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New congruences modulo powers of 2 for k-regular overpartition pairs

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Abstract: Let $\overline{B}_k(n)$ denote the number of k regular overpartition pairs where a k-regular overpartition pair of n is a pair of k-regular overpartitions (a, b) in which the sum of all the parts is n. Naika and Shivasankar (2017) proved infinite families of congruences for $\overline{B}_3(n)$ and $\overline{B}_4(n)$. In this paper, we prove infinite families of congruences modulo powers of 2 for $\overline{B}_{3\gamma}(n)$, $\overline{B}_{4\gamma}(n)$ and $\overline{B}_{6\gamma}(n)$.

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1 Introduction

A partition of a non-negative integer n is a non-increasing sequence of positive integers called parts, whose sum is equal to n. The number of partitions of a non-negative integer n is usually denoted by $p(n)$ (with $p(0) = 1$) and the generating function is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1}$$

where, for any complex number a,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1. \tag{2}$$

We will use the notation, for any positive integer t,

$$E_t := (q^t;q^t)_\infty. \tag{3}$$

An overpartition of a non-negative integer n is a partition of n in which the first occurrence of each part may be overlined. For example, there are 14 overpartitions of 4, namely

$$\begin{aligned}&4, \quad \overline{4},\\ &3+1, \quad \overline{3}+1, \quad 3+\overline{1}, \quad \overline{3}+\overline{1},\\ &2+2, \quad \overline{2}+2, \quad 2+1+1, \quad \overline{2}+1+1, \quad 2+\overline{1}+1, \quad \overline{2}+\overline{1}+1,\\ &1+1+1+1, \quad \overline{1}+1+1+1.\end{aligned}$$

If $\overline{p}(n)$ denotes the number of overpartition of n, then the generating function of $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{4}$$

An overpartition pair of a positive integer n was defined by Lovejoy [12] as a pair of overpartitions (a,b) where the sum of all the parts is n. Let $\overline{pp}(n)$ denote the total number of overpartition pair of n with $\overline{pp}(0)=1$, then the generating function is given by

$$\sum_{n=0}^{\infty}\overline{pp}(n)q^n = \frac{(-q;q)_\infty^2}{(q;q)_\infty^2}. \tag{5}$$

An account of $\overline{pp}(n)$ can be found in [3, 9, 11].

For any positive integer k, a k-regular partition of n is a partition in which no part is divisible by k. If $b_k(n)$ denotes the number of k-regular partitions of n (with $b_k(0)=1$), then

$$\sum_{n=0}^{\infty}b_k(n)q^n = \frac{E_k}{E_1}. \tag{6}$$

One may see [4, 8, 10] and the references therein for some properties of $b_k(n)$.

A k-regular overpartition pair of n is an overpartition pair (a,b) such that no part of a and b is divisible by k and their sum is n. If $\overline{B}_k(n)$ counts the number of k-regular overpartition pair of n, then its generating function [14] is given by

$$\sum_{n=0}^{\infty}\overline{B}_k(n)q^n = \frac{E_2^2E_k^4}{E_1^4E_{2k}^2}. \tag{7}$$

Naika and Shivasankar [14] proved infinite families of congruences modulo $3, 8, 16, 36, 48, 96$ for $B_3(n)$ and modulo $3, 16, 64, 96$ for $B_4(n)$. In this paper, we prove infinite families of congruences modulo powers of 2 for $\overline{B}_{3\gamma}(n)$, $\overline{B}_{4\gamma}(n)$ and $\overline{B}_{6\gamma}(n)$, where γ is any positive integer.

2 Some q-series identities

Ramanujan's general theta-function $f(a,b)$ [2, p. 34, (18.1)] is defined by

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \qquad |ab| < 1. \tag{8}$$

Three important special cases of $f(a,b)$ [2, p. 36, Entry 22 (i)–(iii)] are the theta-functions $\phi(q)$, $\psi(q)$ and $f(q)$ defined by

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \tag{9}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{10}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{11}$$

