Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2024, Volume 30, Number 4, 665–680
DOI: 10.7546/nntdm.2024.30.4.665-680

Algorithms for representing positive odd integers as the sum of arithmetic progressions

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Dedicated to Professor Chung-Wu Ho
on the occasion of his 85th birthday!

Received: 22 May 2024 **Accepted:** 3 October 2024 **Online First:** 5 November 2024

Abstract: This paper delves into the historical and recent developments in this area of mathematical inquiry, tracing the evolution from Wheatstone's representation of powers of an integer as sums of arithmetic progressions to extensions of Sylvester's Theorem (Sylvester and

Franklin, [14]). Sylvester's Theorem, a result that determines the representability of positive integers as sums of consecutive integers, has been the foundation for numerous extensions, including the representation of integers as sums of specific arithmetic progressions and powers of such progressions. The recent works of Ho et al. [3] and Ho et al. [4] have further expanded on Sylvester's Theorem, offering a procedural approach to compute the representability of positive integers in the context of arithmetic progressions. In this paper, efficient algorithms to compute the number of ways to represent an odd positive integer as sums of powers of arithmetic progressions are presented.
Keywords: Arithmetic progressions, Balancing, Fermat, Fibonacci, Geometric, Jacobsthal, Leonardo, Lucas, Mersenne, Padovan, Pell, Perrin sequences.
2020 Mathematics Subject Classification: 11B39, 11A25.

1 Introduction

Since 1844, there has been an interest in representing numbers as the sum of a sequence of consecutive integers. Initially, Sir Charles Wheatstone [15] represented certain powers of an integer as sums of arithmetic progressions. Then, Sylvester and Franklin [14] published a result to determine the number of ways a positive integer can be represented as the sum of a sequence of consecutive integers; this result has since been called Sylvester's Theorem, and there have been many attempts to extend this theorem to sums of different types of sequences, such as sums of certain arithmetic progressions (Munagi and de Vega [10], Munagi and Shonhiwa [9]) and sums of powers of arithmetic progressions (Shiue et al. [12], Shiue et al.[13]).

Recently, the manuscript duology of Ho et al. [3] and Ho et al. [4] extended Sylvester's Theorem to describe a procedure to compute the number of ways a positive integer can be represented as a sum of arithmetic progressions. They also extended Wheatstone's original work by studying certain relationships among the representations of different powers of an integer as sums of arithmetic progressions; this is done by using the method delineated in Junaidu et al. [6].

Furthermore, Long et al. [7] studied six equations of the form $n^2 + (n + a)\, x - (n + a) = 0$, $a = \mp r,\ \mp 2r$, corresponding to 6 permutations of 3 integers in arithmetic progressions with common difference $r > 0$, depending on solutions of the Pell equation $u^2 - 5v^2 = -4r^2$ (Nagell, [11]). These produced results which depended on Fibonacci and Lucas numbers. They were generalizations of earlier work by Mahanthappa [8] with three similar equations, where the coefficients were integers in arithmetic progression with common difference 1. Another link between arithmetic progressions and the Fibonacci numbers was that of Atanassov et al. [1] who considered the sequence $a, a + f\,(1)\,,\ a + f\,(2)\,, \ldots, a + f\,(k)\,, \ldots,\ (f : N \to R)$, as a pseudo arithmetic progression from which one can generate an ordinary arithmetic progression and the ordinary Fibonacci sequence, as well as auto-generations of extensions of the concept *'arithmetic progression'*, linked to equalities in Hoggatt Jr [5]. Extensions with pairs of Fibonacci numbers (Atanassov et al. [2]) had similar development in its use of the same section of Hoggatt Jr [5].

In this paper, we present theorems and algorithms that enable us to represent positive odd integers m as arithmetic progressions of the form $m = a + (a + d) + \cdots + (a + (r - 1)d)$. In Sections 2 and 3, we explore the case where r is odd and even, respectively. Two corollaries follow that outline how many ways powers of primes can be written as arithmetic progressions under certain conditions. We provide computationally efficient algorithms corresponding to the theorems and corollaries mentioned in their respective sections.

