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Quotients of sequences under the binomial convolution

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Abstract: This paper gives expressions for the solution $\{a(n)\}$ of the equation

$$\sum_{k=0}^{n} \binom{n}{k} a(k)b(n-k) = c(n),\ n = 0, 1, 2, \ldots,$$

where $b(0) \neq 0$, that is, of the equation $a \circ b = c$ in a, where $\circ$ is the binomial convolution. These expressions are classified as recursive, explicit, determinant, exponential generating function and convolutional expressions. These expressions are compared with those under the usual Cauchy convolution. Several special cases and examples of combinatorial nature are also discussed.
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1 Introduction

Let $\{a(n)\}$ and $\{b(n)\}$, $n = 0, 1, 2, \ldots$, be sequences of complex numbers. Their binomial convolution ([5, Chapter 7]) is defined by

$$(a \circ b)(n) = \sum_{k=0}^{n} \binom{n}{k} a(k)b(n-k), \quad n = 0, 1, 2, \ldots$$

It is easy to see that the binomial convolution is associative and commutative, and the sequence $\{e(n)\}$, defined by $e(0) = 1$ and $e(n) = 0$ for $n = 1, 2, \ldots$, serves as the identity under the binomial convolution. Further, it is easy to see that a sequence $\{a(n)\}$ possesses an inverse if and only if $a(0) \neq 0$. Thus, the set of sequences $\{a(n)\}$ with $a(0) \neq 0$ forms an Abelian group under the binomial convolution.

The binomial convolution arises naturally from the product of the exponential generating functions ([5, Chapter 7], [6]). In fact,

$$\left(\sum_{n=0}^{\infty} a(n)\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} b(n)\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}(a \circ b)(n)\frac{x^n}{n!}.$$

The binomial convolution can also be treated under the usual generating functions, see [4]. For a recent paper on binomial and related convolutions we mention [2], see also [9]. There is also another binomial convolution in number theoretic literature, see [7]. We, however, confine ourselves to that presented in [5]. A generalized binomial convolution is introduced in [14]. The binomial convolution often appears implicitly in mathematical papers without an explicit reference to binomial convolution. For example, Batır and Sofo [3, Eq. (1.15)] actually evaluate the second power of the sequence of the Cauchy numbers of the first kind under the binomial convolution.

The purpose of this paper is to give expressions for the solution of the equation $a \circ b = c$ in a, where $\{b(n)\}$ and $\{c(n)\}$ are given sequences with $b(0) \neq 0$. The solution of the equation $a \circ b = c$ may be referred to as the quotient of sequences $\{c(n)\}$ and $\{b(n)\}$. Our expressions for quotients of sequences are classified as recursive, explicit, determinant, exponential generating function and convolutional expressions. We also consider the special case $c = e$. We then obtain expressions for the inverse of $\{b(n)\}$. We consider in particular the case $b(n) = 1$ for all $n = 0, 1, 2, \ldots$ Finally, the expressions for quotients under the binomial convolution are briefly compared with those under the usual Cauchy convolution.

Although the expressions in this paper are very obvious and elementary, these general results have not previously been published in the literature. Some special cases in this direction can be found in many combinatorial books, see e.g. [11], Section 2.4. Roots of sequences under the binomial convolution is considered in [6].

This paper is similar to [8], where we studied quotients of arithmetical functions under the Dirichlet convolution. Such quotients have also been studied e.g. in [10] and [15].

