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Double domination number of graphs generated from unary products

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Abstract: A subset S of $V(G)$ is a double dominating set of a graph G if S dominates every vertex of G at least twice. The minimum cardinality of a double dominating set denoted by $\gamma_{2\times}(G)$, is the double domination number of G. In this paper, we identified the double domination number of graphs generated by applying various unary operations on standard graph classes.
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1 Introduction

All graphs we considered are finite, simple and undirected. Let $G = (V, E)$ be a simple graph with no isolated vertices. The open neighborhood of a vertex $v \in V(G)$ is the set $N(v) = \{u \in V(G) | uv \in E(G)\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup v$. A vertex of degree 1 is called a pendant vertex and the vertex adjacent to a pendant vertex is called a support vertex. A vertex that is adjacent to all other vertices of a graph is called a universal vertex of the graph. For graph theory terminology and notation, we in general follow [3, 14].

A subset D of $V(G)$ is a dominating set of G if every vertex in $V-D$ is adjacent to some vertex in D. The domination number of G denoted by $\gamma(G)$ is the minimum cardinality of a dominating set of G [11]. For more details on domination in graphs, we refer the book, Fundamentals of domination in graphs [11]. Frank Harary and Teresa Haynes [8, 9] introduced the concept of double domination and double domination number of graphs. A subset S of $V(G)$ is a double dominating set of a graph G if S dominates every vertex of G at least twice. Equivalently, if $|N[v] \cap S| \geq 2 \; \forall v \in V(G)$, then S is a double dominating set of G. The minimum cardinality of a double dominating set denoted by $\gamma_{2\times}(G)$ is the double domination number of G.

A detailed survey on double domination in graphs can be found in [10]. One of its chapters, Multiple domination by Hansberg and Volkmann provides an overview of all the relevant research results on double domination that have been found up to 2020. Abel Cabrera Martinez and Juan Alberto Rodriguez Velazquez [4] have improved some results on double domination included in the book [10]. Aysun Aytan and Aysen Mutlu [1] have studied the double domination of shadow graphs of some graphs such as cycle, path, star, complete bipartite, and wheel. In [2] the authors have examined the double domination parameter for some shadow distance related graphs. Abel Cabrera Martinez and Alejandro Estrada Moreno [5] have explored this domination parameter in the rooted product of graphs. Up to our knowledge, no one had explored the double domination number on unary product of graphs. This motivated us to explore how the double domination number gets affected if we apply unary operations on various classes of graphs.

The objective of this study is to analyze this domination parameter in the unary product of graphs. In section 2 we provide some preliminary results. In section 3 we obtain some main results on double domination number of unary product of graphs.

2 Some preliminary results

The following table shows the double domination number of various standard graph classes.

Table 1. Double domination number of various standard graph classes

Graph class	$\gamma_{2\times}(G)$	**Reference**
C_n	$\lceil \frac{2n}{3} \rceil$	C. Xuegang and S. Liang [15]
P_n	$\lceil \frac{2(n+1)}{3} \rceil$	C. Xuegang and S. Liang [15]
$K_{1,n}$	$1+n$	C. Xuegang and S. Liang [15]
$K_{m,n}$	4	F. Harary and T. W. Haynes [8]
K_n	2	F. Harary and T. W. Haynes [8]

Observation 2.1. [6] *The double dominating set of a graph G must contain all pendant vertices and support vertices of G.*

Observation 2.2. [9] *Let G be a graph of order $n \geq 2$ without isolated vertices. Then $\gamma_{2\times}(G) = 2$ if and only if G has two universal vertices.*

Observation 2.3. *Let $G = S_{r,t}$ be a double star graph with $r + t$ vertices. Then $\gamma_{2\times}(G) = r + t$.*

Proof. Since, every vertex of G is either a pendant vertex or a support vertex, the proof follows from Observation 1. □

Theorem 2.1. *Let $G = K_{p,q,r}$ be a complete tripartite graph of order $p+q+r$ where, $p \leq q \leq r$. Then*

$$\gamma_{2\times}(G) = \begin{cases} 2, & \text{if } p = 1; q = 1; r \geq 1 \\ 1 + q, & \text{if } p = 1; q, r > 1 \\ p + 1, & \text{if } p, q, r \geq 2 \end{cases}$$

Proof. Let G be a complete tripartite graph with $p+q+r$ vertices, $\{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q, w_1, w_2, \ldots, w_r\}$.

Case 1. $\gamma_{2\times}(G) = 2$; when $p = 1, q = 1, r > 1$

In this case, G has two universal vertices u_1, v_1. The proof of this follows from Observation 2.

Case 2. $\gamma_{2\times}(G) = 1 + q; p = 1; q, r > 1$

In this case, G has one universal vertex u_1. Consider the set $S = \{u_1, v_1, v_2, \ldots, v_q\}$. This S dominates every vertex of G at least twice. Therefore, S is a double dominating set of G. Here, $|S| = 1 + q$. Now to prove S is the minimum double dominating set of G. Let $|S| < 1 + q$ that is $|S| = q$. Without loss of generality, let $S = \{u_1, v_1, v_2, \ldots, v_{q-1}\}$ since, u_1 is a universal vertex S must contain u_1. This set dominates every vertex of G at least twice except v_q since, v_q is dominated only once by u_1. So, S must contain v_q. Thus, $S = \{u_1, v_1, \ldots, v_q\}$. Hence, $\gamma_{2\times}(G) = 1 + q$.

Case 3. $\gamma_{2\times}(G) = p + 1; p, q, r \geq 2$

In this case, G has no universal vertex. Consider the set $S = \{u_1, u_2, \ldots, u_p, v_i\}$ for any $i = 1, 2, \ldots, q$ or $S = \{u_1, u_2, \ldots, u_p, w_k\}$ for any $k = 1, 2, \ldots, r$. This set dominates every vertex of G at least twice. Therefore, S is a double dominating set of G. Here, $|S| = p + 1$. Now to prove S is the minimum double dominating set of G. Let $|S| < p + 1$ that is $|S| = p$. Without loss of generality, let $S = \{u_1, u_2, \ldots, u_p\}$, this S dominates every vertex of G at least twice except $\{u_i\} \forall i = 1, 2, \ldots, p$ since, $|N(u_i \cap S)| < 2$. Therefore S must contain either v_i for any $i = 1, 2, \ldots, q$; or w_k for any $k = 1, 2, \ldots, r$ to dominate $\{u_1, u_2, \ldots, u_p\}$ at least twice. Thus, $|S| = p + 1$. Hence, $\gamma_{2\times}(G) = p + 1$. □

In the next section we identify the impact of various unary operations on double domination number of some standard classes of graphs.

