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On the number of partitions of a number into distinct divisors

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Abstract: Let $p_{\text{dsd}}(n)$ be the number of partitions of n into distinct squarefree divisors of n. In this note, we find a lower bound for $p_{\text{dsd}}(n)$, as well as a sequence of n for which $p_{\text{dsd}}(n)$ is unusually large.
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1 Introduction

A *partition* of n is a representation of n as an unordered sum of positive integers. We let $p(n)$ and $p_d(n)$ be the number of partitions of n and the number of partitions of n into distinct parts. In 1918, Hardy and Ramanujan [5, §1.4, 7.1] proved two of the seminal results on partitions, obtaining asymptotic formulae for $p(n)$ and $p_d(n)$.

Theorem 1.1. *As $n \to \infty$, we have*

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \quad p_d(n) \sim \frac{1}{4\sqrt[4]{3n^3}} \exp\left(\pi\sqrt{\frac{n}{3}}\right).$$

Seven decades later, Bowman submitted a problem to the American Mathematical Monthly, asking for an asymptotic formula for the number of partitions of n into *divisors* of n. Erdős and Odlyzko then found precise bounds for this quantity, which we call $p_{\text{div}}(n)$ [8, Sequence A018818]. From here on, we also let $d(n)$ be the number of divisors of n.

Theorem 1.2 ([1]). *As $n \to \infty$, we have*

$$n^{(1+O(1/\log\log n))(d(n)/2-1)} \leq p_{div}(n) \leq n^{(1+o(1))d(n)/2}.$$

In this note, we consider another type of partition. We let $p_{\text{dd}}(n)$ and $p_{\text{dsd}}(n)$ be the number of partitions of n into distinct divisors and distinct *squarefree* divisors of n, respectively. Though these quantities appear in the Online Encyclopedia of Integer Sequences [8, Sequences A033630, A225245], we are unaware of any published research on them.

The functions $p_{\text{dd}}(n)$ and $p_{\text{dsd}}(n)$ have very erratic behavior. Let $\sigma(n)$ be the sum of the divisors of n. We say that n is *abundant* if $\sigma(n) > 2n$, *deficient* if $\sigma(n) < 2n$, and *perfect* if $\sigma(n) = 2n$. If n is deficient, then $p_{\text{dd}}(n) = 1$ and if n is perfect, then $p_{\text{dd}}(n) = 2$. While the perfect numbers are sparse, the deficient numbers have a density of approximately 0.7524 [6], which implies that $p_{\text{dd}}(n) = 1$ about $3/4$ of the time. (The fact that deficient numbers even have a density was independently proved by Chowla [2], Davenport [3], and Erdős [4]. It is also a consequence of the Erdős-Wintner Theorem [10, Theorem III.4.1].) Even still, we show that $p_{\text{dd}}(n)$ and $p_{\text{dsd}}(n)$ can be quite large.

Theorem 1.3. *For a given i, let p_i be the i-th prime. If $n = p_1 p_2 \cdots p_k$ for some k, then*

$$p_{dd}(n) = p_{dsd}(n) \geq (1+o(1))\frac{2^{d(n)/4}\log n}{n\log\log n} = \exp\left(\exp\left((\log 2 + o(1))\frac{\log n}{\log\log n}\right)\right).$$

This result is very close to being optimal in the sense that $p_{\text{dd}}(n)$ cannot be substantially larger than the bound in Theorem 1.3. Because n has $d(n)$ divisors, there are $2^{d(n)}$ sets of divisors of n, which implies that $p_{\text{dd}}(n) \leq 2^{d(n)}$. In addition [10, Theorem I.5.4],

$$d(n) \leq 2^{(1+o(1))\log n/\log\log n},$$

which implies that

$$p_{\text{dd}}(n) \leq \exp\left(\exp\left((\log 2 + o(1))\frac{\log n}{\log\log n}\right)\right)$$

for all n.

We can actually get a slightly better upper bound than $2^{d(n)}$. A classic theorem of Sárközy and Szemerédi [9] states that for a given m, k, a set of k real numbers has at most

$$(1+o(1))\frac{8}{\sqrt{\pi}}\frac{2^k}{k^{3/2}}$$

subsets with sum m and that this bound is optimal. (Recent results such as [7, Theorem 2.1] suggest that one can get a better bound if the elements of the set do not lie in a small number of arithmetic progressions. It would be interesting to know if this condition applies to the divisors of a given n.) Hence,

$$p_{\mathrm{dd}}(n) \leq (1+o(1))\frac{8}{\sqrt{\pi}}\frac{2^{d(n)}}{d(n)^{3/2}}.$$

We also find some additional lower bounds on $p_{\mathrm{dd}}(n)$ and $p_{\mathrm{dsd}}(n)$ which may be of independent interest. In particular, if n is a multiple of a large power of 2, we can obtain the following result.