2 Expressions for the solution of $a \circ b = c$

2.1 A recursive expression

Theorem 2.1. *If $\{b(n)\}$ and $\{c(n)\}$ are given sequences with $b(0) \neq 0$, then the solution $\{a(n)\}$ of the equation $a \circ b = c$ is unique and is given by*

$$\begin{cases} a(0) = \dfrac{c(0)}{b(0)}, \\ a(n) = \dfrac{1}{b(0)} \left[c(n) - \displaystyle\sum_{k=0}^{n-1} \binom{n}{k} a(k) b(n-k) \right], \quad n = 1, 2, 3, \ldots \end{cases} \tag{1}$$

Proof. Theorem 2.1 follows directly from the equations

$$\sum_{k=0}^{n} \binom{n}{k} a(k) b(n-k) = c(n), \quad n = 0, 1, 2, \ldots$$

□

2.2 An explicit expression

Theorem 2.2. *If $\{b(n)\}$ and $\{c(n)\}$ are given sequences with $b(0) \neq 0$, then the solution $\{a(n)\}$ of the equation $a \circ b = c$ is given by*

$$a(n) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{b(0)^i} \sum_{\substack{k_1+k_2+\cdots+k_i=n \\ k_2,k_3,\ldots,k_i>0}} \frac{n!}{k_1!k_2!\cdots k_i!} c(k_1)b(k_2)\cdots b(k_i), \quad n = 0, 1, 2, \ldots \tag{2}$$

Remark 2.1. In (2), the summation over i is finite. In fact, it suffices that i runs through the integers from 1 to $n+1$.

Proof. We proceed by induction on n. For $n = 0$, the right-hand side of (2) is $c(0)/b(0)$ and hence (2) holds.

Assume that (2) holds for $n < m$. Then, by (1),

$$\begin{aligned} a(m) &= b(0)^{-1} \Big[c(m) - \sum_{\substack{k+j=m \\ k<m}} \frac{m!}{k!j!} a(k) b(j) \Big] \\ &= b(0)^{-1} \Big[c(m) - \sum_{\substack{k+j=m \\ k<m}} \frac{m!}{k!j!} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{b(0)^i} \sum_{\substack{k_1+k_2+\cdots+k_i=k \\ k_2,k_3,\ldots,k_i>0}} \frac{k!}{k_1!k_2!\cdots k_i!} c(k_1)b(k_2)\cdots b(k_i)b(j) \Big] \\ &= b(0)^{-1}c(m) - \sum_{i=1}^{\infty} \frac{(-1)^{i+2}}{b(0)^{i+1}} \sum_{\substack{k_1+k_2+\cdots+k_{i+1}=m \\ k_2,k_3,\ldots,k_{i+1}>0}} \frac{m!}{k_1!k_2!\cdots k_{i+1}!} c(k_1)b(k_2)\cdots b(k_{i+1}) \\ &= b(0)^{-1}c(m) - \sum_{i=2}^{\infty} \frac{(-1)^{i+1}}{b(0)^{i}} \sum_{\substack{k_1+k_2+\cdots+k_i=m \\ k_2,k_3,\ldots,k_i>0}} \frac{m!}{k_1!k_2!\cdots k_i!} c(k_1)b(k_2)\cdots b(k_i) \\ &= \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{b(0)^{i}} \sum_{\substack{k_1+k_2+\cdots+k_i=m \\ k_2,k_3,\ldots,k_i>0}} \frac{m!}{k_1!k_2!\cdots k_i!} c(k_1)b(k_2)\cdots b(k_i). \end{aligned}$$

This completes the proof of Theorem 2.2. □

2.3 A determinant expression

We can write the equation $a \circ b = c$ as a system of linear equations

$$\begin{cases} a(0)b(0) = c(0), \\ a(0)b(1) + a(1)b(0) = c(1), \\ a(0)b(2) + 2a(1)b(1) + a(2)b(0) = c(2), \\ \quad \vdots \\ a(0)b(n) + na(1)b(n-1) + \cdots + \binom{n}{k} a(k)b(n-k) + \cdots + a(n)b(0) = c(n) \end{cases}$$

for $n = 0, 1, 2, \dots$, or

$$\mathbf{B}_n \mathbf{a}_n = \mathbf{c}_n \quad \text{for } n = 0, 1, 2, \dots,$$