2.1 Unary operations on graphs

Graph operations are operations which produce new graphs from initial graphs. They include both unary and binary operations. Unary operations create a new graph from a single graph. Although there are various unary operations in the literature, we concentrate on total graph of a graph, generalised corona of a graph, Myceilskian of a graph, duplication of a vertex and subdivision of edges of a graph.

2.2 Total graph operation

Definition 2.1. [7] **(Total graph, $T(G)$)** *The total graph, $T(G)$ of a graph G is the graph whose set of vertices is the union of the set of vertices and of the set of edges of G, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident.*

For example, consider the cycle graph C_4.

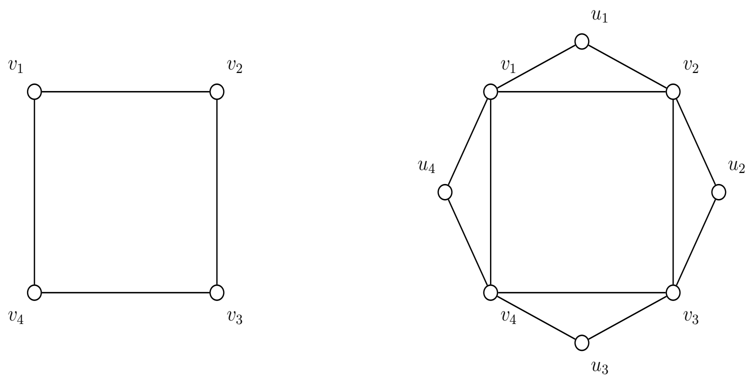

Figure 1. C_4 and $T(C_4)$

Table 2. Impact of total graph operation on double domination number of connected graphs

Graph class	$\gamma_{2\times}(G)$	$\gamma_{2\times}(T(G))$	$\lvert V(G)\rvert$	$\lvert V(T(G))\rvert$
C_n	$\lceil \frac{2n}{3} \rceil$	n	n	$2n$
P_n	$\lceil \frac{2(n+1)}{3} \rceil$	n	n	$2n-1$
W_n	$\lceil \frac{n}{3} \rceil + 1$	$n+1$	$n+1$	$3n+1$
$K_{m,n}$	4	$m+n$	$m+n$	$m(n+1)+n$
K_n	2	n	n	$\frac{2n+n(n-1)}{2}$
$K_{1,n}$	$1+n$	$1+n$	$n+1$	$2n+1$
$S_{r,t}$	$r+t$	$r+t$	$r+t$	$2(r+t)-2$

Theorem 2.2. *Let G be a connected graph with n vertices, then $\gamma_{2\times}(T(G)) = n$.*

Proof. Let G be a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges $\{e_1, e_2, \ldots, e_m\}$. For every edge $\{e_i; 1 \leq i \leq m\}$, add a new vertex $\{u_i; 1 \leq i \leq m\}$ and join the new vertex to the end vertices of the corresponding edge $\{e_i; 1 \leq i \leq m\}$ to get $T(G)$. Observe that, $\deg(u_i) = 2$ for any i such that $1 \leq i \leq m$. Therefore, to dominate every vertex of G at least twice all $\{v_j; 1 \leq j \leq n\}$ must be included. This implies that the set $S = \{v_1, v_2, \ldots, v_n\}$ is a double dominating set of $T(G)$ and $|S| = n$. Now to prove S is a minimum double dominating set of $T(G)$. Let $|S| < n$ that is $|S| = n - 1$. Without loss of generality let $S = \{v_1, v_2, \ldots, v_{n-1}\}$. Then, the vertices u_{m-1}, u_m are dominated only once so, we should include either u_m or v_n. Thus, S is a minimum double dominating set of $T(G)$. Hence, $\gamma_{2\times}(T(G)) = n$. □

2.3 Generalised corona of a graph

Definition 2.2. [13] **(Generalised corona of a graph, $G \circ H_i$).** *Given the simple graphs G, $H_1, \ldots, H_n$, where $n = |V(G)|$, the generalized corona, denoted $G \circ H_i; 1 \leq i \leq n$ is the graph obtained by taking one copy of graphs $G, H_1, \ldots, H_n$ and joining the i-th vertex of G to every vertex of H_i.*

For example, consider the complete bipartite graph $K_{1,3}$.

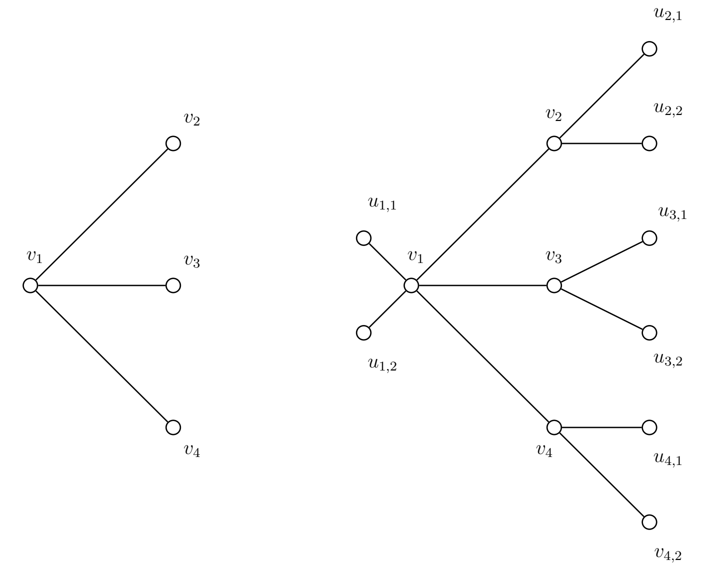

Figure 2. $(K_{1,3})$ and $(K_{1,3} \circ K_1)$

Table 3. Impact of generalised corona operation on double domination number of connected graphs

