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Evaluation of the real parts of polylogarithm expressions containing complex arguments via certain logarithmic integrals

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Abstract: We consider a polylogarithm expression containing complex arguments, namely

$$\mathcal{P}_{\pm}(n) = \Re\left(\mathrm{Li}_n\left(\frac{1 \pm i}{2}\right)\right).$$

The central notion of the present paper is to evaluate the real parts of $\mathcal{P}_{\pm}(n)$ for first four orders, specifically $n = 1,\ 2,\ 3,$ and $4,$ by constructing certain logarithmic integrals. To extract the real parts, we demonstrate an organized approach, and the proofs solely rely on the calculation of the logarithmic integrals. Additionally, we present a potential closed form of $\mathcal{P}_{\pm}(5)$.

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1 Introduction

In the mathematical literature, various special functions (see [7, pp. 859–1046]) are introduced. Among these special functions, the polylogarithm is one that is customarily defined by

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \tag{1}$$

for any arbitrary order $s > 1$ and complex argument z with $|z| \leq 1$. The first published study of the function was due to A. Jonquière in 1889, leading to its common designation as the *Jonquière function*. Lewin [8] conducted a comprehensive study of the function, compiling and addressing numerous intriguing identities and formulas. The evaluation of the real parts of the polylogarithm expressions, which is the core of this paper, is merely the particular cases of formulas that can be found in [8].

For case $n = 1$, the real parts of the following polylogarithm expression

$$\mathcal{P}_{\pm}(n) = \Re\left(\mathrm{Li}_n\left(\frac{1 \pm i}{2}\right)\right), \tag{2}$$

reduce to the natural cases $\Re\left(\log(1 \pm i)\right)$, which are trivial to evaluate. Likewise, for $n = 2$ and $n = 3$, the real parts of (2) can be calculated by using Landen's well-known functional equations. They are dilogarithm identity [8, p. 5, Eqn. (1.12)]

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{-z}{1-z}\right) = -\frac{1}{2}\log^2(1-z), \tag{3}$$

and trilogarithm identity [8, p. 155, Eqn. (6.10)]

$$\begin{aligned} \mathrm{Li}_3(z) + \mathrm{Li}_3\left(\frac{-z}{1-z}\right) &= \frac{\log^3(1-z)}{6} + \zeta(3) + \zeta(2)\log(1-z) \\ &\quad - \frac{\log^2(1-z)\log(z)}{2} - \mathrm{Li}_3\left(1-z\right). \end{aligned} \tag{4}$$

By specializing $x = \pm i$ in the aforementioned formulas and following the routine simplifications to extract the real values, we obtain their respective real parts. A creative approach can be found in [15], particularly for the case $n = 3$. Also, we suggest looking in [5] to the interested readers, which deals with the case $n = 3$ and its closely related polylogarithm expression. For further insights, especially concerning the real parts of dilogarithm, trilogarithm, and tetralogarithm, one can explore [17, p. 36].

Now we collect some basic tools and definitions that will be used repeatedly throughout this paper. For $s \in \mathbb{C}$, a generalized harmonic number $H_n^{(s)}$ is defined by

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}, \quad \text{and} \quad H_n^{(1)} = H_n$$

is the nth harmonic number, and obeys the following generating function [7, Entry 1.513.6]

$$\sum_{n=0}^{\infty} x^n H_n = \frac{\log(1-x)}{1-x}, \quad x \in [-1, 1), \tag{5}$$

and it satisfies the recurrence relation $H_{n+1} = H_n + \frac{1}{n+1}$. The beta and the gamma functions denoted by $B(a, b)$ for $\Re(a) > 0$, $\Re(b) > 0$ and $\Gamma(z)$ for $\Re(z) > 0$, respectively, are defined by

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx, = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \text{and} \quad \Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x}\, dx. \tag{6}$$

Besides the definition of the gamma function, we note that

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)} \text{ and } \Gamma(1+z)=z\Gamma(z), \tag{7}$$

where the former equation is the reflection formula [7, Entry 8.334.3] of the gamma function and the latter one is the fundamental property of the gamma function. The Riemann zeta function and the alternating zeta function for $\Re(s)>1$ are defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \text{ and } \eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\left(1-2^{1-s}\right)\zeta(s), \tag{8}$$

respectively. The latter series is famously known as Dirichlet series, which is the alternating sum to the Dirichlet series expansion of the Riemann zeta function and they are the particular cases $\operatorname{Li}_s(1)$ and $-\operatorname{Li}_s(-1)$, respectively, of the polylogarithmic function (1). In a similar fashion, the Dirichlet beta function, which is also known as the Catalan beta function, is closely related to the Riemann zeta function. The Dirichlet beta function is defined by

$$\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}=\frac{(-4)^s}{\Gamma(s)}\left(\psi^{(s-1)}\left(\frac{1}{4}\right)-\psi^{(s-1)}\left(\frac{3}{4}\right)\right), \tag{9}$$

where $\psi^{(m)}(z)=\frac{d^m}{dz^m}\psi_0(z)=(-1)^{m+1}m!\zeta(m+1,z)$ is the polygamma function where $m>0$ and $\Re(z)>0$. For $s=2,\ s=3$, and $s=4$ in (1), we have

$$\operatorname{Li}_2(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2},\ \operatorname{Li}_3(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^3},\ \text{and}\ \operatorname{Li}_4(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^4} \tag{10}$$

dilogarithm, trilogarithm, and tetralogarithm functions, respectively. Some special values include

$$\operatorname{Li}_2(1)=\zeta(2)=\frac{\pi^2}{6},\ \operatorname{Li}_3(1)=\zeta(3),\ \text{and}\ \operatorname{Li}_4(1)=\zeta(4)=\frac{\pi^4}{90}, \tag{11}$$

$$\beta(2)=G,\ \beta(3)=\frac{\pi^3}{32},\ \text{and,}\ \beta(5)=\frac{5\pi^5}{1536}, \tag{12}$$

$$\eta(1)=\log(2),\ \eta(2)=\frac{\pi^2}{12},\ \text{and}\ \eta(3)=\frac{3}{4}\zeta(3), \tag{13}$$

where G is famously known as Catalan's constant. For more intriguing identities associated with Catalan's constant, we refer interested readers to the papers [1] and [4], and we suggest looking at the references given therein for more identities. These preliminary concepts are instrumental in the paper's analysis.

