r-1}\,, \quad \text{for } n>r,$$

with initial conditions

$$u_0 = a\,, u_1 = x\,, \ldots\,, u_r = x^r\,.$$

This sequence was studied in [1,6,25], setting $a=s+1$ and other values. The distance Fibonacci polynomials sequence considered recently in [5] is a particular case of this generalized bivariate Fibonacci sequence, as well. Other particular examples can be found in [7].

Taking into account (3), we obtain

$$u_{n-1} = \det\begin{pmatrix} a & x & \cdots & x^r & & & \\ -1 & 0 & \cdots & 0 & y & & \\ & -1 & \ddots & \vdots & & \ddots & \\ & & \ddots & 0 & & & y \\ & & & -1 & x & & \\ & & & & \ddots & \ddots & \\ & & & & & -1 & x \end{pmatrix}_{n\times n} \tag{6}$$

$$= \det\begin{pmatrix} a & (1-a)x & & & & & \\ -1 & x & & & y & & \\ & -1 & \ddots & & & \ddots & \\ & & \ddots & x & & & y \\ & & & -1 & x & & \\ & & & & \ddots & \ddots & \\ & & & & & -1 & x \end{pmatrix}_{n\times n}\,, \tag{7}$$

where the second equality is derived from the application of elementary operations over the first $r+1$ columns of the Hessenberg matrix defined in (6). Now, from Theorem 1.2, $\frac{a+(1-a)xz}{1-xz-yz^{r+1}}$ is the generating function of (u_n), i.e.,

$$\sum_{n=0}^{\infty} u_n z^n = \frac{a+(1-a)xz}{1-xz-yz^{r+1}},$$

which corresponds to [1, Theorem 2].

Furthermore, if we make $a = y = 1$ and $r + 1 = k$, we get the generating function of the so-called distance Fibonacci polynomials considered recently in [6].

The same approach can naturally be generalized to

$$v_n = x\, v_{n-s} + y\, v_{n-r-1}\,, \quad \text{for } n > r > s > 0,$$

with initial conditions

$$v_0 = a\,, v_1 = x\,, \ldots\,, v_r = x^r\,.$$

In this general case, we have

$$\frac{a + xz + \cdots + x^{s-1}z^{s-1} + x(x^{s-1} - a)z^s + x^2(x^{s-1} - 1)z^{s+1} + \cdots + x^{r-s}(x^{s-1} - 1)z^r}{1 - xz^s - yz^{r+1}}$$

as the generating function of (v_n). We assume that this identity is probably not new and that it can be found in the literature.

At last, if to the first row of the matrix defined in (7), we add a times the second row and after apply a simple property of the determinants, we have

$$\begin{aligned} u_{n-1} &= \det\begin{pmatrix} 0 & x & & & ay & & \\ -1 & x & & & y & & \\ & -1 & \ddots & & & \ddots & \\ & & \ddots & x & & & y \\ & & & -1 & x & & \\ & & & & \ddots & \ddots & \\ & & & & & -1 & x \end{pmatrix}_{n\times n} \\ &= \det\begin{pmatrix} x & & & ay & & \\ -1 & \ddots & & & y & \\ & \ddots & x & & & \ddots \\ & & -1 & x & & & y \\ & & & -1 & x & & \\ & & & & \ddots & \ddots & \\ & & & & & -1 & x \end{pmatrix}_{(n-1)\times(n-1)} . \end{aligned}$$

The last equality is the same as the one obtained in [6, Theorem 2.5], after transposition. We observe that the other matricial identities that we find in [6] can be established by the right and left product of the above matrix with convenient diagonal matrices with powers on the the imaginary unit on the main diagonal.

4 A complex Fibonacci k-step sequence

For a given integer number $k \geqslant 2$, the k-step (or k-generalized) Fibonacci sequence $(F_n^{(k)})$ is defined by letting $F_0^{(k)} = \cdots = F_{k-2}^{(k)} = 0$, $F_{k-1}^{(k)} = 1$ and the remaining terms according to the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)}\,, \quad \text{for } n \geqslant k.$$

Obviously, for $k = 2$ we have the Fibonacci numbers.