Lemma 2.1. *The following* 2*-dissections hold:*

$$\frac{1}{E_1^2} = \frac{E_8^5}{E_2^5 E_{16}^2} + 2q\frac{E_4^2 E_{16}^2}{E_2^5 E_8}, \tag{12}$$

$$\frac{1}{E_1^4} = \frac{E_4^{14}}{E_2^{14} E_8^4} + 4q\frac{E_4^2 E_8^4}{E_2^{10}}, \tag{13}$$

$$E_1^2 = \frac{E_2 E_8^5}{E_4^2 E_{16}^2} - 2q\frac{E_2 E_{16}^2}{E_8}, \tag{14}$$

$$\frac{E_3^3}{E_1} = \frac{E_4^3 E_6^2}{E_2^2 E_{12}} + q\frac{E_{12}^3}{E_4}. \tag{15}$$

Equation (12) can be derived from the 2-dissection of $\phi(q)$ [6, (1.9.4)]. Equation (13) can be derived from $\phi(q)^2$ [6, (1.10.1)]. Equation (14) can be derived from Equation (12) by substituting q by $-q$, respectively. Equation (15) can be derived from [6, (22.1.14)].

Lemma 2.2. [7] *The following* 3*-dissections hold:*

$$\frac{E_2}{E_1^2} = \frac{E_6^4 E_9^6}{E_3^8 E_{18}^3} + 2q\frac{E_6^3 E_9^3}{E_3^7} + 4q^2\frac{E_6^2 E_{18}^3}{E_3^6}. \tag{16}$$

Lemma 2.3. [15, Lemma 2.3] *We have*

$$E_1^3 = P(q^3) - 3qE_9^3, \tag{17}$$

where

$$P(q) = \sum_{m=-\infty}^{\infty} (-1)^m (6m+1) q^{m(3m+1)/2} = f(-q)\varphi(q)\varphi(q^3) + 4qf(-q)\psi(q^2)\psi(q^6).$$

Lemma 2.4. [2, p.36, Entry 22(iii)]

$$E_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{18}$$

Lemma 2.5. [1, Lemma 2.3] *For any prime $p \geq 3$, we have*

$$E_1^3 = \sum_{\substack{k=0 \\ k \neq (\pm p-1)/2}}^{(p-1)} (-1)^k q^{(k(k+1))/2} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn \cdot \frac{pn+2k+1}{2}}$$

$$+p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} E_{p^2}^3. \tag{19}$$

Furthermore, if $k \neq \dfrac{(p-1)}{2}, \quad 0 \leq k \leq p-1$, *then*

$$\frac{(k^2+k)}{2} \not\equiv \frac{(p^2-1)}{8} \pmod{p}.$$

Lemma 2.6. [4, Theorem 2.2] *For any prime $p \geq 5$, we have*

$$E_1 = \sum_{\substack{k=-(p-1)/2 \\ k \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right)$$
$$+ (-1)^{\frac{(\pm p-1)}{6}} q^{\frac{(p^2-1)}{24}} E_{p^2}, \tag{20}$$

where

$$\frac{\pm p-1}{6} = \begin{cases} \dfrac{(p-1)}{6}, & \text{if } p \equiv 1 \pmod{6}, \\ \dfrac{(-p-1)}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases}$$

Furthermore, if

$$\frac{-(p-1)}{2} \leq k \leq \frac{(p-1)}{2} \quad \text{and} \quad k \neq \frac{(\pm p-1)}{2},$$

then

$$\frac{(3k^2+k)}{2} \not\equiv \frac{(p^2-1)}{24} \pmod{p}.$$

In addition to the above $q-$series identities, we will be using the following congruence properties which follow from the binomial theorem: For any positive integer t and m,

$$E_t^{2m} \equiv E_{2t}^m \pmod{2}, \tag{21}$$

$$E_t^{4m} \equiv E_{2t}^{2m} \pmod{4}. \tag{22}$$

3 Congruences for $\overline{B}_{k\gamma}(n)$

3.1 Congruences for $\overline{B}_{3\gamma}(n)$

Theorem 3.1. *For all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{B}_{3\gamma}(2^{2\alpha}(3n+2)) q^n \equiv 4\frac{E_6^3}{E_2} \pmod{8}, \tag{23}$$

$$\overline{B}_{3\gamma}(2^{2\alpha}(6n+5)) \equiv 0 \pmod{8}. \tag{24}$$

Proof. Setting $k = 3\gamma$ in (7), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{3\gamma}(n)q^n = \frac{E_2^2 E_{3\gamma}^4}{E_1^4 E_{6\gamma}^2}. \tag{25}$$