We organize the remaining work of the paper into different sections. In Section 2, we introduce key logarithmic integrals, which serve as the main framework of this paper along with several generating functions essential for proving the even-indexed alternating harmonic sums. Section 3 highlights several intermediate findings, namely lemmas and propositions. Section 4 contains the major results and their corresponding proofs. Section 5 presents proofs for two integrals derived during the calculation of Proposition 3.1. In Section 6, we discuss an open problem related to the subject of the paper and closely associated identities.

2 Logarithmic integrals and a few generating functions

In this section, we introduce three distinct logarithmic integrals. For natural numbers a and b, we define

$$\mathcal{B}(a,b)=\int_0^1 \frac{x\log^a x\log^b(1-x)}{1+x^2}\,dx.$$

Specifically, we focus on the integrals $\mathcal{B}(1,1)$, $\mathcal{B}(2,1)$, and $\mathcal{B}(1,2)$, which are expressed as follows:

$$\begin{aligned}\mathcal{B}(1,1)&=\int_0^1 \frac{x\log x\log(1-x)}{1+x^2}\,dx,\\ \mathcal{B}(2,1)&=\int_0^1 \frac{x\log^2 x\log(1-x)}{1+x^2}\,dx,\\ \mathcal{B}(1,2)&=\int_0^1 \frac{x\log x\log^2(1-x)}{1+x^2}\,dx.\end{aligned}$$

The first integral, $\mathcal{B}(1,1)$, can be found in references such as [15, pp. 97–100] and [17, p. 100, QLI (12;5)]. Similarly, the second integral and the third integral, $\mathcal{B}(2,1)$ and $\mathcal{B}(1,2)$, are discussed in [17, p. 103, QLI (122;5)] and [17, p. 103, QLI (112;5)], respectively. These integrals are presented as propositions, and we will evaluate them in detail in Section 3. More advanced types of integrals, similar to those discussed, can be found in [17].

During the calculation of $\mathcal{B}(1,1)$ and $\mathcal{B}(2,1)$, we encounter several other intriguing logarithmic integrals such as

$$\begin{aligned}&\int_0^1 \frac{\log x\log(1+x^2)}{1-x}\,dx=2\zeta(3)-\frac{\pi}{2}G-\frac{3\pi^2}{16}\log(2),\\ &\int_0^1 \frac{\log(1+x^2)\log(1-x)}{x}\,dx=\frac{23}{32}\zeta(3)-\frac{\pi}{2}G,\\ &\int_0^1 \frac{\mathrm{Li}_2(x)\log(1+x^2)}{x}\,dx=\frac{35}{32}\zeta(3)\log(2)-\frac{23\pi^4}{2304}+\frac{5}{4}\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{5\pi^2}{96}\log^2(2)+\frac{5}{96}\log^4(2),\\ &\int_0^1 \frac{\log(1+x^2)\log^2(x)}{1-x}\,dx=2G^2+\frac{35}{16}\zeta(3)\log(2)-\frac{199}{5760}\pi^4,\\ &\int_0^1 \frac{\log(x)\log(1-x)\log(1+x^2)}{x}\,dx=G^2+\frac{35}{32}\zeta(3)\log(2)+\frac{5}{96}\log^4(2)+\frac{5}{4}\mathrm{Li}_4\left(\frac{1}{2}\right)\\ &\qquad\qquad-\frac{5}{96}\pi^2\log^2(2)-\frac{119}{5760}\pi^4.\end{aligned}$$

The detailed computation of these integrals can be found in Section 3. The final two integrals are particularly notable, as their closed forms include the term G^2. Several more such integrals can be found in [17]. The computation of integral $\mathcal{B}(1,2)$ involves some difficult harmonic sums, which are briefly highlighted below. In order to derive corresponding results, we list Vălean's (see [14, p. 422] and [16, p. 3–4]) generalized alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(p)}}{n}=p\zeta(p+1)-\frac{1}{2^{p+1}}\sum_{k=1}^{p}\eta(k)\eta(p-k+1)-\sum_{k=1}^{p}\beta(k)\beta(p-k+1),$$

where p in a natural number. For $x \in [-1,1)$, a few generating functions associated with harmonic numbers are as follows:

$$\sum_{n=1}^{\infty} \frac{H_n}{n} x^n = \operatorname{Li}_2(x) + \frac{\log^2(1-x)}{2},$$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} x^n = 2\zeta(4) - \frac{\log x \log^3(1-x)}{3} + \frac{(\operatorname{Li}_2(x))^2}{2} - \log^2(1-x)\operatorname{Li}_2(1-x) + \operatorname{Li}_4(x) + 2\log(1-x)\operatorname{Li}_3(1-x) - 2\operatorname{Li}_4(1-x), \quad x \neq 0$$

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n} x^n = \frac{\log^3(1-x)\log(x)}{6} - \frac{\log^4(1-x)}{24} + \frac{\log^2(1-x)\operatorname{Li}_2(1-x)}{2} - \zeta(4) - \operatorname{Li}_4\left(\frac{x}{1-x}\right) - \log(1-x)\operatorname{Li}_3(1-x) + \operatorname{Li}_4(1-x),$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} x^n = -2\operatorname{Li}_4\left(\frac{x}{x-1}\right) + 2\operatorname{Li}_4(1-x) - \operatorname{Li}_4(x) + \frac{\operatorname{Li}_2^2(x)}{2} + 2\log(1-x)\operatorname{Li}_3(x) - 2\zeta(4) - \frac{\log^4(1-x)}{12} - \zeta(2)\log^2(1-x) - 2\zeta(3)\log(1-x) + \frac{\log(x)\log^3(1-x)}{3},$$
$$x \neq 0.$$