2 Main results for m odd and r odd

Let $m > 1$ be a positive odd integer represented as a sum of an arithmetic progression, i.e.,

$$m = a + (a + d) + \cdots + (a + (r - 1)d), \tag{1}$$

where $a, d \in \mathbb{N}$. In this section, we present algorithms for computing the number of ways that m can be represented as (1) when r is an odd integer.

Theorem 2.1. *Let $m > 1$ be a positive odd integer, not a prime, and let*

$$m = a + (a + d) + \cdots + (a + (r - 1)d), \tag{2}$$

where $a, d \in \mathbb{N}$ and r is odd ≥ 3. Then,

(i). $r \mid m$, $1 \leq d \leq \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor$, *and* $a = \frac{m}{r} - \frac{r-1}{2}d$;

(ii). $3 \leq r \leq \left\lfloor \frac{-1+\sqrt{1+8m}}{2} \right\rfloor \leq \lfloor \sqrt{2m} \rfloor$;

(iii). There are

$$S = \sum_{r|m} \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor \tag{3}$$

number of ways to write m as (2).

Proof. (i). By (2), we have $m = r\left(a + \left(\frac{r-1}{2}\right)d\right)$. Then, $2m = r(2a + (r-1)d)$, or $\frac{2m}{r} = 2a + (r-1)d$. Since r is odd, we have $r \mid m$ and $\frac{m}{r} = a + \frac{r-1}{2}d$. Solving for a, then $a = \frac{m}{r} - \frac{r-1}{2}d$. Since $a \geq 1$, we have $\frac{m}{r} - \frac{r-1}{2}d \geq 1$, which implies $1 \leq d \leq \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor$.

(ii). From $\frac{2(m-r)}{r(r-1)} \geq 1$, we have $2(m-r) \geq r(r-1)$. Then $2m - 2r \geq r^2 - r$. Hence, $r^2 + r - 2m < 0$. Since $r \geq 3$, we have $3 \leq r \leq \left\lfloor \frac{-1+\sqrt{1+8m}}{2} \right\rfloor$. Since m is positive, $4\sqrt{2m} \geq 0$. Then $4\sqrt{2m} + 8m + 1 = \left(2\sqrt{2m} + 1\right)^2 \geq 8m + 1$. Then $2\sqrt{2m} + 1 \geq \sqrt{8m+1}$. Simplifying this, we have $\left\lfloor \frac{-1+\sqrt{1+8m}}{2} \right\rfloor \leq \lfloor \sqrt{2m} \rfloor$.

(iii). For each $r \mid m$, each d between 1 and $\left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor$ is a way to represent m as an arithmetic progression. To find total number of ways, the sum is taken. □

Note that if $r = 1$, we have the arithmetic progression reduced to $m = a$. Hence, r is assumed to be odd and ≥ 3.

When $m = p$, where p is an odd prime number, we have the following corollary.

Corollary 2.1. *Let $m = p^k$, $p \geq 3$ a prime number and $k > 1$ an integer, and let*

$$p^k = a + (a + d) + \cdots + (a + (r - 1)d), \tag{4}$$

where $a, d \in \mathbb{N}$, $r \geq 3$ odd. Then, *(i).* $r = p^j$, $1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor$;

(ii). $1 \leq d \leq 2 \left\lfloor \frac{p^{k-j}-1}{p^j-1} \right\rfloor$ *and* $a = p^{k-j} - \frac{1}{2}\left(p^j - 1\right) d$;

(iii). There are

$$S = \sum_{j=1}^{\left\lfloor \frac{k}{2} \right\rfloor} 2 \left\lfloor \frac{\left(p^{k-j} - 1\right)}{p^j - 1} \right\rfloor \tag{5}$$

number of ways to write p^k *as* (4).