where $\mathbf{B}_n$ is the $(n+1) \times (n+1)$ matrix, whose i, j-entry is $\binom{i-1}{j-1} b(i-j)$ with $b(i-j) = 0$ for $i < j$, and $\mathbf{a}_n$ and $\mathbf{c}_n$ are the column vectors $(a(0), a(1), \dots, a(n))^T$ and $(c(0), c(1), \dots, c(n))^T$, respectively. Since $\binom{i-1}{j-1} b(i-j) = 0$ for $i < j$, $\mathbf{B}_n$ is a lower triangular matrix. Therefore $\det(\mathbf{B}_n) = b(0)^{n+1}$. Assuming $b(0) \neq 0$, we can write

$$\mathbf{a}_n = \mathbf{B}_n^{-1} \mathbf{c}_n \quad \text{for } n = 0, 1, 2, \dots$$

and further, by Cramer's rule, we obtain

Theorem 2.3. *If $\{b(n)\}$ and $\{c(n)\}$ are given sequences with $b(0) \neq 0$, then the solution $\{a(n)\}$ of the equation $a \circ b = c$ is given by*

$$a(n) = \frac{1}{b(0)^{n+1}} \det(M_n), \quad n = 0, 1, 2, \dots, \tag{3}$$

where M_n is the $(n+1) \times (n+1)$ matrix obtained by replacing the $(n+1)$-th column of $\mathbf{B}_n$ with $\mathbf{c}_n$.

2.4 An exponential generating function expression

Let $a(x)$, $b(x)$ and $c(x)$ denote the exponential generating functions of the sequences $\{a(n)\}$, $\{b(n)\}$ and $\{c(n)\}$, respectively. The solution $\{a(n)\}$ of the equation $a \circ b = c$ with $b(0) \neq 0$ in terms of exponential generating functions is then simply

$$a(x) = c(x)/b(x). \tag{4}$$

2.5 Convolutional expression

Let $\{b(n)\}$ and $\{c(n)\}$ be given sequences with $b(0) \neq 0$. Then $\{b(n)\}$ possesses an inverse under the binomial convolution. Therefore $a \circ b = c \iff a = c \circ b^{-1}$, and thus the solution $\{a(n)\}$ of the equation $a \circ b = c$ in a is $a = c \circ b^{-1}$.

3 Expressions for the binomial inverse

If $c = e$, the identity function defined in the introduction, then the solution a of the equation $a \circ b = c$ with $b(0) \neq 0$ is the binomial inverse b^{-1} of b. Thus, from Section 2 we obtain expressions for b^{-1}. In fact, from (1) we obtain

$$\begin{cases} b^{-1}(0) = \dfrac{1}{b(0)}, \\ b^{-1}(n) = -\dfrac{1}{b(0)} \displaystyle\sum_{k=0}^{n-1} \binom{n}{k} b^{-1}(k) b(n-k), \quad n = 1, 2, 3, \ldots, \end{cases} \tag{5}$$

from (2) we obtain

$$b^{-1}(n) = \sum_{i=1}^{n} \frac{(-1)^i}{b(0)^{i+1}} \sum_{\substack{k_1+k_2+\cdots+k_i=n \\ k_1,k_2,\ldots,k_i>0}} \frac{n!}{k_1!k_2!\cdots k_i!} b(k_1)b(k_2)\cdots b(k_i), \quad n = 1, 2, 3, \ldots, \tag{6}$$

and from (3) we obtain

$$b^{-1}(n) = \frac{1}{b(0)^{n+1}} \det(M_n), \quad n = 0, 1, 2, \ldots, \tag{7}$$

where M_n is the $(n+1) \times (n+1)$ matrix obtained by replacing the $(n+1)$-st column of $\mathbf{B}_n$ with the $(n+1)$ vector $(1, 0, 0, \ldots, 0)^T$. The exponential generating function of the binomial inverse b^{-1} of b is equal to $1/b(x)$.