Graph class	$\gamma_{2\times}(G)$	$\gamma_{2\times}(G \circ K_1)$	$\lvert V(G)\rvert$	$\lvert V(G \circ K_1)\rvert$
C_n	$\lceil \frac{2n}{3} \rceil$	$n(k+1)$	n	$n(k+1)$
P_n	$\lceil \frac{2(n+1)}{3} \rceil$	$n(k+1)$	n	$n(k+1)$
W_n	$\lceil \frac{n}{3} \rceil + 1$	$(n+1)(k+1)$	$n+1$	$(n+1)(k+1)$
$K_{m,n}$	4	$(m+n)(k+1)$	$m+n$	$(m+n)(k+1)$
K_n	2	$n(k+1)$	n	$n(k+1)$
$K_{1,n}$	$1+n$	$(1+n)(k+1)$	$n+1$	$(n+1)(k+1)$
$S_{r,t}$	$r+t$	$(r+t)(k+1)$	$r+t$	$(r+t)(k+1)$

Theorem 2.3. *Let G be a connected graph with n vertices, then $\gamma_{2\times}((G \circ K_1)) = n(k+1)$ where, k is the number of copies of K_1 added to each vertex of G.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be n vertices of G. Add k number of vertices to each vertex $\{v_i; i = 1, 2, \ldots, n\}$ to get $(G \circ K_1)$. Let the new vertices added be $\{u_{1,1}, u_{1,2}, \ldots, u_{1,k}, u_{2,1}, u_{2,2}, \ldots, u_{2,k}, \ldots, u_{n,1}, u_{n,2}, \ldots, u_{n,k}\}$. Thus, $(G \circ K_1)$ contains $n(k+1)$ vertices. Every vertex of $(G \circ K_1)$ is either a pendant vertex or a support vertex. Thus, from Observation 1, a double dominating set of $(G \circ K_1)$ must contain every vertex of $(G \circ K_1)$. Hence, $\gamma_{2\times}((G \circ K_1)) = n(k+1)$. □

2.4 Myceilskian operation

Definition 2.3. [14] **(Myceilskian operation, $\mu(G)$).** *Let G be a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$. The Mycielskian graph, $\mu(G)$ contains G itself as a subgraph, together with $n+1$ additional vertices; a vertex $u_i; 1 \leq i \leq n$ corresponding to each $v_i; 1 \leq i \leq n$ and an extra vertex w. Each u_i is connected to w by an edge.*

For example, consider a path graph P_4.

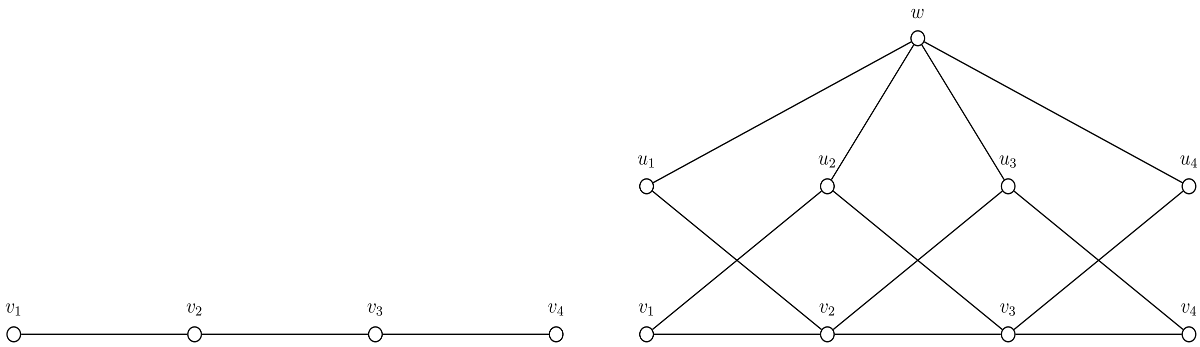

Figure 3. (P_4) and $\mu(P_4)$

Table 4. Impact of Myceilskian operation on double domination number of connected graphs

Graph class	$\gamma_{2\times}(G)$	$\gamma_{2\times}(\mu(G))$	$\lvert V(G)\rvert$	$\lvert V(\mu(G))\rvert$
C_n	$\lceil\frac{2n}{3}\rceil$	$\lceil\frac{2n}{3}\rceil + 2$	n	$2n+1$
P_n	$\lceil\frac{2(n+1)}{3}\rceil$	$\lceil\frac{2(n+1)}{3}\rceil+2$	n	$2n+1$
W_n	$\lceil\frac{n}{3}\rceil + 1$	4	$n+1$	$2n+3$
$K_{m,n}$	4	5	$m+n$	$2(m+n)+1$
K_n	2	4	n	$2n+1$
$K_{1,n}$	$1+n$	4	$n+1$	$2n+3$
$S_{r,t}$	$r+t$	5	$r+t$	$2(r+t)+1$

Theorem 2.4. *Let G be a connected graph of order n with a universal vertex, then $\gamma_{2\times}(\mu(G)) = 4$.*

Proof. Let G be a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$. Let v_n be the universal vertex. Add a new vertex $u_i; 1 \leq i \leq n$ to each $v_i; 1 \leq i \leq n$ and connect $u_i; 1 \leq i \leq n$ to the neighbors of corresponding $v_i; 1 \leq i \leq n$. Add an extra vertex w and connect each $u_i; 1 \leq i \leq n$ to w by an edge to get $\mu(G)$. Let S be the minimum double dominating set of $\mu(G)$. Since, v_n is the universal vertex and u_n is the copy of v_n, both are connected to every vertex $v_i; 1 \leq i \leq n-1$ and together double dominate every vertex $v_i; 1 \leq i \leq n-1$ and dominate every vertex $u_i; 1 \leq i \leq n$ once. Let $v_n, u_n \in S$. To double dominate $u_i; 1 \leq i \leq n$, we include w, since w is connected to every $u_i; 1 \leq i \leq n$. Now let $S = \{v_n, u_n, w\}$. This set S dominates every vertex of $\mu(G)$ at least twice except for v_n, because $N(v_n) \neq \{v_n, u_n, w\}$. To double dominate v_n, we include v_i for any $i|1 \leq i \leq n-1$. Thus, $S = \{v_n, u_n, w, v_i\}$ for any $i|1 \leq i \leq n-1$ is a minimum double dominating set of $\mu(G)$ where, $|S| = 4$. Hence, $\gamma_{2\times}(\mu(G)) = 4$. □