Using (16) in (25), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{3\gamma}(n)q^n = \frac{E_{3\gamma}^4}{E_{6\gamma}^2} \left(\frac{E_6^4 E_9^6}{E_3^8 E_{18}^3} + 2q\frac{E_6^3 E_9^3}{E_3^7} + 4q^2 \frac{E_6^2 E_{18}^3}{E_3^6} \right)^2. \tag{26}$$

Extracting the terms involving q^{3n+2} from (26), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{3\gamma}(3n+2)q^n \equiv 4\frac{E_\gamma^4 E_2^6 E_3^6}{E_{2\gamma}^2 E_1^{14}} \pmod 8. \tag{27}$$

Using (21) in (27), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{3\gamma}(3n+2)q^n \equiv 4\left(\frac{E_6^3}{E_2}\right) \pmod 8, \tag{28}$$

which is the $\alpha = 0$ case of equation (23). Suppose that the congruence (23) is true for any integer $\alpha \geq 0$. Extracting the terms involving q^{2n} from both sides of (23), we arrive at

$$\sum_{n=0}^{\infty} \overline{B}_{3\gamma}(2^{2\alpha+1}(3n+1))q^n \equiv 4\frac{E_3^3}{E_1} \pmod 8. \tag{29}$$

Employing (15) in (29) and extracting the terms involving q^{2n+1}, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{3\gamma}(2^{2(\alpha+1)}(3n+2))q^n \equiv 4\frac{E_6^3}{E_2} \pmod 8, \tag{30}$$

which implies that (23) is true for $\alpha + 1$. Thus, by the principle of mathematical induction, (23) is true for all integers $\alpha \geq 0$. Extracting the terms involving q^{2n+1} from (23), we complete the proof of (24). □

Theorem 3.2. *For all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{B}_{3\gamma}(2^{2\alpha}(3n+1))q^n \equiv 4\frac{E_3^3}{E_1} \pmod 8, \tag{31}$$

$$\overline{B}_{3\gamma}(2^{2\alpha+1}(6n+5)) \equiv 0 \pmod 8. \tag{32}$$

Proof. Extracting the terms involving q^{3n+1} from (26), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{3\gamma}(3n+1)q^n \equiv 4\frac{E_\gamma^4 E_2^7 E_3^9}{E_{2\gamma}^2 E_1^{15} E_6^3} \pmod 8. \tag{33}$$

Using (21) in (33), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{3\gamma}(3n+1)q^n \equiv 4\frac{E_3^3}{E_1} \pmod 8, \tag{34}$$

which is the $\alpha = 0$ case of equation (31). Suppose that the congruence (31) is true for any integer $\alpha \geq 0$. Employing (15) in (31) and extracting the terms involving q^{2n+1}, we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{3\gamma}(2^{2\alpha+1}(3n+2))q^n \equiv 4\frac{E_6^3}{E_2} \pmod 8. \tag{35}$$

Extracting the terms involving q^{2n} from (35), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{3\gamma}(2^{2(\alpha+1)}(3n+1))q^n \equiv 4\frac{E_3^3}{E_1} \pmod 8, \tag{36}$$

which implies that (31) is true for $\alpha+1$. Thus, by the principle of mathematical induction, (31) is true for all integers $\alpha \geq 0$. Extracting the terms involving q^{2n+1} from (35), we complete the proof of (32). □

3.2 Congruences for $\overline{B}_{4\gamma}(n)$

Theorem 3.3. *Let $p \geq 5$ be a prime with $\left(\frac{-2}{p}\right) = -1$ and $1 \leq j \leq (p-1)$. Then for all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\overline{B}_{4\gamma}(4n+3) \equiv 0 \pmod{32}, \tag{37}$$

$$\overline{B}_{4\gamma}(8n+7) \equiv 0 \pmod{64}, \tag{38}$$

$$\sum_{n=0}^{\infty}\overline{B}_{4\gamma}\left(p^{2\alpha}(8n+3)\right)q^n \equiv 32E_1E_8 \pmod{64}, \tag{39}$$

$$\overline{B}_{4\gamma}\left(p^{2\alpha+1}(8(pn+j)+3p)\right) \equiv 0 \pmod{64}. \tag{40}$$

Proof. Setting $k=4\gamma$ in (7), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{4\gamma}(n)q^n = \frac{E_2^2E_{4\gamma}^4}{E_1^4E_{8\gamma}^2}. \tag{41}$$