Equation (5) can be found in [13]. A unitary analogue of (6) has been presented by Schinzel [12]. This result gives an expression for the inverse of an arithmetical function f with $f(1) \neq 0$ under the unitary convolution.

4 Expressions for the binomial Möbius sequence

If $c = e$ and $\{b(n)\}$ is the sequence of 1's, then the solution a of the equation $a \circ b = c$ is the binomial inverse of b and it may be referred to as the binomial Möbius sequence [13]. Let $\{\mu(n)\}$ denote this sequence. Its exponential generating function is then equal to e^{-x}. It is thus easy to see that

$$\mu(n) = (-1)^n, \quad n = 0, 1, 2, \ldots$$

Expressions (5)–(7) give more complicated expressions for $\mu(n)$. These expressions, of course, reduce to the simple expression $\mu(n) = (-1)^n$. The reductions of these expressions to the simple expression are quite interesting. Firstly, (5) becomes

$$\mu(0) = 1, \ \mu(n) = -\sum_{k=0}^{n-1} \binom{n}{k} \mu(k), \quad n = 1, 2, 3, \ldots \tag{8}$$

If we assume inductively that $\mu(k) = (-1)^k$, $0 \leq k < n$, then by the binomial formula $((-1) + 1)^n = 0$, we obtain from (8) the expression $\mu(n) = (-1)^n$. Secondly, (6) becomes

$$\mu(n) = \sum_{i=1}^{n} (-1)^i \sum_{\substack{k_1+k_2+\cdots+k_i=n \\ k_1,k_2,\ldots,k_i>0}} \frac{n!}{k_1!k_2!\cdots k_i!}, \quad n = 1, 2, 3, \ldots \tag{9}$$

Application of the inclusion-exclusion principle and the multinomial theorem gives

$$\begin{aligned}\sum_{\substack{k_1+k_2+\cdots+k_i=n\\k_1,k_2,\ldots,k_i>0}} \frac{n!}{k_1!k_2!\cdots k_i!} &= \sum_{k_1+k_2+\cdots+k_i=n} \frac{n!}{k_1!k_2!\cdots k_i!} - \sum_{1\le j\le i} \sum_{\substack{k_1+k_2+\cdots+k_i=n\\k_j=0}} \frac{n!}{k_1!k_2!\cdots k_i!}\\ &\quad + \sum_{1\le j<m\le i} \sum_{\substack{k_1+k_2+\cdots+k_i=n\\k_j=k_m=0}} \frac{n!}{k_1!k_2!\cdots k_i!} - \cdots\\ &= \sum_{k=0}^{i}(-1)^k\binom{i}{k}(i-k)^n\\ &= \sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}k^n.\end{aligned}$$

It is well known that this last expression gives the number of surjections from $\{1,2,\ldots,n\}$ onto $\{1,2,\ldots,i\}$ and that it is equal to $i!\,S(n,i)$ ([1, p. 97, 3.39]), where $S(n,i)$'s are the Stirling numbers of the second kind. Thus

$$\mu(n) = \sum_{i=1}^{n}(-1)^i\, i!\, S(n,i).$$

Therefore, application of the formula ([1, Proposition 3.24])

$$x^n = \sum_{i=1}^{n} S(n,i)[x]_i$$

with $x=-1$ gives $\mu(n) = (-1)^n$.

From (7) we obtain

$$\mu(n) = (-1)^n \det\left[\binom{i-1}{j-1}_{\substack{i=2,3,\ldots,n+1\\j=1,2,\ldots,n}}\right], \quad n=1,2,3,\ldots \tag{10}$$

Application of Pascal's identity shows that the determinant is equal to 1. Thus $\mu(n) = (-1)^n$.

5 Examples

In Section 4 we gave expressions for the binomial inverse of the sequence of 1's. In this section, we derive further examples of binomial inverses and of binomial quotients. We also point out the effectivity of exponential generating functions in deriving the expressions.