Theorem 1.4. *Let $n = 2^a m$ with $m > 1$ odd. If $2^{a+1} > \sigma(m)$, then*

$$p_{dd}(n) \geq \left\lfloor \frac{2^{a+1}-1}{\sigma(m)-1} \right\rfloor^{d(m)-1}.$$

As $m \to \infty$, we have $\sigma(m) = m^{1+o(1)}$ (which we discuss in more detail later). Fix $\epsilon > 0$. If $n = 2^a m$ is a sufficiently large number and $m < n^{(1/2)-\epsilon}$, then $2^{a+1} > \sigma(m)$, allowing us to use the previous theorem. Thus, the number of $n \leq x$ satisfying the conditions of Theorem 1.4 is equal to $x^{(1/2)+o(1)}$.

A similar result holds for squarefree divisors. From here on, $\mathrm{rad}(m)$ is the radical of m, i.e., the largest squarefree divisor of m.

Theorem 1.5. *Let $n = q_1 q_2 \cdots q_k m$ where the q_i's are an increasing sequence of primes with $q_1 = 2$ and $q_{i+1} \leq \sigma(q_1 \cdots q_i) + 1$. If $q_1, \ldots, q_k \nmid m$ and*

$$\sigma(q_1 \cdots q_k) < n < \sigma(q_1 \cdots q_k)(\sigma(rad(m)) - 1),$$

then

$$p_{dsd}(n) \geq \left\lfloor \frac{\sigma(q_1 \cdots q_k)}{\sigma(rad(m))-1} \right\rfloor^{d(rad(m))-1}.$$

Note that if p_i is the i-th prime, then Euclid's proof of the infinitude of primes implies that $p_{i+1} \leq p_1 \cdots p_i + 1$. So, our bound applies to $n = p_1 \cdots p_k m$, where m is a number whose prime factors are greater than p_k. Though the q_i's in Theorem 1.5 do not have to be consecutive, they cannot grow too quickly either.

2 The main result

We begin this section by showing that for certain numbers m, we can write all numbers less than or equal to $\sigma(m)$ as the sum of distinct squarefree divisors of m. (Note that in the following lemma, m is already squarefree, making each of its divisors squarefree as well.)

Lemma 2.1. *Let $m = q_1 q_2 \cdots q_k$, where the q_i's are distinct primes, $q_1 = 2$, and $q_{i+1} \leq \sigma(q_1 q_2 \cdots q_i) + 1$ for all $i < k$. Then we can express every number less than or equal to $\sigma(m)$ as a sum of distinct divisors of m.*

Proof. We prove this result by induction on k. Clearly, it holds for $k = 1$ because we can express 1, 2, and 3 as sums of distinct divisors of 2. Suppose $k > 1$ and we already have the result for $k-1$. For a given $q_k \leq (q_1+1)\cdots(q_{k-1}+1)+1$, we have that

$$\bigcup_{i,j \leq (q_1+1)\cdots(q_{k-1}+1)} \{iq_k + j\} = [0, (q_1+1)\cdots(q_k+1)].$$

By assumption, we can express every number less than or equal to $(q_1+1)\cdots(q_{k-1}+1)$ as a sum of distinct divisors of $q_1\cdots q_{k-1}$. In particular, for any $i,j \le (q_1+1)\cdots(q_{k-1}+1)$, we can write

$$i = \sum_{d\in S_1} d, \quad j = \sum_{d\in S_2} d$$

for some sets S_1 and S_2 of divisors of $q_1\cdots q_{k-1}$. So,

$$iq_k + j = \sum_{d\in S_1} dq_k + \sum_{d\in S_2} d,$$

which is a sum of distinct divisors of m. □

Using this result, we prove our main theorem.

Proof of Theorem 1.3. Let $n = p_1\cdots p_k$ and let $C > e^\gamma - \epsilon$ for a small positive ϵ, where γ is the Euler–Mascheroni constant. In addition, let q be the prime closest to $C\log\log n$. Because the ratio of consecutive primes goes to 1, we have $q \sim C\log\log n$ as $n\to\infty$. If n is sufficiently large, then $q < p_k \sim \log n/\log\log n$, and so we have $q|n$. From here on, we let $n = qp_km$.