We can find in the literature several matricial representations and explicit formulas for $(F_n^{(k)})$ [10, 20, 22]. Using (3) and then some elementary row operations, we have

$$\begin{aligned} F_{n-1}^{(k)} &= \det \begin{pmatrix} 0 & \cdots & 0 & 1 & & & \\ -1 & 0 & \cdots & 0 & 1 & & \\ & \ddots & \ddots & \vdots & 1 & \ddots & \\ & & -1 & 0 & \vdots & \ddots & 1 \\ & & & -1 & 1 & \ddots & 1 \\ & & & & \ddots & \ddots & \vdots \\ & & & & & -1 & 1 \end{pmatrix}_{n\times n} \\ &= \det \begin{pmatrix} 1 & \cdots & 1 & & \\ -1 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ & & \ddots & \ddots & \vdots \\ & & & -1 & 1 \end{pmatrix}_{(n-k)\times(n-k)} . \end{aligned}$$

As for the last equality, we are assuming that $n > k$. Therefore, from Theorem 1.2, the generating function of the sequence $(F_n^{(k)})$, after a convenient shift, is

$$\frac{1 + z + \cdots + z^{k-1}}{1 - z - \cdots - z^k} = \sum_{n=0}^{\infty} F_{n+k}^{(k)} z^n .$$

Moreover, from Theorem 1.1, we have the explicit formula

$$F_{n+k}^{(k)} = \sum_{t_1+2t_2+\cdots+kt_k=n} \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} .$$

Remark 4.1. *We note that using the same procedures as in the previous sections, it is not difficult to see that*

$$\frac{z^{k-1}}{1 - z - \cdots - z^k} = \sum_{n=0}^{\infty} F_n^{(k)} z^n .$$

We use the above approach for convenience.

In [9], it is proposed a complex extension for the sequence $(F_n^{(k)})$, say $(Q_n^{(k)})$, defined by the recurrence relation

$$Q_n^{(k)} = iQ_{n-1}^{(k)} + i^2 Q_{n-2}^{(k)} + \cdots + i^k Q_{n-k}^{(k)} ,$$

for $n \geqslant k$, with the same initial conditions as $(F_n^{(k)})$. Thus,

$$Q_{n-1}^{(k)} = \det \begin{pmatrix} i & \cdots & i^k & & \\ -1 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & i^k \\ & & \ddots & \ddots & \vdots \\ & & & -1 & i \end{pmatrix}_{(n-k)\times(n-k)} ,$$

for $n > k$.

Let us set

$$D_n^{\downarrow} = \begin{pmatrix} 1 & & & & & \\ & i & & & & \\ & & -1 & & & \\ & & & -i & & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix} \quad \text{and} \quad D_n^{\uparrow} = \begin{pmatrix} \ddots & & & & & \\ & 1 & & & & \\ & & -i & & & \\ & & & -1 & & \\ & & & & i & \\ & & & & & 1 \end{pmatrix}.$$

If we define

$$A_n^{(k)} = \begin{pmatrix} i & \cdots & i^k & & \\ -1 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & i^k \\ & & \ddots & \ddots & \vdots \\ & & & -1 & i \end{pmatrix} \quad \text{and} \quad H_n^{(k)} = \begin{pmatrix} 1 & \cdots & 1 & & \\ -1 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ & & \ddots & \ddots & \vdots \\ & & & -1 & 1 \end{pmatrix},$$

we claim that

$$(-i)^n \, D_n^{\downarrow} \, A_n^{(k)} \, D_n^{\uparrow} = H_n^{(k)} .$$

Since

$$\det(D_n^{\downarrow} \, D_n^{\uparrow}) = i^{(n-1)n} ,$$

we conclude that

$$\det A_n^{(k)} = (-i)^{(n-1)n} \, i^{n^2} \det H_n^{(k)} = i^n \, \det H_n^{(k)} .$$

Notice that the powers of i in the above equalities do not depend on k. Hence, we may conclude that

$$Q_n^{(k)} = i^{n-k+1} \, F_n^{(k)} .$$

From Theorem 1.1 we also deduce the identity

$$\begin{aligned} Q_n^{(k)} &= \sum_{t_1+2t_2+\cdots+kt_k=n-k+1} \binom{t_1+\cdots+t_k}{t_1,\ldots,t_k} i^{t_1} i^{2t_2} \cdots i^{kt_k} \\ &= i^{n-k+1} \sum_{t_1+2t_2+\cdots+kt_k=n-k+1} \binom{t_1+\cdots+t_k}{t_1,\ldots,t_k} . \end{aligned}$$

5 Conclusion

This note has opened up a wide field of number theory where notation is a 'tool of thought' as described in [15], by linking generalized Fibonacci sequences with Hessenberg matrices. In the more general context of recursive sequences, these results can lead to extensions into the realm of binary sequence matrices and associated recurrence relations in order to stimulate some enrichment exercises and pattern puzzles by developing some new properties of known binary sequences [8], especially the 'good' sequences of Austin and Guy [4]. In this sense, these sorts of results are also useful in providing material for capstone subjects [13]. There are also close links to other generalized Q-matrices [18], and applications in combinatorics [24], and even in medicine [23].

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