Let $D(n)$ denote the number of derangements of $1,2,\ldots,n$. Then $\binom{n}{r}D(n-r)$ is the number of permutations $a_1,a_2,\ldots,a_n$ of $1,2,\ldots,n$ such that $a_i=i$ for exactly r values of i. Therefore,

$$\sum_{r=0}^{n}\binom{n}{r}D(n-r) = n!. \tag{11}$$

In terms of exponential generating functions, (11) can be written as

$$D(x)e^x = \frac{1}{1-x}$$

or

$$D(x) = \frac{1}{e^x(1-x)}.$$

Further,

$$\frac{1}{D(x)} = e^x(1-x) = \sum_{n=0}^{\infty}(1-n)\frac{x^n}{n!}.$$

This means that the binomial inverse $\{D^{-1}(n)\}$ of the sequence $\{D(n)\}$ is the sequence $\{1-n\}$. Plainly,

$$\frac{\mu(x)}{D(x)} = 1-x;$$

hence the quotient sequence $\{\mu \circ D^{-1}(n)\}$ is given by $\mu \circ D^{-1}(0) = 1$, $\mu \circ D^{-1}(1) = -1$ and $\mu \circ D^{-1}(n) = 0$ for $n > 1$. Note that $\mu \circ D^{-1}(n) = m(n)$, where $\{m(n)\}$ is the sequence to be defined in Section 6.

Let $\{B(n)\}$ denote the sequence of Bernoulli numbers. Then

$$\frac{1}{B(x)} = \frac{e^x - 1}{x} = \sum_{n=0}^{\infty}\frac{1}{n+1}\frac{x^n}{n!}.$$

Thus the binomial inverse $\{B^{-1}(n)\}$ of the sequence $\{B(n)\}$ is the sequence $\{\frac{1}{n+1}\}$. Further,

$$\frac{\mu(x)}{B(x)} = \frac{1-e^{-x}}{x} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}\frac{x^n}{n!};$$

hence the quotient sequence $\{\mu \circ B^{-1}(n)\}$ is the sequence $\{(-1)^{n+1}/(n+1)\}$.

Consider next two basic binomial inverse formulas [5]. Take first the formula

$$c(n) = \sum_{k=0}^{n}\binom{n}{k}a(k) \iff a(n) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}c(k).$$

This formula can be written in terms of the binomial convolution simply as

$$c = a \circ u \iff a = c \circ \mu,$$

where u is the sequence of 1's. In terms of exponential generating functions this goes as

$$c(x) = a(x)e^x \iff a(x) = \frac{c(x)}{e^x} = c(x)e^{-x}.$$

Consider then the binomial self-inverse formula

$$c(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}a(k) \iff a(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}c(k).$$

This can be written in terms the binomial convolution as

$$\begin{aligned} c(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}a(k) &\iff c = \mu a \circ u \\ &\iff \mu a = c \circ \mu \\ &\iff (-1)^n a(n) = \sum_{k=0}^{n}\binom{n}{k}c(k)(-1)^{n-k} \\ &\iff c(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}a(k). \end{aligned}$$

In terms of exponential generating functions this goes simply as

$$c(x) = a(-x)e^x \iff a(x) = c(-x)e^x.$$

6 The Cauchy convolution

In this section we briefly consider quotients under the usual Cauchy convolution. The Cauchy convolution of the sequences $\{a(n)\}$ and $\{b(n)\}$ is defined by

$$(a * b)(n) = \sum_{k=0}^{n} a(k)b(n-k), \quad n = 0, 1, 2, \dots,$$

and it arises naturally from the product of the ordinary generating functions:

$$\left(\sum_{n=0}^{\infty} a(n)x^n\right)\left(\sum_{n=0}^{\infty} b(n)x^n\right) = \sum_{n=0}^{\infty}(a * b)(n)x^n,$$