Using (13) in (41), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{4\gamma}(n)q^n = \frac{E_2^2E_{4\gamma}^4}{E_{8\gamma}^2}\left(\frac{E_4^{14}}{E_2^{14}E_8^4}+4q\frac{E_4^2E_8^4}{E_2^{10}}\right). \tag{42}$$

Extracting the terms involving q^{2n+1} from (42), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{4\gamma}(2n+1)q^n = 4\frac{E_{2\gamma}^4E_2^2E_4^4}{E_{4\gamma}^2E_1^8}. \tag{43}$$

Using (13) in (43), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{4\gamma}(2n+1)q^n = 4\frac{E_{2\gamma}^4E_2^2E_4^4}{E_{4\gamma}^2}\left(\frac{E_4^{14}}{E_2^{14}E_8^4}+4q\frac{E_4^2E_8^4}{E_2^{10}}\right)^2. \tag{44}$$

Extracting the terms involving q^{2n+1} from (44), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{4\gamma}(4n+3)q^n = 32\frac{E_\gamma^4E_2^{20}}{E_{2\gamma}^2E_1^{22}}. \tag{45}$$

Employing (21) in (45), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}(4n+3)q^n \equiv 32E_2^9 \pmod{64} \tag{46}$$

from (46), we arrive at (37). Extracting the terms involving q^{2n+1} from (46), we complete the proof of (38).

Again, extracting the terms involving q^{2n} from (46) and using (21), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}(8n+3)q^n \equiv 32E_1E_8 \pmod{64}. \tag{47}$$

Congruence (47) is the $\alpha = 0$ case of (39). Suppose that congruence (39) is true for all $\alpha \geq 0$. Utilizing (20) in (39), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{B}_{4\gamma}\left(p^{2\alpha}(8n+3)\right)q^n \\
&\equiv 32\Bigg\{ \sum_{\substack{k=-(p-1)/2 \\ k\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right) \\
&\qquad + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} E_{p^2} \Bigg\} \\
&\times \Bigg\{ \sum_{\substack{m=-(p-1)/2 \\ m\neq(p-1)/6}}^{(p-1)/2} (-1)^m q^{4(3m^2+m)} f\left(-q^{4(3p^2+(6m+1)p)}, -q^{4(3p^2-(6m+1)p)}\right) \\
&\qquad + (-1)^{(\pm p-1)/6} q^{8(p^2-1)/24} E_{8p^2} \Bigg\} \pmod{64}.
\end{aligned} \tag{48}$$

Consider the congruence

$$\frac{(3k^2+k)}{2} + 4(3m^2+m) \equiv \frac{3(p^2-1)}{8} \pmod{p},$$

which is equal to

$$(6k+1)^2 + 2(12m+2)^2 \equiv 0 \pmod{p}.$$

For $\left(\frac{-2}{p}\right) = -1$, the above congruence has only solution $k = m = \left(\frac{\pm p - 1}{6}\right)$. Therefore, extracting the terms involving $q^{pn+3(p^2-1)/8}$ from both sides of (48), dividing throughout by $q^{3(p^2-1)/24}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}\left(p^{2\alpha+1}(8n+3p)\right)q^n \equiv 32E_pE_{8p} \pmod{64}. \tag{49}$$

Extracting the terms involving q^{pn} from (49) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}\left(p^{2\alpha+2}(8n+3)\right)q^n \equiv 32E_1E_8 \pmod{64}, \tag{50}$$

which is the $\alpha + 1$ case of (39). Thus, by the principle of mathematical induction, we arrive at (39). Extracting the coefficients of terms involving q^{pn+j} for $1 \leq j \leq p-1$, from both sides of (49), we complete the proof of (40). □

Remark 3.4. *Setting* $\gamma = 1$ *in* (46) *and extracting the terms involving* q^{2n+1}*, we arrive at the congruence* $\overline{B}_4(8n+7) \equiv 0 \pmod{64}$*, due to Naika and Shivasankar,* [14].