The generating functions mentioned above, along with their proofs, can be found in [9, pp. 71–85] and [14, pp. 398–405]. Next, we highlight some even-indexed alternating harmonic sums, which are crucial for computing Proposition 3.3. They are as follows:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_{2n}}{n} = \frac{5\pi^2}{48} - \frac{\log^2(2)}{4}, \tag{14}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_{2n}^{(3)}}{n} = \frac{199\pi^4}{11520} - \frac{3}{32}\log(2)\zeta(3) - G^2, \tag{15}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_{2n}^{(2)}}{n^2} = 2G^2 - \frac{353\pi^4}{5760} - \frac{5\pi^2}{24}\log^2(2) + \frac{35}{8}\log(2)\zeta(3) + \frac{5}{24}\log^4(2) + 5\operatorname{Li}_4\left(\frac{1}{2}\right), \tag{16}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_{2n}^2}{n^2} = 2G^2 - \frac{5}{48}\log^4(2) + \frac{\pi^2}{6}\log^2(2) - \frac{35}{16}\log(2)\zeta(3) + \frac{77\pi^4}{960} - \pi G\log(2) - 2\pi\Im\left(\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right) - \frac{5}{2}\operatorname{Li}_4\left(\frac{1}{2}\right), \tag{17}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_{2n}H_{2n}^{(2)}}{n} = \frac{5}{96}\log^4(2) + \frac{35}{64}\log(2)\zeta(3) - \frac{\pi^2}{16}\log^2(2) - \frac{137\pi^4}{11520} + \frac{\pi G}{4}\log(2) + \frac{\pi}{2}\Im\left(\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right) + \frac{5}{4}\operatorname{Li}_4\left(\frac{1}{2}\right). \tag{18}$$

Using Vălean's generalized result for the case $p = 3$ and after some computation, the conclusion of (15) (see [14, p. 425] and [16, p. 5]) follows. The other series (14), (16) (see [14, p. 450]), (17), and (18) can be obtained by substituting $x = i$ in the last three generating functions and extracting the real parts. The extraction of the real parts is elementary and the details are left to the reader.

While the main results of this paper are acknowledged in the mathematical literature, this paper offers an organized approach by introducing specific classes of logarithmic integrals, which are specified above. Before proving our results, we compile some intermediate findings that will aid in the analysis of both propositions and theorems.

3 Lemmas, propositions, and proofs

Lemma 3.1. *The following relation holds:*

$$\int_0^1 x^p \log^q(x)\,dx = (-1)^q \frac{\Gamma(q+1)}{(p+1)^{q+1}}, \quad \Re(p) > -1, \Re(q) > -1.$$

Proof. A proof of the aforementioned result can be found in [13, pp. 57–58], demonstrated using the recurrence method. However, making the substitution $x = e^y$ leads to

$$\int_0^1 x^p \log^q(x)\,dx = \int_{-\infty}^0 y^q e^{y(p+1)}\,dy = \frac{(-1)^q}{(p+1)^{q+1}} \int_0^\infty e^{-t} t^q\,dt.$$

By using the definition of the gamma function in the latter integral, the conclusion follows. □

Lemma 3.2. *For* $\Re(m) > -1$, *the following equality holds:*

$$\int_0^1 \frac{x \log^m(x)}{1+x^2}\,dx = \frac{(-1)^m}{2^{m+1}} \left(1 - \frac{1}{2^m}\right) \zeta(m+1)\Gamma(m+1).$$

Proof. Since $\dfrac{x}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k+1}$, and employing Lemma 3.1, we have

$$\begin{aligned}
\int_0^1 \frac{x \log^m(x)}{1+x^2}\,dx &= \sum_{k=0}^{\infty} (-1)^k \int_0^1 x^{2k+1} \log^m(x)\,dx \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{(-1)^m \Gamma(m+1)}{(2k+2)^{m+1}} \\
&= \frac{(-1)^m}{2^{m+1}} \Gamma(m+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{m+1}} \\
&= \frac{(-1)^m}{2^{m+1}} \Gamma(m+1) \sum_{k=1}^{\infty} \left(\frac{1}{k^{m+1}} - \frac{2}{(2k)^{m+1}} \right) \\
&= \frac{(-1)^m}{2^{m+1}} \left(1 - \frac{1}{2^m}\right) \zeta(m+1)\Gamma(m+1),
\end{aligned}$$

which is the desired result. □

Next, we establish a lemma associated with the polylogarithm expression, $\mathcal{P}_{\pm}(n)$, and we give its equivalent integral form.

Lemma 3.3. [An integral representation of $\mathcal{P}_{\pm}(n)$] *If $n \geq 1$ is a positive integer, then*

$$\mathcal{P}_{\pm}(n) = \frac{(-1)^{n-1}}{\Gamma(n)} \int_0^1 \frac{x \log^{n-1}(1-x)}{1+x^2}\, dx = \Re\left(\mathrm{Li}_n\left(\frac{1 \pm i}{2}\right)\right),$$

where $\Re(.)$ denotes the real part and $i = \sqrt{-1}$ is the imaginary unit.