Theorem 2.5. *Let G be a cycle graph of order n. Then $\gamma_{2\times}(\mu(G)) = \gamma_{2\times}(G) + 2 = \lceil \frac{2n}{3} \rceil + 2$.*

Proof. Let G be a cycle with n vertices $\{v_1, v_2, \ldots, v_n\}$. Add a new vertex $u_i; 1 \leq i \leq n$ corresponding to each $v_i; 1 \leq i \leq n$ and connect $u_i; 1 \leq i \leq n$ to the neighbors of corresponding $v_i; 1 \leq i \leq n$. Add an extra vertex w and connect each $u_i; 1 \leq i \leq n$ to w by an edge to get $\mu(G)$. Therefore, $\mu(G)$ has $(2n + 1)$ vertices $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, w\}$. Let S be the minimum double dominating set of G where, $|S| = \lceil \frac{2n}{3} \rceil$. Let S' be the double dominating set of $\mu(G)$. Then $S \subseteq S'$. Now S' dominates every vertex $v_i; 1 \leq i \leq n$ at least twice and dominates $u_i; 1 \leq i \leq n$ at least once. To double dominate every u_i and w, we include w, u_i; for any $i|1 \leq i \leq n$. Thus, $S' = S \cup \{u_i, w\}$ for any $i|1 \leq i \leq n$. Hence, $\gamma_{2\times}(\mu(G)) = \lceil \frac{2n}{3} \rceil + 2$. □

Theorem 2.6. *Let G be a path graph of order n. Then $\gamma_{2\times}(\mu(G)) = \gamma_{2\times}(G) + 2 = \lceil \frac{2(n+1)}{3} \rceil + 2$.*

Proof. Let G be a path graph with n vertices $\{v_1, v_2, \ldots, v_n\}$. Add a new vertex $u_i; 1 \leq i \leq n$ corresponding to each $v_i; 1 \leq i \leq n$ and connect $u_i; 1 \leq i \leq n$ to the neighbors of corresponding $v_i; 1 \leq i \leq n$. Add an extra vertex w and connect each $u_i; 1 \leq i \leq n$ to w by an edge to get $\mu(G)$. Therefore, $\mu(G)$ has $(2n + 1)$ vertices $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, w\}$. Let S be the minimum double dominating set of G where, $|S| = \lceil \frac{2(n+1)}{3} \rceil$. Then $S \subseteq S'$. Now S' dominates every vertex $v_i; 1 \leq i \leq n$ at least twice and dominates $u_i; 1 \leq i \leq n$ at least once. To double dominate every u_i and w, we include w, u_i; for any $i|1 \leq i \leq n$. Thus, $S' = S \cup \{u_i, w\}$ for any $i|1 \leq i \leq n$. Hence, $\gamma_{2\times}(\mu(G)) = \lceil \frac{2(n+1)}{3} \rceil + 2$. □

Theorem 2.7. *Let G be a complete bipartite graph of order $m + n$. Then $\gamma_{2\times}(\mu(G)) = 5$.*

Proof. Let G be a complete graph with $m + n$ vertices $\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}$. Add a new vertex v'_i and $u'_j|\ 1 \leq i \leq m; 1 \leq j \leq n$ for each $v_i; 1 \leq i \leq m$ and $u_j; 1 \leq j \leq n$, respectively, and connect them to the neighbors of corresponding v_i and u_j, respectively. Add an extra vertex w and connect each $v'_i, u'_j; 1 \leq i \leq m; 1 \leq j \leq n$ to w by an edge to get $\mu(G)$. Consider the set $S = \{v_p, v'_q, u_r, u'_t, w\}$ for any $1 \leq p, q \leq m; 1 \leq r, t \leq n$. This set dominates every vertex of $\mu(G)$ at least twice. Therefore, S is a double dominating set of $\mu(G)$ where, $|S| = 5$. Now to prove S is a minimum double dominating set, let $|S| < 5$. Without loss of generality, let $S = \{v_p, v'_q, u_r, u'_t\}$ for any $1 \leq p, q \leq m; 1 \leq r, t \leq n$. This S dominates every vertex of $\mu(G)$ at least twice but w is dominated only once. To double dominate w, we include it in S. Thus, S is a minimum double dominating set of $\mu(G)$. Hence, $\gamma_{2\times}(\mu(G)) = 5$. □

Theorem 2.8. *Let G be a double star graph of order $r + t$. Then $\gamma_{2\times}(\mu(G)) = 5$.*

Proof. Let G be a double star graph with $r + t$ vertices $\{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_t\}$. Let v_r, u_t be the two non-pendant vertices. Add a new vertex v'_i and u'_j; $1 \leq i \leq r; 1 \leq j \leq t$ for each $v_i; 1 \leq i \leq r$ and $u_j; 1 \leq j \leq t$, respectively, and connect them to the neighbors of corresponding v_i and u_j, respectively. Add an extra vertex w and connect each $v'_i, u'_j; 1 \leq i \leq r; 1 \leq j \leq t$ to w by an edge to get $\mu(G)$. Consider the set $S = \{v_r, u_t, v'_r, u'_t, w\}$. This set dominates every vertex of $\mu(G)$ at least twice. Therefore, S is a double dominating set of $\mu(G)$ where, $|S| = 5$. Since, every vertex $\{v_i, u_j, v'_i, v'_j; 1 \leq i \leq r - 1; 1 \leq j \leq t - 1\}$ are of degree 2 and their neighbors are $\{v_r, u_t, v'_r, v'_t, w\}$, S must contain these vertices and therefore S is a minimum double dominating set of $\mu(G)$. Hence, $\gamma_{2\times}(\mu(G)) = 5$. □

2.5 Duplication of a vertex

Definition 2.4. [12, 14] **(Duplication of a vertex, $D(vG)$).** *Duplication of a vertex v of graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$.*

For example, consider the cycle graph with 4 vertices C_4.