There are $2^{d(m)}$ sets of distinct divisors of m, each of which has a sum of at most $\sigma(m)$. By the Pigeonhole Principle, there exists some $a \le \sigma(m)$ which has at least $2^{d(m)}/\sigma(m)$ representations as a sum of distinct divisors of m. In addition, $\sigma(m) - a$ also has at least $2^{d(m)}/\sigma(m)$ representations because we can simply take the complement of any subset of the set of divisors of m which add up to a. If we let $A = \max(a, \sigma(m)-a)$, we have a number $\ge \sigma(m)/2$ which we can write as a sum of distinct divisors of m in at least $2^{d(m)}/\sigma(m)$ different ways.

At this point, we show that the set $\{p_1, p_2, \dots, p_k\}\backslash\{q\}$ satisfies the conditions of Lemma 2.1. If $p_r < q$, then Euclid's proof of the infinitude of the primes shows that $p_r \le p_1\cdots p_{r-1}+1$. Suppose $p_r > q$ with $r \le k$. As $q\to\infty$, the product of the primes $< p_r$ excluding q is still asymptotic to e^{p_r}, which is much larger than p_r.

Define $B = n - qA$ with A, q, and n as above, implying that $n = qA + B$. We already know that we can express A as a sum of distinct divisors of m in at least $2^{d(m)}/\sigma(m)$ ways. If we can show that $B \ge 0$ and that it is possible to express B as a sum of distinct divisors of p_km, then we will be able to find at least $2^{d(m)}/\sigma(m)$ expressions for n as a sum of distinct divisors of n. Simply take each sum for A and multiply every element by q, then add the sum for B. If $a_1, a_2, \dots, a_s$ are distinct divisors of m with sum A and $b_1, b_2, \dots, b_t$ are distinct divisors of p_km with sum B, then

$$n = q(a_1+\cdots+a_s) + (b_1+b_2+\cdots+b_t).$$

Each qa_i and b_j is a divisor of m. We already know that the a_i's are distinct and that the b_j's are distinct. In addition, $qa_i \ne b_j$ for all i and j because $q \nmid p_km$ and each b_j divides p_km. We have already established that there are $2^{d(m)}/\sigma(m)$ tuples $(a_1,\dots,a_s)$. We still need to show that there is at least one tuple $(b_1,\dots,b_t)$.

We prove that $B \in [0, \sigma(p_km)]$. From there, Lemma 2.1 implies that B is a sum of distinct divisors of p_km. In order to prove that $B \ge 0$, we need to show that $qA \le n$. Note that $qA \le q\sigma(m)$. Mertens' Theorem [10, Theorem I.1.12] gives us

$$q\sigma(m) = qm \prod_{\substack{p \le p_{k-1} \\ p \ne q}} \left(1 + \frac{1}{p}\right) \sim e^{\gamma} \frac{n}{p_k} \log p_{k-1} \sim e^{\gamma} \frac{n(\log\log n)^2}{\log n}.$$

If n is sufficiently large, then $qA < n$.

We now show that $B \le \sigma(p_k m)$. Because A is positive, $B = n - qA < n$. We prove that $B \le \sigma(p_k m)$ by showing that $\sigma(p_k m) > n$. We apply Mertens' Theorem again, obtaining

$$\sigma(p_k m) \sim e^{\gamma} p_k m \log\log(p_k m) \sim n(e^{\gamma} \log\log n)/q \sim (e^{\gamma}/(e^{\gamma} - \epsilon))n.$$

Putting everything together gives us $p_{\text{dsd}}(n) \ge 2^{d(m)}/\sigma(m)$. In addition,

$$\sigma(m) \sim e^{\gamma} m \log\log m \sim e^{\gamma} \frac{n \log\log n}{q p_k} \sim \frac{e^{\gamma}}{e^{\gamma} - \epsilon} \frac{n \log\log n}{\log n}.$$

We also have $d(m) = d(n)/4$. Letting $\epsilon \to 0$ gives us our desired result. $\square$

3 Proofs of Theorems 1.4 and 1.5

In order to prove Theorems 1.4 and 1.5, we provide alternate characterizations of $p_{\text{dd}}(n)$ and $p_{\text{dsd}}(n)$ in terms of lattice points. From here on, we let $\mathcal{D}(k)$ be the set of divisors of an integer k.