Proof. Lemma 3.3 is a particular case ($a = 1$) of the general result that can be found in [14, p. 11]. Since $x^2 + 1 = (x+i)(x-i)$, we have

$$\begin{aligned}\int_0^1 \frac{x \log^{n-1}(1-x)}{1+x^2}\, dx &= \int_0^1 \frac{x \log^{n-1}(1-x)}{(x+i)(x-i)}\, dx \\ &= \frac{1}{2}\int_0^1 \log^{n-1}(1-x)\left(\frac{1}{x+i} + \frac{1}{x-i}\right) dx \\ &= \frac{1}{2}\int_0^1 \log^{n-1}(x)\left(\frac{1}{1-x+i} + \frac{1}{1-x-i}\right) dx.\end{aligned}$$

Enforcing the substitution $x \to 1-x$, we obtain the last integral. Furthermore, we note that $\frac{1}{1 \pm i - x} = \frac{1}{(1\pm i)(1 - x/(1\pm i))} = \frac{1}{1\pm i}\sum_{k=0}^{\infty} \left(x/(1\pm i)\right)^k$. Thus,

$$\int_0^1 \frac{x \log^{n-1}(1-x)}{1+x^2}\, dx = \frac{1}{2}\sum_{k=0}^{\infty}\int_0^1 \log^{n-1}(x)\left(\frac{x^k}{(1+i)^{k+1}} + \frac{x^k}{(1-i)^{k+1}}\right) dx. \tag{19}$$

Next, we interchange the sum and the integral in (19), which is justifiable by the dominated convergence theorem. Finally, invoking Lemma 3.1, we obtain

$$\int_0^1 \frac{x \log^{n-1}(1-x)}{1+x^2}\, dx = \frac{(-1)^{n-1}\Gamma(n)}{2}\sum_{k=0}^{\infty}\left(\frac{1/(1+i)^{k+1}}{(k+1)^n} + \frac{1/(1-i)^{k+1}}{(k+1)^n}\right).$$

Shifting the index k to $k-1$, and in view of (1), we have

$$\begin{aligned}\int_0^1 \frac{x \log^{n-1}(1-x)}{1+x^2}\, dx &= \frac{(-1)^{n-1}\Gamma(n)}{2}\sum_{k=1}^{\infty}\left(\frac{1/(1+i)^{k}}{k^n} + \frac{1/(1-i)^{k}}{k^n}\right) \\ &= \frac{(-1)^{n-1}\Gamma(n)}{2}\left(\mathrm{Li}_n\left(\frac{1}{1+i}\right) + \mathrm{Li}_n\left(\frac{1}{1-i}\right)\right) \\ &= \frac{(-1)^{n-1}\Gamma(n)}{2}\left(\mathrm{Li}_n\left(\frac{1-i}{2}\right) + \mathrm{Li}_n\left(\frac{1+i}{2}\right)\right) \\ &= \frac{(-1)^{n-1}\Gamma(n)}{2}\left(\mathrm{Li}_n\left(\frac{\overline{1+i}}{2}\right) + \mathrm{Li}_n\left(\frac{1+i}{2}\right)\right) \\ &= (-1)^{n-1}\Gamma(n)\Re\left(\mathrm{Li}_n\left(\frac{1\pm i}{2}\right)\right).\end{aligned}$$

We note that the conjugate of $1+i$ is $1-i$ and $\Re\left(\mathrm{Li}_n(z)\right) = \Re\left(\mathrm{Li}_n(\overline{z})\right)$, where $\overline{z}$ is the conjugate of $z \in \mathbb{C}$. Utilizing these facts and dividing both sides by $(-1)^{n-1}\Gamma(n)$, we establish the desired closed form between integral and polylogarithm expression. Furthermore, by equating the obtained result with (2), the desired conclusion follows. □

Lemma 3.4. *For all $x \in (-1,1)$, we have*

$$2\sum_{n=1}^{\infty}(-1)^{n+1}H_{2n}x^{2n-1} = \frac{2\tan^{-1}x}{1+x^2} + \frac{\log(1+x^2)}{x(1+x^2)}.$$

Proof. We start with the series representations of $\tanh^{-1}x$ [7, Entry 1.643.2] and $\log(1-x^2)$ [7, Entry 1.513.4] as follows

$$-\tanh^{-1}x\log(1-x^2) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}\sum_{n=0}^{\infty}\frac{x^{2n+2}}{n+1} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^{2n+3}}{(k+1)(2n-2k+1)}.$$

We obtain the latter double sum by applying the Cauchy product. Additionally, by performing the partial fraction decomposition of the summand, we get

$$\begin{aligned} -\tanh^{-1}x\log(1-x^2) &= \sum_{n=0}^{\infty}\frac{2}{2n+3}\sum_{k=0}^{n}\left(\frac{1}{2k+2}+\frac{1}{2n-2k+1}\right)x^{2n+3} \\ &= \sum_{n=0}^{\infty}\frac{2}{2n+3}\sum_{k=1}^{2n+2}\frac{1}{k}x^{2n+3} = 2\sum_{n=0}^{\infty}\frac{H_{2n+2}}{2n+3}x^{2n+3} \\ &= 2\sum_{n=1}^{\infty}\frac{H_{2n}}{2n+1}x^{2n+1}. \end{aligned} \tag{20}$$

We shifted the index n to $n-1$, to achieve (20). Further, differentiating both sides of (20) with respect to x gives us

$$2\sum_{n=1}^{\infty}H_{2n}x^{2n} = -\frac{d}{dx}\left(\tanh^{-1}x\log(1-x^2)\right) = \frac{2x\tanh^{-1}x - \log(1-x^2)}{1-x^2}.$$

Finally, replacing x with ix, and taking into account $ix\tanh^{-1}(ix) = -x\tan^{-1}x$, along with routine simplification, leads us to the desired result. □

Lemma 3.5. *For all $n \in \mathbb{N}$, the following relation holds:*

$$\int_0^1 x^{n-1}\log x\log^2(1-x)\,dx = \frac{2\zeta(3)}{n} + \frac{2\zeta(2)H_n}{n} - \frac{H_n^{(2)}}{n^2} - \frac{H_n^2}{n^2} - \frac{2H_nH_n^{(2)}}{n} - \frac{2H_n^{(3)}}{n}.$$

Proof. It is well-known that (see [9, p. 114], [13, p. 2])

$$\int_0^1 x^{n-1}\log^2(1-x)\,dx = \frac{H_n^2 + H_n^{(2)}}{n}.$$

We note that $H_n = \psi_0(n+1)+\gamma$ and $H_n^{(2)} = \sum_{k=1}^{n}1/k^2 = \zeta(2) - \psi_1(n+1)$, where γ, $\psi_0(z)$, and $\psi_1(z)$ are the Euler–Mascheroni constant, the digamma function, and the trigamma function, respectively. By substituting the values into the above relation and taking the partial derivatives of it with respect to n, we obtain

$$\int_0^1 x^{n-1}\log x\log^2(1-x)\,dx = \frac{\partial}{\partial n}\left(\frac{(\psi_0(n+1)+\gamma)^2 + \zeta(2) - \psi_1(n+1)}{n}\right) = \frac{\partial}{\partial n}\frac{f(n)}{g(n)}.$$