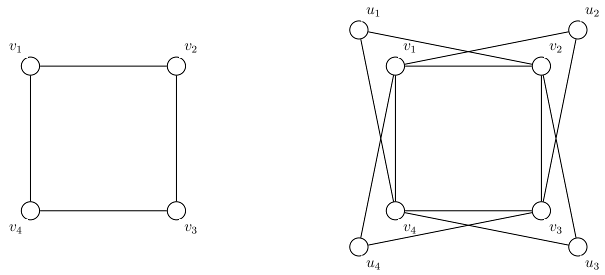

Figure 4. (C_4) and $D(vC_4)$

Table 5. Impact of duplication of a vertex operation on double domination number of connected graphs

Graph class	$\gamma_{2\times}(G)$	$\gamma_{2\times}(D(vG))$	$\lvert V(G)\rvert$	$\lvert V(D(vG))\rvert$
C_n	$\lceil \frac{2n}{3} \rceil$	n	n	$2n$
P_n	$\lceil \frac{2(n+1)}{3} \rceil$	$n+2$	n	$2n$
W_n	$\lceil \frac{n}{3} \rceil + 1$	$\lceil \frac{n}{2} \rceil + 1$	$n+1$	$2(n+1)$
$K_{m,n}$	4	4	$m+n$	$2(m+n)$
K_n	2	3	n	$2n$
$K_{1,n}$	$1+n$	$n+3$	$n+1$	$2(n+1)$
$S_{r,t}$	$r+t$	$r+t+2$	$r+t$	$2(r+t)$

Theorem 2.9. *Let $G = C_n$ be a cycle of order n. Then, $\gamma_{2\times}(D(vG)) = n$.*

Proof. Let G be a cycle with n vertices $\{v_1, v_2, \ldots, v_n\}$. For each $v_i; 1 \leq i \leq n$ add a new vertex $u_i; 1 \leq i \leq n$ and join $u_i; 1 \leq i \leq n$ to the neighbors of corresponding $v_i; 1 \leq i \leq n$ to get $D(vG)$. Consider the set $S = \{v_i\} \forall i = 1, 2, \ldots, n$. This set dominates every vertex of $D(vG)$ at least twice because, $N(u_i) = \{v_i, v_{i+1}\}, \forall i = 1, 2, \ldots, n$. Therefore, S is a double dominating set of $D(vG)$ and $|S| = n$. Also, note that $\deg(u_i) = 2, \forall i = 1, 2, \ldots, n$. Therefore, it is clear that S is a minimum double dominating set of $D(vG)$. Hence, $\gamma_{2\times}(D(vG)) = n$. □

Theorem 2.10. *Let $G = P_n$ be a path of order n. Then, $\gamma_{2\times}(D(vG)) = n + 2$.*

Proof. Let G be a path graph with n vertices $\{v_1, v_2, \ldots, v_n\}$. For each $v_i; 1 \leq i \leq n$ add a new vertex $u_i; 1 \leq i \leq n$ and join u_i to the neighbors of corresponding v_i to get $D(vG)$. Here, v_1 and v_n are pendant vertices, therefore, the new vertices u_1 and u_2 are also pendant vertices. A double dominating set of a graph must contain all the pendant vertices and support vertices. Consider the set $S = \{v_1, v_2, v_{n-1}, v_n, u_1, u_n, v_3, \ldots, v_{n-2}\}$. This set dominates every vertex of $D(vG)$ at least twice. Therefore, S is a double dominating set of $D(vG)$. Here, $|S| = n + 2$. Now to prove,

S is a minimum double dominating set of $D(vG)$. Let $|S| < n+2$, that is $|S| = n+1$. Without loss of generality, let $S = \{v_1, v_2, v_{n-1}, v_n, u_1, u_n, v_3, \dots, v_{n-3}\}$. This set dominates every vertex of $D(vG)$ at least twice but, u_{n-1} and u_{n-3} are dominated only once because $|N(u_{n-1} \cup S)| < 2$ and $|N(u_{n-3} \cup S)| < 2$. Thus, $S = \{v_1, v_2, \dots, v_n, u_1, u_n\}$ is a minimum double dominating set of $D(vG)$. Hence, $\gamma_{2\times}(D(vG)) = n+2$. □

Theorem 2.11. *Let $G = W_n$ be a wheel graph of order $n+1$. Then $\gamma_{2\times}(D(vG)) = \lceil \frac{n}{2} \rceil + 1$.*

Proof. Let G be a wheel graph with $n+1$ vertices $\{v_1, v_2, \dots, v_{n+1}\}$. Let us assign the vertex v_{n+1} to be the universal vertex in the center and $\{v_1, v_2, \dots, v_n\}$ to be the vertices of the cycle. Add new vertex $u_i; 1 \leq i \leq n+1$ for each $v_i; 1 \leq i \leq n+1$ and join u_i to the neighbors of corresponding v_i. Here, $N(u_1) = \{v_{n+1}, v_2, v_n\}, N(u_2) = \{v_{n+1}, v_1, v_3\}, N(u_3) = \{v_{n+1}, v_2, v_4\}, \dots, N(u_{n-1}) = \{v_{n+1}, v_{n-2}, v_n\}, N(u_n) = \{v_{n+1}, v_{n-1}, v_1\}, N(u_{n+1}) = \{v_i\} \forall i = 1, 2, \dots, n$. Since, v_{n+1} is a universal vertex it is connected to all $v_i; 1 \leq i \leq n$ and so connected to all $u_i; 1 \leq i \leq n$. Note that $\{u_i, u_{i+2}\}; 1 \leq i \leq n$ have a common neighbor $v_{i+1}; 1 \leq i \leq n$. Therefore, to dominate every vertex of $D(vG)$ at least twice, minimum double dominating set S must contain v_{n+1} and $v_{i+1}; 1 \leq i \leq n$. Therefore, $|S| = \lceil \frac{n}{2} \rceil + 1$. Thus, $\gamma_{2\times}(D(vG)) = \lceil \frac{n}{2} \rceil + 1$. □