see, e.g., [5, Chapter 7]. Expressions for quotients under the Cauchy convolution are similar in character to those for quotients under the binomial convolution. Such expressions are easily obtained by substituting the number 1 for the binomial and multinomial coefficients in the expressions (1)–(3) for quotients under the binomial convolution. To be more precise, the analogues of the expressions (1)–(3) are the following. If $\{b(n)\}$ and $\{c(n)\}$ are given sequences with $b(0) \neq 0$, then the solution $\{a(n)\}$ of the equation $a * b = c$ is given by

$$\begin{cases} a(0) = \dfrac{c(0)}{b(0)}, \\ a(n) = \dfrac{1}{b(0)}\left[c(n) - \displaystyle\sum_{k=0}^{n-1} a(k)b(n-k)\right], \quad n = 1, 2, 3, \dots \end{cases} \tag{12}$$

or

$$a(n) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{b(0)^i} \sum_{\substack{k_1+k_2+\cdots+k_i=n \\ k_2,k_3,\dots,k_i>0}} c(k_1)b(k_2)\cdots b(k_i), \quad n = 0, 1, 2, \dots, \tag{13}$$

or

$$a(n) = \frac{1}{b(0)^{n+1}} \det(M_n), \quad n = 0, 1, 2, \dots, \tag{14}$$

where M_n is the $(n+1) \times (n+1)$ matrix obtained by replacing the $(n+1)$-st column of $\mathbf{B}_n$ with $\mathbf{c}_n$, $\mathbf{B}_n$ being the $(n+1) \times (n+1)$ matrix, whose i, j-entry is $b(i-j)$ with $b(i-j) = 0$ for $i < j$, and $\mathbf{c}_n$ being the column vector $(c(0), c(1), \dots, c(n))^T$.

Note that the exponential generating functions may be replaced by the ordinary generating functions. The convolutional expression is, of course, with respect to the Cauchy convolution. The analogues of the expressions (5)–(7) for the binomial inverse are obtained in a similar way. We do not present the details here.

Finally, we consider the analogue of the binomial Möbius sequence $\{\mu(n)\}$. Let $\{m(n)\}$ denote the inverse of the sequence of 1's under the Cauchy convolution. Then $\{m(n)\}$ may be referred to as the Möbius sequence under the Cauchy convolution. It is well known that its ordinary generating function is $m(x) = 1 - x$ and that its elements are simply given by

$$m(0) = 1, \; m(1) = -1, \; m(n) = 0, \; n = 2, 3, \dots$$

The analogues of (8)–(10) give more complicated expressions. In fact, the analogue of the recursive expression (8) is

$$m(0) = 1,\ m(n) = -\sum_{k=0}^{n-1} m(k),\ \ n = 1, 2, 3, \ldots, \tag{15}$$

and that of the explicit expression (9) is

$$m(n) = \sum_{i=1}^{n} (-1)^i \sum_{\substack{k_1+k_2+\cdots+k_i=n \\ k_1,k_2,\ldots,k_i>0}} 1, \quad n = 1, 2, 3, \ldots \tag{16}$$

The equation (16) implies that for $n \geq 1$,

$$\begin{aligned} m(n) &= \sum_{i=1}^{n} (-1)^i \binom{i+n-i-1}{n-i} = \sum_{i=0}^{n-1} (-1)^{n-i} \binom{n-1}{i} \\ &= -((-1)+1)^{n-1} = -0^{n-1}, \end{aligned}$$

that is, $m(1) = -1,\ m(n) = 0$ for $n = 2, 3, \ldots$ The determinant expression is

$$m(n) = \det(M_n), \quad n = 0, 1, 2, \ldots, \tag{17}$$

where M_n is the $(n+1) \times (n+1)$ matrix, whose i, j-entry is $= 1$ for $i = 1, 2, \ldots, n+1$; $j = 1, 2, \ldots, n$ with $i \geq j$ and for $i = 1; j = n+1$, and $= 0$ otherwise.

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