Since $\psi_m(z) = \frac{d^m}{dz^m}\psi_0(z)$ and

$$\frac{\partial}{\partial n} f(n) = 2\psi_1(n+1)(\gamma + \psi_0(n+1)) - \psi_2(n+1) = 2(\zeta(2) - H_n^{(2)})H_n + 2\zeta(3, n+1),$$

using the quotient rule for derivatives, i.e., $\frac{\partial}{\partial n}\frac{f(n)}{g(n)} = \frac{g(n)\frac{\partial}{\partial n}f(n) - f(n)\frac{\partial}{\partial n}g(n)}{(g(n))^2}$, simplying, and rearranging the terms, the conclusion follows. □

Next, we calculate the logarithmic integral, which are highlighted in Section 2.

Proposition 3.1. *The following integral equality holds:*

$$\mathcal{B}(1,1) = \int_0^1 \frac{x\log(x)\log(1-x)}{1+x^2}\,dx = \frac{41}{64}\zeta(3) - \frac{3\pi^2}{32}\log(2).$$

Proof. The proof of the integral $\mathcal{B}(1,1)$ can be found in [13, pp. 97–100] and [17, p.100, QLI(12;5)]. In this paper, we provide a different approach of $\mathcal{B}(1,1)$. It can be observed that

$$\int_0^1 \frac{x\log(x)\log(1-x)}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \left(\frac{\log(x)}{1-x} - \frac{\log(1-x)}{x}\right)\log(1+x^2)\,dx. \tag{21}$$

In order to prove it, we consider a dilogarithm integral as follows

$$\mathcal{A} = \int_0^1 \frac{x\operatorname{Li}_2(x)}{1+x^2}\,dx.$$

With the aid of $\mathcal{A}$, we evaluate the former integral of (21). We proceed by applying integration by parts to the integral $\mathcal{A}$, and using the value of $\operatorname{Li}_2(1)$ leads to

$$\begin{aligned}\int_0^1 \frac{x\operatorname{Li}_2(x)}{1+x^2}\,dx &= \frac{1}{2}\log(2)\operatorname{Li}_2(1) + \frac{1}{2}\int_0^1 \frac{\log(1+x^2)\log(1-x)}{x}\,dx\\ &= \frac{\pi^2}{12}\log(2) + \frac{1}{2}\int_0^1 \frac{\log(1+x^2)\log(1-x)}{x}\,dx. \end{aligned} \tag{22}$$

In addition, using the integral form of the dilogarithm function, namely $-\operatorname{Li}_2(x) = x\int_0^1 \frac{\log(y)}{1-xy}\,dy$ (see [17, p. 46]), $\mathcal{A}$ can be expressed in other direction as follows

$$\mathcal{A} = -\int_0^1\int_0^1 \frac{x^2\log(y)}{(1+x^2)(1-xy)}\,dy\,dx = \int_0^1 \left(\int_0^1 \frac{x^2\log(1/y)}{(1+x^2)(1-xy)}\,dx\right)dy.$$

To compute the latter double integral, we decompose the integrand into partial fractions as follows:

$$\begin{aligned}\frac{x^2}{(1+x^2)(1-xy)} &= \frac{x^2+x^2y^2}{(1+x^2)(1+y^2)(1-xy)} = \frac{(1+x^2)-(1-x^2y^2)}{(1+x^2)(1+y^2)(1-xy)}\\ &= \frac{1}{(1+y^2)(1-xy)} - \frac{1}{(1+x^2)(1+y^2)} - \frac{xy}{(1+x^2)(1+y^2)}. \end{aligned} \tag{23}$$

Thus, upon integration, we find that

$$\begin{aligned}\mathcal{A} &= \int_0^1 \left(\frac{\log(1-y)}{y(1+y^2)} + \frac{\pi}{4(1+y^2)} + \frac{y\log(2)}{2(1+y^2)}\right)\log(y)\,dy\\ &= \int_0^1 \frac{\log(y)\log(1-y)}{y(1+y^2)}\,dy - \frac{\pi}{4}G - \frac{\pi^2}{96}\log(2). \end{aligned}$$

Setting $m = 1$ in Lemma 3.2, we obtain that $\int_0^1 \frac{x\log(x)}{1+x^2}\,dx = -\frac{\pi^2}{48}$. And $\int_0^1 \frac{\log(y)}{1+y^2}\,dy = -\eta(1) = -G$. Substituting these values, we obtain, using partial fractions and integration by parts

$$\begin{aligned}\mathcal{A} &= -\frac{\pi}{4}G - \frac{\pi^2}{96}\log(2) + \int_0^1 \frac{\log(y)\log(1-y)}{y(1+y^2)}\,dy \\ &= -\frac{\pi}{4}G - \frac{\pi^2}{96}\log(2) + \int_0^1 \frac{\log(y)\log(1-y)}{y}\,dy - \int_0^1 \frac{y\log(y)\log(1-y)}{1+y^2}\,dy \\ &= -\frac{\pi}{4}G - \frac{\pi^2}{96}\log(2) + \int_0^1 \frac{\mathrm{Li}_2(y)}{y}\,dy - \int_0^1 \frac{x\log(x)\log(1-x)}{1+x^2}\,dx.\end{aligned}$$