Proof. Since $m = p^k$, where p is a prime, then $r \mid m$ means that $r \mid p^k$. Hence, we have $r = p^j$, $1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor$. Substitute $m = p^k$ into the results in Theorem 2.1, we obtain the results for this corollary. □

Another way of listing the ways that a positive odd number can be represented in the form of (2) is as follows:

Remark 2.2. *Theorem 2.1 (i) and (iii) can also be proved in the following manner.*

Proof. Recall that $m = a + (a + d) + \cdots + (a + (r-1)d)$, where r is odd ≥ 3. We may set up

$$m = \left(\frac{m}{r} - \frac{r-1}{2}d\right) + \cdots + \frac{m}{r} + (M + d) + \cdots + \left(\frac{m}{r} + \frac{r-1}{2}d\right). \tag{6}$$

Since $\frac{m}{r} - \frac{r-1}{2}d \geq 1$, we have $\frac{m}{r} - 1 \geq \frac{r-1}{2}d$. Since $d \geq 1$, we have

$$1 \leq d \leq \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor.$$

Thus, we have

$$S = \sum_{r \mid m} \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor.$$

We may use (6) to list all possible ways. □

2.1 Algorithms

By using Theorem 2.1, we have the following Algorithm 1.

Algorithm 1 Finding the number of ways S to write m as a sum of arithmetic progression when r is odd and $3 \leq r \leq \left\lfloor \sqrt{2m} \right\rfloor$

Input: Positive odd integer m (not prime)
Output: S

Define integers $q = \left\lfloor \sqrt{2m} \right\rfloor$, $r = 3$, $S = 0$
while $r \leq q$ **do**
  if $m \equiv 0 \pmod{r}$ **then**
    $g = \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor$
    $S = S + g$
  end if
  $r = r + 2$
end while
There are S number of ways to write the given m as the form (2) when $r \geq 3$ is odd.

The overall time complexity of this algorithm is determined by the number of iterations of the while loop. In the worst case, the while loop iterates until r exceeds q. Therefore, the time complexity is $O(\frac{q}{2}) = O(q)$.

The next algorithm computes the number of ways an integer $m = p^k$ can be represented as a sum of arithmetic progressions based on Corollary 2.1.

Algorithm 2 Finding the number of ways S to write $m = p^k$ as a sum of arithmetic progression when $r \geq 3$ is odd

Input: Positive odd integer $m = p^k$, p prime, and $k > 1$ integer
Output: S

Define integers $q = \lfloor \frac{k}{2} \rfloor$, $j = 1$, $S = 0$
while $j \leq q$ **do**
    $g = 2 \left\lfloor \frac{p^{k-j}-1}{p^j-1} \right\rfloor$
    $S = S + g$
    $j = j + 1$
end while
There are S number of ways to write the given m as the form (2) when $r \geq 3$ is odd and $m = p^k$, where p is a prime and $k > 1$ integer.

The next algorithm lists all possible ways to write m as sums of arithmetic progressions for a given m and r from Algorithm 1.

Algorithm 3 Listing the possible ways to write m as a sum of arithmetic progressions

Input: Positive odd integer m not prime.
Output: Prints all S ways

Define integers $q = \lfloor \sqrt{2m} \rfloor$, $r = 3$
while $r \leq q$ **do**
    $g = \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor$
    if $m \equiv 0 \pmod r$ and $g \neq 0$ **then**
        for $d = 1, \ldots, g$ **do**
            $a = \frac{m}{r} - \frac{(r-1)}{2} d$
            Print '$m = a + (a+d) + \cdots + (a + (r-1)d)$'
        end for
    end if
    $r = r + 2$
end while

Similarly, this algorithm also has the time complexity $O(q \cdot g) = O(q)$. Although there is a for loop inside the while loop, since the operations take $O(1)$ and g is a constant factor, we can factor it out of the big-O notation.

2.2 Examples

Example 2.1. *Let $m = 65$. In how many ways can we write m as a sum of arithmetic progressions when $r \geq 3$ odd?*

We use Algorithm 1. First, $q = \lfloor \sqrt{2m} \rfloor = \lfloor \sqrt{130} \rfloor = 11$ and $3 \leq r \leq 11$. Because $m = 65$, then $r = 5$. Computing the quantity $\sum_{r|m} \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor$ for $r = 5$, we have $S = 6$. Thus, there are 6 ways to write $m = 65$ as a sum of arithmetic progressions when $r \geq 3$ is odd.