Theorem 2.12. *Let $G = K_{m,n}$ be a complete bipartite graph of order $m+n$ where, $m \leq n$ and $m, n \geq 2$. Then $\gamma_{2\times}(D(vG)) = 4$.*

Proof. Let G be a complete bipartite graph with $m+n$ vertices $\{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n\}$. For every $v_i; 1 \leq i \leq m$ and for every $u_j; 1 \leq j \leq n$ add a new vertex $v'_i; 1 \leq i \leq m$ and $u'_j; 1 \leq j \leq n$, respectively. Join v'_i and u'_j to the neighbors of corresponding $v_i; 1 \leq i \leq m$ and $u_j; 1 \leq j \leq n$, respectively, to get $D(vG)$. Consider the set $S = \{v_i, v_k, u_j, u_l\}$ for any i, j, k, l; $1 \leq i, k \leq m; 1 \leq j, l \leq n; i \neq k, j \neq l$. This set dominates every vertex of $D(vG)$ at least twice. Therefore, S is a double dominating set of $D(vG)$ and $|S| = 4$. Now to prove S is a minimum double dominating set, let $|S| < 4$. Without loss of generality, let us assume that $S = \{v_i, v_k, u_j\}$ for any $i, j, k | 1 \leq i, k \leq m; 1 \leq j \leq n; i \neq k$. This S dominates every vertex of $D(vG)$ at least twice but $\{v_i; 1 \leq i \leq m\}$ are dominated only once. Thus, S is a minimum double dominating set of $D(vG)$. Hence, $\gamma_{2\times}(D(vG)) = 4$. □

Theorem 2.13. *Let $G = K_{1,n}$ be a star graph of order $1+n$. Then $\gamma_{2\times}(D(vG)) = n+3$.*

Proof. Let G be a star graph with $n+1$ vertices $\{v_1, v_2, \dots, v_n, u\}$ where, u is the universal vertex and $\{v_1, v_2, \dots, v_n\}$ are all pendant vertices. Let $v'_i; 1 \leq i \leq n$ and u' be the new vertices added to each $v_i; 1 \leq i \leq n$ and u, respectively. Make v'_i adjacent to the neighbors of corresponding v_i and make u' adjacent to the neighbors of u. In $D(vG)$, all $v'_i; 1 \leq i \leq n$ are pendant vertices and u is their support vertex. Therefore, from Observation 1, the double dominating set S must contain u and $v'_i; 1 \leq i \leq n$. Consider $S = \{u, u', v_1, v'_i | i = 1, 2, \dots, n\}$, this set dominates every vertex of $D(vG)$ at least twice. Now we have to prove S is a minimum double dominating set of $D(vG)$. Being a copy of universal vertex, u' is adjacent to all $v_i; 1 \leq i \leq n$. Note that $\deg(v_i) = 2; \forall i = 1, 2, \dots, n$ in $D(vG)$ that is $N(v_i) = \{u, u'\} \forall i = 1, 2, \dots, n$. Therefore, S must contain u and u'. But u' is dominated only once by itself. so, we include any of $v_i; 1 \leq i \leq n$ to dominate u' at least twice. This implies, S is a minimum double dominating set of $D(vG)$ where, $|S| = n+3$. Thus, $\gamma_{2\times}(D(vG)) = \gamma_{2\times}(G) + 2 = n+3$. □

Theorem 2.14. *Let $G = K_n$ be a complete graph of order n. Then for $n \geq 3$, $\gamma_{2\times}(D(vG)) = 3$.*

Proof. Let G be a complete graph with n vertices $\{v_1, v_2, \ldots, v_n\}$. For every vertex $v_i; 1 \leq i \leq n$ add a new vertex $u_i; 1 \leq i \leq n$ and join u_i to the neighbors of corresponding v_i. Consider $S = \{v_i, v_j, v_k\}$ for any $i, j, k | 1 \leq i, j, k \leq n; i \neq j \neq k\}$, this set dominates every vertex of $D(vG)$ at least twice. Thus, S is a double dominating set of $D(vG)$ and $|S| = 3$. Now to prove S is the minimum double dominating set of $D(vG)$, let $|S| < 3$, that is $|S| = 2$. Without loss of generality, let $S = \{v_i, v_j\}$ for any $i, j | 1 \leq i, j \leq n; i \neq j$. This set dominates every vertex of $D(vG)$ at least twice but the copies of v_i, v_j for any $i, j | 1 \leq i, j \leq n; i \neq j$ that is u_i, u_j for any $i, j | 1 \leq i, j \leq n; i \neq j$ are dominated only once. Therefore we include v_k for any $k = 1, 2, \ldots, n | k \neq i, j$. Thus S is a minimum double dominating set of $D(vG)$. Hence, $\gamma_{2\times}(D(vG)) = 3$. □

Theorem 2.15. *Let $G = S_{r,t}$ be a double star graph with $r + t$ vertices. Then $\gamma_{2\times}(D(vG)) = r + t + 2$.*

Proof. Let G be a double star graph with $r + t$ vertices $\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_t\}$. A double star graph has exactly two non pendant vertices and let them be u_r and v_t. Let $\{u_i; 1 \leq i \leq r-1\}$ be pendant vertices connected to u_r and $\{v_j; 1 \leq j \leq t-1\}$ be pendant vertices connected to v_t. Now for every vertex $\{u_1, u_2, \ldots, u_r\}$ and $\{v_1, v_2, \ldots, v_t\}$ add a new vertex $\{u'_1, u'_2, \ldots, u'_r\}$ and $\{v'_1, v'_2, \ldots, v'_t\}$, respectively, and join them to the neighbors of corresponding $\{u_i; 1 \leq i \leq r\}$ and $\{v_j; 1 \leq j \leq r\}$. Consider the set $S = \{u'_i, u_r, v_r, v'_r; 1 \leq i \leq r\}$. This set dominates every vertex of $D(vG)$ at least twice. Therefore, S is double dominating set of $D(vG)$ where, $|S| = r + t + 2$. Now, we have to prove S is a minimum double dominating set. Here, all $u'_i; 1 \leq i \leq r-1$ are pendant vertices and u_r, v_r are their support vertices. So, from Observation 1, S must contain these vertices. Also, every $\{u_i; 1 \leq i \leq r-1\}$ and $\{v_j; 1 \leq j \leq t-1\}$ are vertices of degree 2 with common neighbors u_r, u'_r and v_r, v'_r, respectively. So, to double dominate $\{u_i; 1 \leq i \leq r-1\}$ and $\{v_j; 1 \leq j \leq t-1\}$ u_r, u'_r and v_r, v'_r are needed. This shows that S is a minimum double dominating set of $D(vG)$. Hence, $\gamma_{2\times}(D(vG)) = r + t + 2$. □