Now using the relation $\int_0^1 \frac{\mathrm{Li}_2(y)}{y}\,dy = \mathrm{Li}_3(1) = \zeta(3)$, we find using (21):

$$\begin{aligned}\mathcal{A} = -\frac{\pi}{4}G - \frac{\pi^2}{96}\log(2) + \zeta(3) - \int_0^1 \frac{x\log(x)\log(1-x)}{1+x^2}\,dx = \zeta(3) - \frac{\pi}{4}G \\ - \frac{\pi^2}{96}\log(2) - \frac{1}{2}\int_0^1 \frac{\log(1+x^2)\log(x)}{1-x}\,dx + \frac{1}{2}\int_0^1 \frac{\log(1+x^2)\log(1-x)}{x}\,dx.\end{aligned} \tag{24}$$

Equating (22) and (24), we conclude

$$\frac{1}{2}\int_0^1 \frac{\log(1+x^2)\log(x)}{1-x}\,dx = \zeta(3) - \frac{\pi}{4}G - \frac{3\pi^2}{32}\log(2). \tag{25}$$

Next, we demonstrate that the latter integral of (21) or integral in (22) holds

$$\frac{1}{2}\int_0^1 \frac{\log(1+x^2)\log(1-x)}{x}\,dx = \frac{23}{64}\zeta(3) - \frac{\pi}{4}G. \tag{26}$$

We proceed to prove (26) by using the logarithmic series [7, Entry 1.551] of $\log(1+x^2)$

$$\frac{1}{2}\int_0^1 \frac{\log(1+x^2)\log(1-x)}{x}\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1 x^{2n-1}\log(1-x)\,dx \tag{27}$$

$$= -\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_{2n}}{4n^2}. \tag{28}$$

The series (28) is an immediate consequence of the integral $\int_0^1 x^{n-1}\log(1-x)\,dx = -\frac{H_n}{n}$ for $n > 0$ (see [13, p. 2]), which we obtain by shifting index n to $2n$ and employing in (27).

Taking advantage of Lemma 3.4, we calculate (28). We multiply both sides of Lemma 3.4 by the factor $1/2$ to get

$$-\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_{2n}}{4n^2} = -\int_0^1 \frac{1}{y}\int_0^y \left(\frac{\tan^{-1}(x)}{1+x^2} + \frac{\log(1+x^2)}{2x(1+x^2)}\right)dx\,dy. \tag{29}$$

First, performing the inner integration, we have

$$\int_0^y \left(\frac{\tan^{-1}(x)}{1+x^2} + \frac{\log(1+x^2)}{2x(1+x^2)}\right)dx = \frac{\left(\tan^{-1}(y)\right)^2}{2} - \frac{\mathrm{Li}_2(-y^2)}{4} - \frac{\log^2(1+y^2)}{8}. \tag{30}$$

Substituting the result (30) back into (29), we obtain

$$
\begin{aligned}
-\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_{2n}}{4n^2} &= \int_0^1 \frac{1}{y}\left(\frac{\mathrm{Li}_2(-y^2)}{4}+\frac{\log^2(1+y^2)}{8}-\frac{\left(\tan^{-1}(y)\right)^2}{2}\right)dy \\
&= \frac{\mathrm{Li}_3(-y^2)}{8}\bigg|_0^1 + \frac{\zeta(3)}{64} - \frac{1}{2}\int_0^1 \frac{\left(\tan^{-1}(y)\right)^2}{y}\,dy \\
&= -\frac{3\zeta(3)}{32} + \frac{\zeta(3)}{64} - \frac{\pi}{4}G + \frac{7\zeta(3)}{16} \\
&= \frac{23}{64}\zeta(3) - \frac{\pi}{4}G.
\end{aligned}
$$

In the course of the evaluation, we make use of well-known results, $\mathrm{Li}_3(-1) = -3\zeta(3)/4$ and $\int_0^1 \frac{\log^2(1+y^2)}{y}\,dy = \frac{\zeta(3)}{8}$, $\int_0^1 \frac{\left(\tan^{-1}(y)\right)^2}{y}\,dy = \frac{\pi}{2}G - \frac{7\zeta(3)}{8}$ (for proofs, see Section 5). Putting the obtained value back into (28), proves (26). Finally, collecting the results (25) and (26), and substituting them into (21) produces the announced result of the integral. □

Proposition 3.2. *The following relation holds:*

$$
\mathcal{B}(2,1) = \int_0^1 \frac{x\log^2(x)\log(1-x)}{1+x^2}\,dx = -\frac{5}{4}\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{13\pi^4}{3840} - \frac{5}{96}\log^4(2) + \frac{5\pi^2}{96}\log^2(2).
$$

Proof. Integral $\mathcal{B}(2,1)$ [17, p. 103, QLI(122;5)] can be evaluated by using integration by parts, and we obtain integrals that can be found in [10, 17]. Now, we begin as follows:

$$
\int_0^1 \frac{x\log^2(x)\log(1-x)}{1+x^2}\,dx = \int_0^1 \left(\frac{\log(x)}{2(1-x)} - \frac{\log(1-x)}{x}\right)\log(x)\log(1+x^2)\,dx.
$$

Using the linearity of the integral, we compute the latter integral as follows:

$$
\begin{aligned}
\int_0^1 \frac{\log(x)\log(1-x)\log(1+x^2)}{x}\,dx &= \int_0^1 \frac{\mathrm{Li}_2(x)\log(1+x^2)}{x}\,dx + 2\int_0^1 \frac{x\log(x)\,\mathrm{Li}_2(x)}{1+x^2}\,dx \\
&= I + 2J \qquad (31)
\end{aligned}
$$

The former integral

$$
\begin{aligned}
I = \int_0^1 \frac{\mathrm{Li}_2(x)\log(1+x^2)}{x}\,dx &= \frac{5}{4}\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{23\pi^4}{2304} + \frac{35}{32}\zeta(3)\log(2) \\
&\quad - \frac{5\pi^2}{96}\log^2(2) + \frac{5}{96}\log^4(2), \qquad (32)
\end{aligned}
$$

which is given in [17, p. 82]. It states that the value of integral is equal to **QPLI4(1;4;2)**, and its corresponding result is tabulated in [17, p. 107]. Alternatively, closed form (32) can be obtained by transforming the integral into an infinite sum. To do so, we use the logarithmic series of $\log(1+x^2)$, and then the integral relation $\int_0^1 x^{2n-1}\,\mathrm{Li}_2(x)\,dx = \frac{\pi^2}{12n} - \frac{H_{2n}}{4n^2}$ (see [8, p. 22]). The general result can be found in [9, p. 194]. After some calculations, we get

$$
\int_0^1 \frac{\mathrm{Li}_2(x)\log(1+x^2)}{x}\,dx = \frac{\pi^4}{144} - \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_{2n}}{4n^3}.
$$

The obtained series is the toughest harmonic series that can be reduced to closed form by using the generating function $\sum_{n=1}^{\infty} \frac{H_n}{n^3} x^n$ (see [12, p. 84] and [9, p. 76]). However, an approach due to Vălean by real analysis can be found in [10].