To list the possible ways, we use Algorithm 3. For $r = 5$, $g(r) = \left\lfloor \frac{2(65-5)}{5(5-1)} \right\rfloor = 6$. We iterate over each d from 1 to 6. Then

$$d = 1 \text{ gives } a = 11$$
$$d = 2 \text{ gives } a = 9$$
$$d = 3 \text{ gives } a = 7$$
$$d = 4 \text{ gives } a = 5$$
$$d = 5 \text{ gives } a = 3$$
$$d = 6 \text{ gives } a = 1$$

Then, we have the following list:

$$65 = 11 + 12 + \cdots + 15$$
$$65 = 9 + 11 + \cdots + 17$$
$$65 = 7 + 10 + \cdots + 19$$
$$65 = 5 + 9 + \cdots + 21$$
$$65 = 3 + 8 + \cdots + 23$$
$$65 = 1 + 7 + \cdots + 25$$

Example 2.2. *Let $m = 1125$. In how many many ways can we write m as a sum of arithmetic progressions when $r \geq 3$ odd?*

We use Algorithm 1. First, $q = \lfloor \sqrt{2m} \rfloor = \lfloor \sqrt{2250} \rfloor = 47$. Then, $3 \leq r \leq 47$. Since $m = 1125$, $r = 3, 5, 9, 15, 25, 45$. Now, let $g(r) = \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor$. Then,

r	$g(r)$
3	374
5	112
9	31
15	10
25	3
45	1
S	531

There are 531 ways to write $m = 1125$ as a sum of arithmetic progressions when $r \geq 3$ is odd. To list the possible ways, we use Algorithm 3.

For $r=3$*, there is a total of* 374 *ways* $(374+1-1=374)$*:*

$$1125 = 374 + 375 + 376$$
$$1125 = 373 + 375 + 377$$
$$\vdots$$
$$1125 = 2 + 375 + 748$$
$$1125 = 1 + 375 + 749$$

For $r=5$*, there is a total of* 112 *ways* $\left(\frac{223-1}{2}+1=112\right)$*:*

$$1125 = 223 + 224 + \cdots + 227$$
$$1125 = 221 + 223 + \cdots + 229$$
$$\vdots$$
$$1125 = 3 + 114 + \cdots + 447$$
$$1125 = 1 + 113 + \cdots + 449$$

For $r=9$*, there is a total of* 31 *ways* $\left(\frac{121-1}{4}+1=31\right)$*:*

$$1125 = 121 + 122 + \cdots + 129$$
$$1125 = 117 + 119 + \cdots + 133$$
$$\vdots$$
$$1125 = 5 + 35 + \cdots + 245$$
$$1125 = 1 + 32 + \cdots + 249$$

For $r=15$*, there is a total of* 10 *ways* $\left(\frac{68-5}{7}+1=10\right)$*:*

$$1125 = 68 + 69 + \cdots + 82$$
$$1125 = 61 + 63 + \cdots + 89$$
$$\vdots$$
$$1125 = 12 + 21 + \cdots + 138$$
$$1125 = 5 + 15 + \cdots + 145$$

For $r=25$*, there is a total of* 3 *ways:*

$$1125 = 33 + 34 + \cdots + 57$$
$$1125 = 21 + 23 + \cdots + 69$$
$$1125 = 9 + 12 + \cdots + 81$$

For $r=45$*, there is only* 1 *way:*

$$1125 = 3 + 4 + \cdots + 47.$$

Example 2.3. *Let $m = 6125$. In how many ways can we write m as a sum of arithmetic progressions when $r \geq 3$ odd?*

We use Algorithm 1. First, $q = \lfloor \sqrt{2m} \rfloor = \lfloor \sqrt{12250} \rfloor = 110$. Then, $3 \leq r \leq 110$. Since $m = 6125$, $r = 5, 7, 25, 35, 49$. Now, let $g(r) = \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor$. Then,

r	$g(r)$
5	612
7	291
25	20
35	10
49	5
S	938

There are 938 ways to write $m = 6125$ as a sum of arithmetic progressions when $r \geq 3$ is odd. To list the possible ways, we use Algorithm 3.