2.6 Subdivision operation

Definition 2.5. [14] **(Subdivision operation, $S(G)$).** *$S(G)$ is obtained by splitting each edge of G by introducing a new vertex.*

For example, consider a complete bipartite graph $K_{2,3}$.

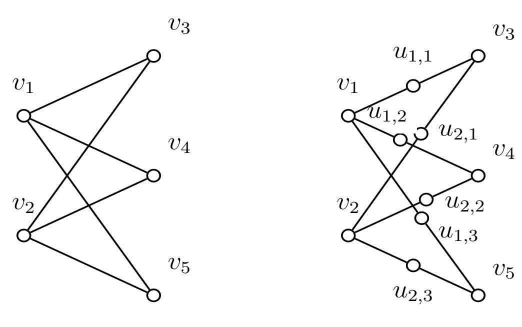

Figure 5. $(K_{2,3})$ and $(S(K_{2,3}))$

Table 6. Impact of subdivision operation on double domination number of a graph

Graph class	$\gamma_{2\times}(G)$	$\gamma_{2\times}(S(G))$	$\lvert V(G)\rvert$	$\lvert V(S(G))\rvert$
C_n	$\lceil\frac{2n}{3}\rceil$	$\lceil\frac{4n}{3}\rceil$	n	$2n$
P_n	$\lceil\frac{2(n+1)}{3}\rceil$	$\lceil\frac{4n}{3}\rceil$	n	$2n-1$
W_n	$\lceil\frac{n}{3}\rceil+1$	$2n$	$n+1$	$3n+1$
$K_{m,n}$	4	$2n+m$	$m+n$	$m(n+1)+n$
K_n	2	$n+\lceil\frac{n}{2}\rceil$	n	$n+\frac{2n+n(n-1)}{2}$
$K_{1,n}$	$1+n$	$2n$	$n+1$	$2n+1$
$S_{r,t}$	$r+t$	$2(r+t)-2$	$r+t$	$3(r+t)-2$

Theorem 2.16. *Let $G = C_n$ be a cycle of order n. Then $\gamma_{2\times}(S(G)) = \lceil\frac{4n}{3}\rceil$.*

Proof. Let G be a cycle with n vertices $\{v_1, v_2, \dots, v_n\}$ and n edges $\{e_1, e_2, \dots, e_n\}$. Split each edge $e_i; 1 \le i \le n$ by a new vertex $u_i; 1 \le i \le n$ to get $S(G)$. Then $S(G)$ is again a cycle of order $2n$. Therefore, $\gamma_{2\times}(S(G)) = \lceil\frac{2(2n)}{3}\rceil = \lceil\frac{4n}{3}\rceil$. □

Theorem 2.17. *Let $G = P_n$ be a path of order n. Then $\gamma_{2\times}(S(G)) = \lceil\frac{4n}{3}\rceil$.*

Proof. Let $G = P_n$ be a path graph with n vertices $\{v_1, v_2, \dots, v_n\}$ and $n-1$ edges $\{e_1, e_2, \dots, e_{n-1}\}$. Split each edge $e_i; 1 \le i \le n-1$ by a new vertex $u_i; 1 \le i \le n-1$ to get $S(G)$. The graph $S(G)$ is again a path with $2n-1$ vertices $\{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_{n-1}\}$. Thus, $\gamma_{2\times}(S(G)) = \lceil\frac{2(2n-1+1)}{3}\rceil = \lceil\frac{4n}{3}\rceil$. □

Theorem 2.18. *Let $G = W_n$ be a wheel graph of order $n+1$. Then, $\gamma_{2\times}(S(G)) = 2n$.*

Proof. Let G be a wheel graph with $n+1$ vertices $\{v_1, v_2, \dots, v_{n+1}\}$ and $2n$ edges $\{e_1, e_2, \dots, e_{2n}\}$. Split each edge $e_i; 1 \le i \le 2n$ by a new vertex $u_i; 1 \le i \le 2n$ to get $S(G)$. Consider the set $S = \{v_1, v_2, \dots, v_n, u_{n+1}, u_{n+2}, \dots, u_{2n}\}$. This set dominates every vertex of $S(G)$ at least twice. Therefore, S is a double dominating set of $S(G)$ where, $|S| = 2n$. Now to prove S is a minimum double dominating set of $S(G)$. Let $|S| < 2n$ that is $|S| = 2n-1$. Without loss of generality let $S = \{v_1, v_2, \dots, v_n, u_{n+1}, \dots, u_{2n-1}\}$. This set dominates every vertex of $S(G)$ but, v_n is dominated only once. Therefore, S is the minimum double dominating set of $S(G)$. Hence, $\gamma_{2\times}(S(G)) = 2n$. □

Theorem 2.19. *Let $G = K_{m,n}$ be a complete bipartite graph of order $m+n$ where, $m \le n$. Then, $\gamma_{2\times}(S(G)) = 2n+m$.*