Integral J is a particular ($n = 1$) case of the integral that can be found in [13, p. 19]. We transform the integral J to a double integral,

$$J = \int_0^1 \frac{x \log(x) \operatorname{Li}_2(x)}{1+x^2} \, dx = -\int_0^1 \left(\int_0^1 \frac{x^2 \log(x) \log(y)}{(1+x^2)(1-xy)} \, dx \right) dy, \tag{33}$$

using the identity $-\operatorname{Li}_2(x) = x \int_0^1 \frac{\log(y)}{1-xy} \, dy$. Furthermore, employing (23) in (33) and integrating yields

$$\begin{aligned}
J &= \int_0^1 \frac{\operatorname{Li}_2(y) \log(y)}{y(1+y^2)} \, dy + \int_0^1 \int_0^1 \left(\frac{\log(x)\log(y)}{(1+x^2)(1+y^2)} \, dy + \frac{x \log(x) y \log(y)}{(1+x^2)(1+y^2)} \right) dx \, dy \\
&= \int_0^1 \frac{\operatorname{Li}_2(y) \log(y)}{y} \, dy - \int_0^1 \frac{y \log(y) \operatorname{Li}_2(y)}{1+y^2} \, dy + G^2 + \frac{\pi^4}{2304} \\
&= -\frac{\pi^4}{90} - J + G^2 + \frac{\pi^4}{2304}, \\
2J &= G^2 - \frac{41}{3840}\pi^4.
\end{aligned}$$

The two integrals can be found in [7]. Since the calculations involved are trivial, the details are left to the reader. Thus, substituting the final answers into (31) leads us to a remarkable conclusion as follows:

$$\begin{aligned}
\int_0^1 \frac{\log(x)\log(1-x)\log(1+x^2)}{x} \, dx &= \frac{5}{4} \operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{35}{32}\zeta(3)\log(2) + \frac{5}{96}\log^4(2) \\
&\quad + G^2 - \frac{5}{96}\pi^2 \log^2(2) - \frac{119}{5760}\pi^4.
\end{aligned} \tag{34}$$

In a similar fashion, we can readily conclude that

$$\frac{1}{2} \int_0^1 \frac{\log(1+x^2)\log^2(x)}{1-x} \, dx = \zeta(3)\log(2) - \int_0^1 \int_0^1 \frac{x^2 \log^2(xy)}{(1+x^2)(1-xy)} \, dx \, dy, \tag{35}$$

which is established through integration by parts. The general result of the former integral can be found in [14, p. 38]. Using (23), the final integral reduces to

$$\int_0^1 \int_0^1 \left(\frac{1}{(1+x^2)(1-xy)} - \frac{1}{(1+x^2)(1+y^2)} - \frac{xy}{(1+x^2)(1+y^2)} \right) \log^2(xy) \, dx \, dy.$$

By applying the identity $\log^2(xy) = \log^2(x) + 2\log(x)\log(y) + \log^2(y)$, we deduce that

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{\log^2(xy)(1+xy)}{(1+x^2)(1+y^2)} \, dx \, dy &= \int_0^1 \int_0^1 \frac{\log^2(x) + 2\log(x)\log(y) + \log^2(y)}{(1+x^2)(1+y^2)} \, dx \, dy \\
&\quad + \int_0^1 \int_0^1 \frac{xy(\log^2(x) + 2\log(x)\log(y) + \log^2(y))}{(1+x^2)(1+y^2)} \, dx \, dy \\
&= 2G^2 + \frac{\pi^4}{32} + \frac{3}{16}\zeta(3)\log(2) + \frac{\pi^4}{1152}.
\end{aligned} \tag{36}$$

The integrals involved in the above calculation are straightforward, and we leave the details to the reader to pursue the final result (36). Similarly, we show that

$$\int_0^1\int_0^1 \frac{\log^2(xy)}{(1+x^2)(1-xy)}\,dx\,dy = \frac{\pi^4}{15} - \zeta(3)\log(2) + \int_0^1 \frac{\log^2(x)\log(1+x^2)}{2(1-x)}\,dx. \tag{37}$$

Substituting (36) and (37) into (35), and dividing the obtained result by 2 yields

$$\frac{1}{2}\int_0^1 \frac{\log(1+x^2)\log^2(x)}{1-x}\,dx = G^2 + \frac{35}{32}\zeta(3)\log(2) - \frac{199}{11520}\pi^4. \tag{38}$$

Subtracting (38) from (34) completes the proof. □

Proposition 3.3. *The following integral equality holds:*

$$\mathcal{B}(1,2) = \int_0^1 \frac{x\log x\log^2(1-x)}{1+x^2}\,dx = -\frac{15}{8}\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{167\pi^4}{23040} - \frac{5}{64}\log^4(2) + \frac{\pi^2}{32}\log^2(2).$$