For $r = 5$, there is a total of 612 ways $(1223 - 1 + 1 = 1223)$:

$$6125 = 1223 + 1224 + \cdots + 1227$$
$$6125 = 1221 + 1223 + \cdots + 1229$$
$$\vdots$$
$$6125 = 3 + 614 + \cdots + 2447$$
$$6125 = 1 + 613 + \cdots + 2449$$

For $r = 7$, there is a total of 291 ways $\left(\frac{872-2}{3} + 1 = 291\right)$:

$$6125 = 872 + 873 + \cdots + 878$$
$$6125 = 869 + 871 + \cdots + 881$$
$$\vdots$$
$$6125 = 5 + 295 + \cdots + 1745$$
$$6125 = 2 + 293 + \cdots + 1748$$

For $r = 25$, there is a total of 20 ways $\left(\frac{223-5}{12} + 1 = 20\right)$:

$$6125 = 233 + 234 + \cdots + 257$$
$$6125 = 221 + 223 + \cdots + 269$$
$$\vdots$$
$$6125 = 17 + 36 + \cdots + 473$$
$$6125 = 5 + 25 + \cdots + 485$$

For $r = 35$*, there is a total of* 10 *ways* $\left(\frac{158-5}{17} + 1 = 10\right)$*:*

$$\begin{aligned}
6125 &= 158 + 159 + \cdots + 192 \\
6125 &= 141 + 143 + \cdots + 209 \\
&\vdots \\
6125 &= 22 + 31 + \cdots + 328 \\
6125 &= 5 + 15 + \cdots + 345
\end{aligned}$$

For $r = 45$*, there is a total of* 5 *ways:*

$$\begin{aligned}
6125 &= 101 + 102 + \cdots + 149 \\
6125 &= 77 + 79 + \cdots + 173 \\
6125 &= 53 + 56 + \cdots + 197 \\
6125 &= 29 + 33 + \cdots + 221 \\
6125 &= 5 + 10 + \cdots + 245
\end{aligned}$$

Example 2.4. *Let* $m = 7^5$ *(Ho et al.,* [3]*). In how many ways can we write* m *as a sum of arithmetic progressions when* $r \geq 3$ *odd?*

Using Algorithm 2, we have $p = 7$*,* $k = 5$*,* $r = 7^j$*,* $\left\lfloor \frac{k}{2} \right\rfloor = 2$*, and* $j = 1, 2$*. Let* $g(j) = \left\lfloor \frac{2(7^{5-j}-1)}{7^j-1} \right\rfloor$*. Then,*

j	$g(j)$
1	800
2	14
S	814

There are 814 *number of ways of writing* 7^5 *as Eq.* (4) *when* $r \geq 3$ *is odd. To list the possible ways, we use Algorithm 3.*

For $j = 1$*, there is a total of* 800 *ways* $\left(\frac{2398-1}{3} + 1 = 800\right)$*:*

$$\begin{aligned}
16807 &= 2398 + 2399 + \cdots + 2404 \\
16807 &= 2395 + 2397 + \cdots + 2407 \\
&\vdots \\
16807 &= 4 + 803 + \cdots + 4798 \\
16807 &= 1 + 801 + \cdots + 4801
\end{aligned}$$

For $j = 2$*, there is a total of* 14 *ways* $\left(\frac{319-7}{24} = 14\right)$*:*

$$\begin{aligned}
16807 &= 319 + 320 + \cdots + 367 \\
16807 &= 295 + 297 + \cdots + 391 \\
&\vdots \\
16807 &= 31 + 44 + \cdots + 655 \\
16807 &= 7 + 21 + \cdots + 679
\end{aligned}$$

Example 2.5. *Let $m = 3^{10}$. In how many ways can we write m as a sum of arithmetic progressions when $r \geq 3$ odd?*

Using Algorithm 2, we have $r = 3^j$, $\left\lfloor \frac{10}{2} \right\rfloor = 5$, and $j = 1, 2, \cdots, 5$. Let $g(j) = \left\lfloor \frac{2(7^{5-j}-1}{7^j-1} \right\rfloor$. Then,

j	$g(j)$
1	19682
2	1640
3	168
4	18
5	2
S	21510

Thus, there are 21510 ways of writing 3^{10} as Eq. (4) when $r \geq 3$ is odd.

3 Main results for m odd and r even

Throughout this section, we consider even r. When $r = 2$, m is represented by the sum of two positive integers, which is trivial. When $r > 2$, we have the following result.