Theorem 3.5. *For all integers* $n \geq 0$ *and* $\alpha \geq 0$*, we have*

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}\left(4 \cdot 3^{2\alpha+2}n + 3^{2\alpha+2}\right) q^n \equiv E_1^6 \pmod{16}, \tag{51}$$

$$\overline{B}_{4\gamma}\left(4 \cdot 3^{2\alpha+2}n + 3^{2\alpha+2}\right) \equiv \overline{B}_{4\gamma}(4n+1) \pmod{16}, \tag{52}$$

$$\overline{B}_{4\gamma}\left(4 \cdot 3^{2\alpha+2}n + 7 \cdot 3^{2\alpha+1}\right) \equiv 0 \pmod{16}, \tag{53}$$

$$\overline{B}_{4\gamma}\left(4 \cdot 3^{2\alpha+2}n + 11 \cdot 3^{2\alpha+2}\right) \equiv 0 \pmod{16}. \tag{54}$$

Proof. From (44), we note that

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}(2n+1)q^n \equiv 4\frac{E_4^{32}}{E_2^{26}E_8^8} \pmod{16}. \tag{55}$$

Employing (22) in (55), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}(2n+1)q^n \equiv 4E_2^6 \pmod{16}. \tag{56}$$

Extracting the terms involving q^{2n} from (56), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}(4n+1)q^n \equiv 4(E_1^3)^2 \pmod{16}. \tag{57}$$

Employing (17) in (57), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}(4n+1)q^n \equiv 4(P(q^3) - 3qE_9^3)^2 \pmod{16}. \tag{58}$$

Extracting the terms involving q^{3n+2} from (58), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}(4 \cdot 3n + 3^2)q^n \equiv 4E_3^6 \pmod{16}. \tag{59}$$

Extracting the terms involving q^{3n} from (59), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}(4 \cdot 3^2n + 3^2)q^n \equiv 4E_1^6 \pmod{16}. \tag{60}$$

Congruence (60) is the $\alpha = 0$ case of (51). Suppose that congruence (51) is true for all $\alpha \geq 1$. Utilizing (17) in (51), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}\left(4 \cdot 3^{2\alpha+2}n + 3^{2\alpha+2}\right) q^n \equiv 4(P(q^3) - 3qE_9^3)^2 \pmod{16}. \tag{61}$$

Extracting the terms involving q^{3n+2} from (61), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}\left(4 \cdot 3^{2\alpha+3}n + 3^{2\alpha+2+2}\right) q^n \equiv 4E_3^6 \pmod{16}. \tag{62}$$

Extracting the terms involving q^{3n} from (62), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}\left(4 \cdot 3^{2(\alpha+1)+2}n + 3^{2(\alpha+1)+2}\right) q^n \equiv 4E_1^6 \pmod{16}. \tag{63}$$

which is $\alpha + 1$ case of (51). Thus, by the principle of mathematical induction, we complete the proof of (51). Comparing (57) and (60) we can simply arrive at (52). Suppose that congruence (53) and (54) is true for all $\alpha \geq 1$. Extracting the terms involving q^{3n+1} and q^{3n+2} from (62), we can conclude that (53) and (54) is also true for $\alpha + 1$. Thus, by the principle of mathematical induction, we complete the proof of (53) and (54). □

Remark 3.6. *Setting $\gamma = 1$ in Theorem 3.5, we arrive at the congruence of Theorem 4.3 due to Naika and Shivasankar,* [14].

Theorem 3.7. *Let $p \geq 3$ be a prime and $1 \leq j \leq p - 1$. Then for all integers $\alpha, n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}\left(p^{2\alpha}(8n+1)\right) q^n \equiv 4E_1^3 \pmod{8}, \tag{64}$$

$$\overline{B}_{4\gamma}\left(p^{2\alpha+1}(8(pn+j)+p)\right) \equiv 0 \pmod{8}. \tag{65}$$

Proof. From (55), we note that

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}(2n+1)q^n \equiv 4\frac{E_4^{32}}{E_2^{26}E_8^8} \pmod{8}. \tag{66}$$

Employing (21) in (66), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}(2n+1)q^n \equiv 4E_4^3 \pmod{8}. \tag{67}$$

Extracting the terms involving q^{4n} from (67), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}(8n+1)q^n \equiv 4E_1^3 \pmod{8}. \tag{68}$$

Congruence (68) is the $\alpha = 0$ case of (64). Suppose that congruence (64) is true for all integer $\alpha \geq 0$. Employing (19) in (64), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}\left(p^{2\alpha}(8n+1)\right) q^n \equiv 4\Big\{ \sum_{\substack{k=0 \\ k\neq(p-1)/2}}^{(p-1)} (-1)^k q^{k(k+1))/2} \sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{pn\cdot(pn+2k+1)/2}$$
$$+ p(-1)^{(p-1)/2}q^{(p^2-1)/8}E_{p^2}^3\Big\} \pmod{8}. \tag{69}$$