Lemma 3.1. *If $n = 2^a m$ with m odd, then*

$$p_{dd}(n) = \#\left\{ (x_d)_{d \in \mathcal{D}(m)\setminus\{1\}} : x_d \le 2^{a+1} - 1 \textit{ and } n - 2^{a+1} + 1 \le \sum_{d \in \mathcal{D}(m)\setminus\{1\}} d x_d \le n \right\}.$$

Proof. Let S be a set of divisors of n with sum n. Every element of S has the form $2^b d$ with $b \le a$ and $d|m$. Let x_d be the sum of 2^b for all b satisfying $2^b d \in S$. Then,

$$n = \sum_{s \in S} s = \sum_{d|m} d \sum_{2^b d \in S} 2^b = \sum_{d|m} d x_d.$$

By definition, x_d can be any sum of distinct powers of 2 up to 2^a. Hence, x_d can be any non-negative integer less than 2^{a+1}. We have

$$\sum_{d \in \mathcal{D}(m)\setminus\{1\}} x_d d = n - x_1.$$

Setting d to 1 shows that x_1 can be any non-negative integer less than or equal to $2^{a+1} - 1$. Therefore,

$$n - 2^{a+1} + 1 \le \sum_{d \in \mathcal{D}(m)\setminus\{1\}} x_d d \le n,$$

where the only restriction on the x_d's is that they are also less than or equal to $2^{a+1} - 1$.

Conversely, suppose $(x_d)_{d \in \mathcal{D}(m)\setminus\{1\}}$ is a tuple satisfying the conditions of the lemma. If we let

$$x_1 = n - \sum_{d \in \mathcal{D}(m)\setminus\{1\}} d x_d,$$

then the sum of all dx_d is equal to n. We also observe that $x_1 \geq 0$ because the sum of dx_d over all $d \in \mathcal{D}(m)\backslash\{1\}$ is at most n. In addition, we can break each x_d into powers of 2. For each $d \in \mathcal{D}(m)$, let T_d be the unique set of non-negative integers satisfying

$$x_d = \sum_{i \in T_d} 2^i.$$

By assumption, $x_d \leq 2^{a+1} - 1$ for all $d > 1$. In addition, $x_1 \leq 2^{a+1} - 1$ because the sum of dx_d over all possible $d > 1$ is at most $n - 2^{a+1} - 1$. Therefore, every set T_d consists of elements less than or equal to a. We now have a representation

$$n = \sum_{d \in \mathcal{D}(m)} x_d \sum_{i \in T_d} 2^i.$$

Each number $2^i x_d$ is a distinct divisor of n and the set of pairs $(2^i, x_d)$ is uniquely determined by $(x_d)_{d \in \mathcal{D}(m)\backslash\{1\}}$. Therefore, each tuple corresponds to a different representation of n as a sum of distinct divisors of n. $\square$

Lemma 3.2. *Let $n = q_1 \cdots q_k m$ where $q_1, \ldots, q_k$ is an increasing sequence of primes with $q_1 = 2$ and $q_{i+1} \leq \sigma(q_1 \cdots q_i) + 1$ for all i and $q_1, \ldots, q_k \nmid m$. In addition, let $\mathcal{S} = \mathcal{D}(rad(m))$ be the set of squarefree divisors of m. Then,*

$$p_{dsd}(n) = \#\left\{(x_d)_{d \in \mathcal{S}\backslash\{1\}} : x_d \leq \sigma(q_1 \cdots q_k) \text{ and } n - \sigma(q_1 \cdots q_k) \leq \sum_{d \in \mathcal{S}\backslash\{1\}} dx_d \leq n\right\}.$$

Proof. Our proof is similar to the proof of the previous lemma. The squarefree divisors of m are simply the divisors of rad(m). Every sum of distinct squarefree divisors of n has the form

$$\sum_{d|\text{rad}(m)} d \sum_{s \in S_d} s,$$

where S_d is a set of divisors of $q_1 \cdots q_k$.