Theorem 3.1. *Let $m > 1$ be a positive odd integer expressed as (2), where $a, d \in \mathbb{N}$ and $r = 2t$, with $t > 1$. Then,*

(i). t *and* d *are odd;*

(ii). $m \geq 21$*;*

(iii). $t \mid m$ *and* $1 < t \leq \left\lfloor \frac{-1+\sqrt{1+4m}}{4} \right\rfloor \leq \lfloor \sqrt{m} \rfloor$*;*

(iv). $1 \leq d \leq \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$ *and* $a = \frac{1}{2}\left(\frac{m}{t} - (2t-1)d\right)$*;*

(v). There are

$$S = \sum_{t|m,\ g\in\mathbb{E}} \frac{g}{2} + \sum_{t|m,\ g\in\mathbb{O}} \frac{g+1}{2}, \tag{7}$$

where $g = \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$ ways to write m as the form of (2), where g is either $\mathbb{E}$ (even) or $\mathbb{O}$ (odd).

Proof. (i). From (2), we have $m = r\left(a + \left(\frac{r-1}{2}d\right)\right)$. Next, $\frac{2m}{r} = 2a + (r-1)d$, we have $r \mid 2m$. Let $r = 2t$ with $t > 1$, then $\frac{m}{t} = 2a + (2t-1)d$. If m is odd, then t is odd. Note that $2a$ is even and $\frac{m}{t}$ is odd, so d is odd.

(ii). We consider the smallest numbers $a = 1,\ d = 1$, and $r = 6$. Thus, $m \geq 6(1 + \frac{5}{2}) = 21$.

(iii). From $\frac{m}{t} = 2a + (2t-1)d$, we have $\frac{m-2at}{t(2t-1)} = d$. Since $a \geq 1$, then $\frac{m-2at}{t(2t-1)} \leq \frac{m-2t}{t(2t-1)}$. Hence $1 \leq d \leq \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$. Solving for a, we have $a = \frac{1}{2}\left(\frac{m}{t} - (2t-1)d\right)$.

(iv). From $\frac{m-2t}{t(2t-1)} \geq 1$, we have $t(2t-1) \leq m - 2t$. Then $2t^2 + t - m \leq 0$.
Hence, $t \leq \left\lfloor \frac{-1+\sqrt{1+4m}}{4} \right\rfloor$. To further lower the upper bound, observe that since $0 \leq 8\sqrt{m}$, $0 \leq 8\sqrt{m} + 12m$. Then $4m + 1 \leq 8\sqrt{m} + 16m + 1 = (4\sqrt{m}+1)^2$. Simplifying this, we have $\left\lfloor \frac{-1+\sqrt{1+4m}}{4} \right\rfloor \leq \lfloor \sqrt{m} \rfloor$.

(v). Since d is odd, we need to count the number of odd numbers between 1 and $\left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$, inclusive. Let $g = \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$. If g is even, then there are $\frac{g}{2}$ odd numbers. If g is odd, then there are $\frac{g+1}{2}$ odd numbers. □

Corollary 3.1. *Let $m = p^k$, $p \geq 3$ a prime number and $k \in \mathbb{N}$, and let*

$$p^k = a + (a+d) + \cdots + (a + (r-1)d), \tag{8}$$

where $a, d \in \mathbb{N}$, d odd, and $r > 2$ even. Then,

(i). $r = 2p^j$, $1 \leq j \leq \lfloor \frac{k}{2} \rfloor$;

(ii). $1 \leq d \leq \left\lfloor \frac{p^{k-j}-2}{2p^j-1} \right\rfloor$ and $a = \frac{1}{2}\left(p^{k-j} - (2p^j - 1)\,d\right)$;

(iii). There are

$$S = \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \left\lfloor \frac{1}{2}\left(\left\lfloor \frac{p^{k-j}-2}{2p^j-1} \right\rfloor + 1 \right) \right\rfloor \tag{9}$$

number of ways to write p^k as (4).

Proof. Since p is prime and $m = p^k$, then $r = 2t$ and $t \mid m$ gives $r = 2t = 2p^j$, $1 \leq j \leq \lfloor \frac{k}{2} \rfloor$.