Extracting the term involving $q^{pn+(p^2-1)/8}$ from both sides of (69), dividing throughout by $q^{(p^2-1)/8}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}\left(p^{2\alpha+1}(8n+p)\right) q^n \equiv 4E_p^3 \pmod{8}. \tag{70}$$

Extracting the terms involving q^{pn} from (70) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{4\gamma}\left(p^{2(\alpha+1)}(8n+1)\right) q^n \equiv 4E_1^3 \pmod{8}, \tag{71}$$

which is the $\alpha+1$ case of (64). Thus, by the principle of mathematical induction, we arrive at (64). Extracting the coefficients of terms involving q^{pn+j} for $1 \le j \le p-1$, from both sides of (70), we complete the proof of (65). □

3.3 Congruences for $\overline{B}_{6\gamma}(n)$

Theorem 3.8. *Let $p \geq 3$ be a prime with $\left(\frac{-4}{p}\right) = -1$ and $1 \leq j \leq (p-1)$. Then for all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\overline{B}_{6\gamma}(4n+3) \equiv 0 \pmod{16}, \tag{72}$$

$$\sum_{n=0}^{\infty} \overline{B}_{6\gamma}\left(p^{2\alpha}(8n+5)\right) q^n \equiv E_1^3 E_4^3 \pmod{16}, \tag{73}$$

$$\overline{B}_{6\gamma}\left(p^{2\alpha+1}(8(pn+j)+5p)\right) \equiv 0 \pmod{16}. \tag{74}$$

Proof. Setting $k = 6\gamma$ in (7), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{6\gamma}(n) q^n = \frac{E_2^2 E_{6\gamma}^4}{E_1^4 E_{12\gamma}^2}. \tag{75}$$

Using (13) in (75), extracting the terms involving q^{2n+1} from (76), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{6\gamma}(n) q^n = \frac{E_2^2 E_{6\gamma}^4}{E_{12\gamma}^2} \left(\frac{E_4^{14}}{E_2^{14} E_8^4} + 4q \frac{E_4^2 E_8^4}{E_2^{10}} \right). \tag{76}$$

Extracting the terms involving q^{2n+1} from (76), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{6\gamma}(2n+1) q^n = 4 \frac{E_{3\gamma}^4 E_2^2 E_4^4}{E_{6\gamma}^2 E_1^8}. \tag{77}$$

Using (13) in (77), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{6\gamma}(2n+1) q^n \equiv 4 \frac{E_{3\gamma}^4 E_4^{32}}{E_{6\gamma}^2 E_2^{26} E_8^8} \pmod{16}. \tag{78}$$

Extracting the terms involving q^{2n+1} from (78), we complete the proof of (72). Using (22) in (78) and then extracting the terms involving q^{2n}, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{6\gamma}(4n+1) q^n \equiv 4 \frac{E_2^{32}}{E_1^{26} E_4^8} \pmod{16}. \tag{79}$$

Employing (22) in (79), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{6\gamma}(4n+1) q^n \equiv 4E_1^6 \pmod{16}. \tag{80}$$

Employing (14) in (80), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{6\gamma}(4n+1)q^n \equiv 4\left(\frac{E_4^8E_8}{E_2E_{16}^2}-2q\frac{E_4^{10}E_{16}^2}{E_2E_8^5}\right) \pmod{16}. \tag{81}$$

Extracting the terms involving q^{2n+1} from (81), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{6\gamma}(8n+5)q^n \equiv 8\frac{E_2^{10}E_8^2}{E_1E_4^5} \pmod{16}. \tag{82}$$

Employing (21) in (82), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{6\gamma}(8n+5)q^n \equiv 8E_1^3E_4^3 \pmod{16}. \tag{83}$$

Congruence (83) is the $\alpha=0$ case of (73). Suppose that congruence (73) is true for all $\alpha \geq 0$. Utilizing (19) in (73), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{B}_{6\gamma}\big(p^{2\alpha}(8n+5)\big)q^n \equiv 8\Big\{&\sum_{\substack{k=0\\k\neq(p-1)/2}}^{(p-1)}(-1)^kq^{k(k+1))/2}\sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{pn\cdot(pn+2k+1)/2}\\&+p(-1)^{(p-1)/2}q^{(p^2-1)/8}E_{p^2}^3\Big\}\\ \times\Big\{&\sum_{\substack{m=0\\m\neq(p-1)/2}}^{(p-1)}(-1)^mq^{4m(m+1))/2}\sum_{n=0}^{\infty}(-1)^n(2pn+2m+1)q^{4pn\cdot(pn+2m+1)/2}\\&+p(-1)^{(p-1)/2}q^{4(p^2-1)/8}E_{4p^2}^3\Big\} \pmod{16}.\end{aligned} \tag{84}$$