Using these results, we can bound $p_{\text{dsd}}(n)$ from above. Lemma 2.1 implies that we can express every number up to $\sigma(q_1 \cdots q_k)$ as a sum of distinct divisors of $q_1 \cdots q_k$. So, we can rewrite any sum of distinct divisors of n in the form

$$x_1 + \sum_{d \in \mathcal{S}} dx_d,$$

with each $x_d \leq \sigma(q_1 \cdots q_k)$. If this quantity equals n, the rightmost sum must lie in the interval $[n - \sigma(q_1 \cdots q_k), n]$ because $x_1 \leq \sigma(q_1 \cdots q_k)$. An argument similar to the last paragraph of the previous proof implies that every tuple $(x_d)_{d \in \mathcal{S}\backslash\{1\}}$ corresponds to a unique representation of n as a sum of distinct divisors of n. $\square$

Proof of Theorem 1.4. Let $(x_d)_{d \in \mathcal{D}(m)\backslash\{1\}}$ be a tuple of integers which lie in the interval

$$\left[\frac{n - 2^{a+1} + 1}{\sigma(m) - 1}, \frac{n}{\sigma(m) - 1}\right].$$

We show that this tuple satisfies the conditions of Lemma 3.1.

We note that for each x_d, we have

$$x_d \leq \frac{n}{\sigma(m)-1} = \frac{2^a m}{\sigma(m)-1} \leq 2^a \leq 2^{a+1}-1,$$

which shows that the x_d's are not too large. In addition,

$$\sum_{d\in\mathcal{D}(m)\setminus\{1\}} dx_d \leq \left(\sum_{d\in\mathcal{D}(m)\setminus\{1\}} d\right)\frac{n}{\sigma(m)-1} = n.$$

For the lower bound on this sum, we note that

$$\sum_{d\in\mathcal{D}(m)\setminus\{1\}} dx_d \geq \left(\sum_{d\in\mathcal{D}(m)\setminus\{1\}} d\right)\frac{n-2^{a+1}+1}{\sigma(m)-1} = n-2^{a+1}-1.$$

Thus, (x_d) satisfies the conditions of Lemma 3.1.

In order to obtain our desired result, we simply bound the number of possible tuples from below. By definition, each x_d lies in the interval $[(n-2^{a+1}+1)/(\sigma(m)-1), n/(\sigma(m)-1)]$, which contains at least $\lfloor (2^{a+1}-1)/(\sigma(m)-1)\rfloor$ integers. In addition, $\mathcal{D}(m)\setminus\{1\}$ contains $d(m)-1$ elements. So, the total number of possible tuples is at least

$$\left\lfloor \frac{2^{a+1}-1}{\sigma(m)-1}\right\rfloor^{d(m)-1}. \qquad \square$$

Proof of Theorem 1.5. This proof is similar to the previous one. We now let $(x_d)_{d\in\mathcal{S}\setminus\{1\}}$ be a tuple of integers which lie in the interval

$$\left[\frac{n-\sigma(q_1\cdots q_k)}{\sigma(\mathrm{rad}(m))-1}, \frac{n}{\sigma(\mathrm{rad}(m))-1}\right],$$

where $\mathcal{S} = \mathcal{D}(\mathrm{rad}(m))$. In this case, we need to show that (x_d) satisfies the conditions of Lemma 3.2.

We now have

$$x_d \leq \frac{n}{\sigma(\mathrm{rad}(m))-1} \leq \sigma(q_1\cdots q_k)$$

for all $d \in \mathcal{S}\setminus\{1\}$. In addition,

$$\begin{aligned}\sum_{d\in\mathcal{S}\setminus\{1\}} dx_d &\leq \left(\sum_{d\in\mathcal{S}\setminus\{1\}} d\right)\frac{n}{\sigma(\mathrm{rad}(m))-1} = n,\\ \sum_{d\in\mathcal{S}\setminus\{1\}} dx_d &\geq \left(\sum_{d\in\mathcal{S}\setminus\{1\}} d\right)\frac{n-\sigma(q_1\cdots q_k)}{\sigma(\mathrm{rad}(m))-1} = n-\sigma(q_1\cdots q_k).\end{aligned}$$

To finish the proof, we observe that the interval

$$\left[\frac{n-\sigma(q_1\cdots q_k)}{\sigma(\mathrm{rad}(m))-1}, \frac{n}{\sigma(\mathrm{rad}(m))-1}\right]$$

contains at least $\lfloor \sigma(q_1 \cdots q_k)/(\sigma(\text{rad}(m)) - 1) \rfloor$ integers and that $\mathcal{S}\backslash\{1\}$ has $d(\text{rad}(m)) - 1$ elements. Therefore, there are at least

$$\left\lfloor \frac{\sigma(q_1 \cdots q_k)}{\sigma(\text{rad}(m)) - 1} \right\rfloor^{d(\text{rad}(m))-1}$$

acceptable tuples. □

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