Next, since $r = 2p^j = 2t$ and $m = p^k$, we have

$$\left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor = \left\lfloor \frac{p^k - 2p^j}{p^j(2p^j-1)} \right\rfloor = \left\lfloor \frac{p^{k-j}-2}{2p^j-1} \right\rfloor.$$

Lastly, since d is odd, we have the number of odd numbers in the interval $1 \leq d \leq \left\lfloor \frac{p^{k-j}-2}{2p^j-1} \right\rfloor$ equal to $\left\lfloor \frac{1}{2}\left(\left\lfloor \frac{p^{k-j}-2}{2p^j-1} \right\rfloor + 1 \right) \right\rfloor$. □

3.1 Algorithms

Algorithm 4 Finding the number of ways to write m as a sum of arithmetic progression when $r = 2t$, with $t > 1$ odd

Input: Positive odd integer m (not prime)
Output: S

Define integers $q = \lfloor \sqrt{m} \rfloor$, $t = 3$, $S = 0$
while $t \leq q$ **do**
  if $m \equiv 0 \pmod{t}$ **then**
    $g = \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$
    if $g \pmod{2} \equiv 0$ **then**
      $S_1 = \frac{g}{2}$
    else
      $S_2 = \frac{g+1}{2}$
    end if
    $S = S_1 + S_2$
  end if
  $t = t + 2$
end while
There are S ways to write the given m as the form (2) when $r = 2t$, with $t > 1$ odd.

Similarly, this algorithm should achieve the time complexity $O(q)$.

The next algorithm is similar to Algorithm 2, with $r = 2t$, $t \in \mathbb{N}$ odd.

Algorithm 5 Finding the number of ways S to write $m = p^k$ as a sum of arithmetic progression when $r = 2t$, $t \in \mathbb{N}$ odd

Input: Positive odd integer $m = p^k$, p prime, and $k > 1$ integer
Output: S

Define integers $q = \lfloor \frac{k}{2} \rfloor$, $j = 1$, $S = 0$
while $j \leq q$ **do**
  $g = \left\lfloor \frac{1}{2} \left(\frac{p^{k-j}-2}{2p^j-1} + 1 \right) \right\rfloor$
  $S = S + g$
  $j = j + 1$
end while
There are S number of ways to write the given m as the form (2) when $r > 2$ is even and $m = p^k$, where p is a prime, and $k > 1$ integer.

The next algorithm lists all sums of arithmetic progressions for a given m and $r = 2t$, with $t > 1$ odd, from Algorithm 4.

Algorithm 6 Listing the possible ways to write m, m odd not prime, as a sum of arithmetic progressions

Input: Positive odd integer m not prime.
Output: Prints all S ways

Initialize integers $q = \lfloor \sqrt{m} \rfloor$, $t = 3$
while $t \leq q$ **do**
  $g = \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$
  if $m \equiv 0 \pmod t$ and $g \neq 0$ **then**
    for $d = 1, \ldots, g$ **do**
      $a = \frac{1}{2}\left(\frac{m}{t} - (2t-1)d\right)$
      Print '$m = a + (a+d) + \cdots + (a + (r-1)d)$'
    end for
  end if
  $t = t + 2$
end while

Similarly, this algorithm also has the time complexity $O(q \cdot g) = O(q)$.

3.2 Examples

Example 3.1. *Let $m = 65$. In how many ways can we write m as a sum of a sum of arithmetic progressions when $r > 2$ even?*

We use Algorithm 4. First, $q = \lfloor \sqrt{65} \rfloor = 8$. Then, $3 \leq t \leq 8$. The only t that divides $m = 65$ is $t = 5$. Now, $g = \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor = \left\lfloor \frac{65-2\cdot5}{5\cdot9} \right\rfloor = 1$, i.e., there is only one way to write $m = 65$ as a sum of arithmetic progressions when $t = 5$ or $r = 10$. The parameters are $a = 2$ and $d = 1$, which gives

$$65 = 2 + 3 + \cdots + 11.$$

Thus, combining with the result from Example 2.1, the total number of ways to write $m = 65$ as a sum of arithmetic progressions is 7.