Consider the congruence

$$\frac{(k^2+k)}{2}+\frac{4(m^2+m)}{2}\equiv\frac{5(p^2-1)}{8} \pmod{p},$$

which is equal to

$$(2k+1)^2+4(2m+1)^2\equiv 0 \pmod{p}.$$

For $\left(\frac{-4}{p}\right)=-1$, the above congruence has only solution $m=k=\frac{\pm p-1}{6}$. Therefore, extracting the terms involving $q^{pn+5(p^2-1)/8}$ from both sides of (84), dividing throughout by $q^{5(p^2-1)/8}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{6\gamma}\left(p^{2\alpha+1}(8n+5p)\right)q^n \equiv 8E_p^3E_{4p}^3 \pmod{16}. \tag{85}$$

Extracting the terms involving q^{pn} from (85) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{6\gamma}\left(p^{2\alpha+2}(8n+5)\right)q^n \equiv 8E_1^3E_4^3 \pmod{16}, \tag{86}$$

which is the $\alpha+1$ case of (73). Thus, by the principle of mathematical induction, we complete the proof of (73). Extracting the coefficients of terms involving q^{pn+j} for $1\leq j\leq p-1$, from both sides of (85), we complete the proof of (74). □

Theorem 3.9. *Let $p \geq 3$ be a prime with $\left(\frac{-3}{p}\right) = -1$ and $1 \leq j \leq (p-1)$. Then for all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{B}_{6\gamma}\left(p^{2\alpha}(8n+1)\right) q^n \equiv 4E_1^3 \pmod{8}, \tag{87}$$

$$\overline{B}_{6\gamma}\left(p^{2\alpha+1}(8(pn+j)+p\right) \equiv 0 \pmod{8}. \tag{88}$$

Proof. Using (21) in (80) and extracting the terms involving q^{2n}, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{6\gamma}(8n+1)q^n \equiv 4E_1^3 \pmod{8}. \tag{89}$$

Congruence (89) is the $\alpha = 0$ case of (87). Suppose that congruence (87) is true for all integer $\alpha \geq 0$. Employing (19) in (87), we obtain

$$\sum_{n} \overline{B}_{6\gamma}\left(p^{2\alpha}(8n+1)\right) q^n \equiv 4\Big\{ \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{(p-1)} (-1)^k q^{k(k+1))/2} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn\cdot(pn+2k+1)/2}$$
$$+ p(-1)^{(p-1)/2} q^{(p^2-1)/8} E_{p^2}^3 \Big\} \pmod{8}. \tag{90}$$

Extracting the term involving $q^{pn+(p^2-1)/8}$ from both sides of (90), dividing throughout by $q^{(p^2-1)/8}$ and then replacing q^p by q, we obtain

$$\sum_{n} \overline{B}_{6\gamma}\left(p^{2\alpha+1}(8n+p)\right) q^n \equiv 4E_p^3 \pmod{8}. \tag{91}$$

Extracting the terms involving q^{pn} from (91) and replacing q^p by q, we obtain

$$\sum_{n} \overline{B}_{6\gamma}\left(p^{2(\alpha+1)}(8n+1)\right) q^n \equiv 4E_1^3 \pmod{8}, \tag{92}$$

which is the $\alpha + 1$ case of (87). Thus, by the principle of mathematical induction, we arrive at (87). Extracting the coefficients of terms involving q^{pn+j} for $1 \leq j \leq p-1$, from both sides of (91), we complete the proof of (88). □

4 Conclusion

In this paper, we generalize some of the theorems proved by Naika and Shivasankar in 2017, [14]. Many authors have proved some congruences for $\overline{B}_3$, $\overline{B}_4$ and $\overline{B}_6$. We have further extended these congruences for $\overline{B}_{3\gamma}$, $\overline{B}_{4\gamma}$ and $\overline{B}_{6\gamma}$, where γ is any positive integer.

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