Proof. Let G be a complete bipartite graph with $m+n$ vertices $\{v_1, v_2, \dots, v_m, u_1, u_2, \dots, v_n\}$ and mn edges $\{e_{1,1}, e_{1,2}, \dots, e_{1,n}, e_{2,1}, e_{2,2}, \dots, e_{2,n}, \dots, e_{n,1}, e_{n,2}, \cdots, e_{n,n}\}$. Split each edge $e_{k,l}; 1 \le k \le m; 1 \le l \le n$ by a new vertex $w_{k,l}; 1 \le k \le m; 1 \le l \le n$. Consider $S = \{v_i, u_j, w_{k,l} | 1 \le i \le m; 1 \le j \le n; 1 \le k \le m; 1 \le l \le n; k = l\} \cup \{w_{k,k+1}, w_{k,k+2}, \dots, w_{k,l} | k = m; l = n\}$. This S dominates every vertices of $S(G)$ at least twice. Therefore, S is a double dominating set of $S(G)$ where, $|S| = 2n+m$. Now to prove S is a minimum double dominating set of $S(G)$. Let $|S| < 2n+m$ that is $|S| = 2n+m-1$. Without loss of generality let $S = \{v_i, u_j, w_{k,l} | 1 \leq i \leq m; 1 \leq j \leq n; 1 \leq k \leq m; 1 \leq l \leq n; k = l\} \cup \{w_{k,k+1}, w_{k,k+2}, \ldots, w_{k,l} | k = m; l = n\} - \{w_{1,1}\}$. This set dominates every vertex of $S(G)$ at least twice but the vertices v_1 and u_1 are dominated only once. This implies S is the minimum double dominating set of $S(G)$. Hence, $\gamma_{2\times}(S(G)) = 2n + m$. □

Theorem 2.20. *Let $G = K_n$ be a complete graph. Then, for $n \geq 4$ $\gamma_{2\times}(S(G)) = n + \lceil \frac{n}{2} \rceil$.*

Proof. Let G be a complete graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and $n\frac{(n-1)}{2}$ edges $\{e_{i,j}; 1 \leq i, j \leq n\frac{(n-1)}{2}\}$. Split each edge $\{e_{i,j}; 1 \leq i, j \leq n\frac{(n-1)}{2}\}$ by a new vertex $\{u_{i,j}; 1 \leq i, j \leq n\frac{(n-1)}{2}\}$ to get $S(G)$. Then, $S(K_2)$ is a path of order 3 with $\gamma_{2\times}(S(K_2)) = 3$. Also, $S(K_3)$ is a cycle of order 6 with $\gamma_{2\times}(S(K_3)) = 4$. For $n \geq 4$, let S be the minimum double dominating set of $S(G)$ and let $S = \{v_i; 1 \leq i \leq n\}$. This set dominates every $\{u_{i,j}; 1 \leq i, j \leq n\frac{(n-1)}{2}\}$ at least twice but $\{v_i; 1 \leq i \leq n\}$ are dominated only once. To dominate them at least twice, include the vertices dividing the edges joining every distinct pair of vertices of $S(G)$. Therefore, $|S| = n + \lceil \frac{n}{2} \rceil$. Hence, $\gamma_{2\times}(S(G)) = n + \lceil \frac{n}{2} \rceil$. □

Theorem 2.21. *Let $G = K_{1,n}$ be a star graph. Then $\gamma_{2\times}(S(G)) = 2n$.*

Proof. Let G be a star graph with $n+1$ vertices $\{v_1, v_2, \ldots, , v_n, u\}$ and n edges $\{e_1, e_2, \ldots, e_n\}$. Split each edge $e_i; 1 \leq i \leq n$ by a new vertex $v_i'; 1 \leq i \leq n$ to get $S(G)$. Here, all $\{v_i | \forall i = 1, 2, \ldots, n\}$ are pendant vertices and $\{v_i'; \forall i = 1, 2, \ldots, n\}$ are their support vertices, respectively. Therefore, a double dominating set S of $S(G)$ must contain $\{v_i, v_i' | \forall i = 1, 2, \ldots, n\}$. This S is a minimum double dominating set of $S(G)$. Also, $|S| = 2n$. Thus $\gamma_{2\times}(S(G)) = 2n$. □

Theorem 2.22. *Let $G = S_{r,t}$ be a double star graph with $r + t$ vertices. Then $\gamma_{2\times}S(G) = 2(r + t) - 2$.*

Proof. Let G be a double star graph with $r + t$ vertices $\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_t\}$. A double star graph has exactly two non pendant vertices and let them be u_r and v_t. Let $\{u_i; 1 \leq i \leq r-1\}$ be pendant vertices connected to u_r and $\{v_j; 1 \leq j \leq t-1\}$ be pendant vertices connected to v_t. Also, G has $r + t - 1$ edges $\{e_{r,1}, e_{r,2}, \ldots, e_{r,r-1}, e_{t,1}, e_{t,2}, \ldots, e_{t,t-1}\}$ and $e_{r,t}$. Split each edge $\{e_{r,i}; 1 \leq i \leq r-1\}$ and $\{e_{t,j}; 1 \leq j \leq t-1\}$ and $e_{r,t}$ by a new vertex $\{w_{r,i}; 1 \leq i \leq r-1\}$ and $\{w_{t,j}; 1 \leq j \leq t-1\}$ and $w_{r,t}$ to get $S(G)$. Consider the set $S = \{u_i, v_j, w_{r,i}, w_{t,j}; 1 \leq i \leq r; 1 \leq j \leq t\}$. This set dominates every vertex of $S(G)$ at least twice. Therefore, S is a double dominating set of $S(G)$ where, $|S| = 2(r + t) - 2$. Clearly, S is a minimum double dominating set of $S(G)$ because $\{u_i; 1 \leq i \leq r-1\}$ and $\{v_j; 1 \leq j \leq t-1\}$ are pendant vertices and $\{w_{r,i}, w_{t,j}; 1 \leq i \leq r-1; 1 \leq j \leq t-1\}$ are their support vertices and u_r, v_t are neighbors of a vertex of degree 2. Therefore, a minimum double dominating set must contain all these vertices. This implies, S is a minimum double dominating set of $S(G)$. Hence, $\gamma_{2\times}S(G) = 2(r+t)-2$. □

Conclusion and Future scope

In this paper, we identified the double domination number of graphs that are generated from various graph classes by applying unary operations. Further, we investigated the impact of those unary operations on double domination number. In near future, one can investigate the impact of double domination number by applying other unary products.

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