Example 3.2. *Let $m = 1125$. In how many ways can we write m as a sum of arithmetic progressions when $r > 2$ even?*

We use Algorithm 4. First, $q = \lfloor \sqrt{m} \rfloor = \lfloor \sqrt{1125} \rfloor = 33$. Then, $3 \leq t \leq 33$. Since $m = 1125$, the possible values of t are $3, 5, 9, 15, 25$. Then,

t	$g(t)$
3	74
5	24
9	7
15	2
25	1
S	$54 = \frac{74}{2} + \frac{24}{2} + \frac{7+1}{2} + \frac{2}{2} + \frac{1+1}{2}$

There are 54 *ways to write* $m = 1125$ *as a sum of arithmetic progressions when* $r > 2$ *even. Thus, combining with the result from Example 2.2, there are* 585 *ways.*

Example 3.3. *Let* $m = 6125$. *In how many ways can we write* m *as a sum of arithmetic progressions when* $r > 2$ *even?*

We use Algorithm 4. First, $q = \lfloor\sqrt{m}\rfloor = \lfloor\sqrt{6125}\rfloor = 78$. *Then,* $3 \leq t \leq 78$. *Since* $m = 6125$, $t = 5, 7, 25, 35, 49$. *Then,*

t	$g(t)$
5	135
7	67
25	4
35	2
49	1
S	$106 = \frac{135+1}{2} + \frac{67+1}{2} + \frac{4}{2} + \frac{2}{2} + \frac{1+1}{2}$

There are 106 *ways to write* $m = 6125$ *as a sum of arithmetic progressions when* $r > 2$ *even. Thus, combining with the result from Example 2.3, there are* 1044 *ways.*

Example 3.4. *Let* $m = 7^5$. *In how many ways can we write* m *as a sum of arithmetic progressions when* $r > 2$ *even?*

Using Algorithm 5, we have $q = \left\lfloor\frac{5}{2}\right\rfloor = 2$. *Then* $j = 1, 2$. *Let* $g(j) = \left\lfloor\frac{1}{2}\left(\frac{p^{k-j}-2}{2p^j-1} + 1\right)\right\rfloor$. *Then,*

j	$g(j)$
1	92
2	2
S	94

There are 94 *ways to write* 7^5 *as Eq.* (4) *when* $r > 2$ *is even. Thus, combining with the result from Example 2.4, there is a total of* 908 *ways to write* 7^5 *as Eq.* (4).

Example 3.5. *Let* $m = 3^{10}$. *In how many ways can we write* m *as a sum of arithmetic progressions when* $r > 2$ *even?*

Using Algorithm 5, we have $r = 2 \cdot 3^j$, $j = 1, 2, \cdots, 5$. *Let* $g(j) = \left\lfloor\frac{1}{2}\left(\frac{p^{k-j}-2}{2p^j-1} + 1\right)\right\rfloor$. *Then,*

j	$g(j)$
1	1968
2	193
3	21
4	2
5	0
S	2184

There are 2184 *ways of writing* 3^{10} *as Eq.* (4) *when* $r > 2$ *is even. Thus, combining with the result from Example 2.5, there is a total of* 23694 *ways.*

4 Conclusion

We presented novel theorems and algorithms concerning the representation of a positive odd integer m as arithmetic progressions. The presented results greatly expand upon the work of the manuscript duology of Ho et al. [3] and Ho et al. [4], co-authored by Professor Chungwu Ho, to whom this manuscript is dedicated. Historically, there has been a great interest in representing integers and powers of integers as sums of arithmetic progressions. Notably, Sylvester's Theorem has received much recent attention in the field of number theory. We extended Sylvester's Theorem to include results for all positive odd m.

We solved the problem of counting the total number of ways one can represent m as a sum of arithmetic progressions. In doing so, we considered the two distinct cases of odd r and even r. We found that partitioning the solution in such a way results in convenient mathematical results and highly efficient computational algorithms; the mathematical convenience motivates a further exploration of sums of arithmetic progressions, and the efficient algorithms encourages the adoption of the presented results. Representing positive even integers as the sum of arithmetic progressions will be a focus of future work in this paper.

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