Proof. Let the integral be denoted by $\mathcal{I}$. We note that $\dfrac{x}{1+x^2} = \sum_{n\geq 1}(-1)^{n-1}x^{2n-1}$, and using Lemma (3.5), we have

$$\begin{aligned}
\mathcal{I} &= \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-1}\log x\log^2(1-x)\,dx \\
&= \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2\zeta(3)}{2n} + \frac{2\zeta(2)H_{2n}}{2n} - \frac{H_{2n}^{(2)}}{4n^2} - \frac{H_{2n}^2}{4n^2} - \frac{2H_{2n}H_{2n}^{(2)}}{2n} - \frac{2H_{2n}^{(3)}}{2n}\right) \\
&= \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{\zeta(3)}{n} + \frac{\zeta(2)H_{2n}}{n} - \frac{H_{2n}^{(2)}}{4n^2} - \frac{H_{2n}^2}{4n^2} - \frac{H_{2n}H_{2n}^{(2)}}{n} - \frac{H_{2n}^{(3)}}{n}\right). \\
&= \zeta(3)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} + \zeta(2)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}}{n} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(2)}}{4n^2} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^2}{4n^2} \\
&\quad - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}H_{2n}^{(2)}}{n} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(3)}}{n}.
\end{aligned} \tag{39}$$

By substituting the values of (14), (15), (16), (17), (18), and $\eta(1) = \log 2$ into (39), and after some computations, we obtain the desired result. □

Remark 3.1. Corollary 2.3 in [11] states that for $m \neq -1, -2, \ldots$, and $q \in \mathbb{N}$, the following holds:

$$\int_0^1 x^m\log^q(x)\log^2(1-x)\,dx = 2(-1)^q q!\sum_{n=0}^{\infty}\frac{H_{n+1}}{(n+2)(n+m+3)^{q+1}}.$$

Letting $q = 1$, replacing m with $2m+1$, and then carrying out the sum as $\sum_{m\geq 0}(-1)^m$ yields

$$\mathcal{B}(1,2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+1}H_{n+1}}{(n+2)(n+2m+4)^2} = -\frac{15}{8}\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{167\pi^4}{23040} - \frac{5\log^4(2)}{64} + \frac{\pi^2}{32}\log^2(2).$$

Proposition 3.4. *For a positive integer $m > 1$, the following equality holds:*

$$\int_0^1 \frac{x}{1+x^2} \sqrt[m]{\frac{1-x}{x}}\, dx = \frac{\pi}{\sin\left(\frac{\pi}{m}\right)} \left(\sqrt[2m]{2} \cos\left(\frac{\pi}{4m}\right) - 1 \right). \tag{40}$$

Proof. Let $\mathcal{G}(m)$ represent the integral (40). Using the geometric series expansion of $\frac{x}{1+x^2} = \sum_{k\geq 0}(-1)^k x^{2k+1}$, we simplify $\mathcal{G}(m)$ as follows:

$$\sum_{k=0}^{\infty}(-1)^k \int_0^1 x^{2k+1-1/m}(1-x)^{1/m}\, dx = \Gamma\left(1+\frac{1}{m}\right) \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma\left(2k+2-\frac{1}{m}\right)}{(2k+2)!}. \tag{41}$$

We set $a = 2k+2-1/m$, $b = 1+1/m$ into the definition of the beta function (6). Summing, we get an expression for the former and latter quantities of (41), respectively. Moreover, by the definition of the gamma function (6), the term in the numerator of (41) equals $\Gamma(2k+2-1/m)$, and can be represented as an integral i.e., $\int_0^\infty x^{2k+1-1/m}e^{-x}\, dx$. This simplifies $\frac{\mathcal{G}(m)}{\Gamma(1+1/m)}$ to

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+2)!} \int_0^\infty x^{2k+2} \frac{e^{-x}}{x^{1+1/m}}\, dx &= \int_0^\infty x^{-1-1/m}e^{-x} \left(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+2}}{(2k+2)!} \right) dx \\ &= \int_0^\infty x^{-1-1/m}e^{-x}\left(1-\cos x\right)\, dx. \end{aligned} \tag{42}$$

Further, integrating by parts with $f(x) = e^{-x}(1-\cos x)$, $f'(x) = e^{-x}(\sin x - \cos x + 1)$, $g'(x) = x^{-1-1/m}$ and $g(x) = -mx^{-1/m}$ gives us

$$\int_0^\infty x^{-1-1/m}e^{-x}\left(1-\cos x\right)\, dx = m \int_0^\infty x^{-1/m}e^{-x}\left(\sin x + \cos x - 1\right)\, dx. \tag{43}$$

The equality (42) is achieved by using the Maclaurin series [7, Entry 1.411.3] of

$$\cos x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!},\ x \in \mathbb{R}.$$

By linearity, we have $\int_0^\infty x^{-1/m}e^{-x}\, dx = \Gamma\left(1-\frac{1}{m}\right)$, which is a direct consequence of the gamma function. Now, to find the closed form of the latter integral, we recall the general results $(\mathbf{5})$ and $(\mathbf{6})$ from [7, Entry 3.944], respectively.

$$\int_0^\infty x^{a-1}e^{-bx}\sin(xc)\, dx = \frac{\Gamma(a)}{(c^2+b^2)^{a/2}} \sin\left(a \tan^{-1}\frac{c}{b}\right),\ \Re(a) > 0, \Re(b) > |\Im(c)|, \tag{44}$$

$$\int_0^\infty x^{p-1}e^{-qx}\cos(xr)\, dx = \frac{\Gamma(p)}{(r^2+q^2)^{p/2}} \cos\left(p \tan^{-1}\frac{r}{q}\right),\ \Re(p) > 0, \Re(q) > |\Im(r)|. \tag{45}$$

By setting $a = 1-1/m$, $b = c = 1$ and $p = 1-1/m$, $p = r = 1$ in (44) and (45), respectively, we obtain

$$\int_0^\infty x^{-1/m}e^{-x}\sin x\, dx = \frac{\sqrt[2m]{2}}{\sqrt{2}}\Gamma\left(1-\frac{1}{m}\right) \sin\left[\left(1-\frac{1}{m}\right)\frac{\pi}{4}\right], \tag{46}$$

$$\int_0^\infty x^{-1/m}e^{-x}\cos x\, dx = \frac{\sqrt[2m]{2}}{\sqrt{2}}\Gamma\left(1-\frac{1}{m}\right) \cos\left[\left(1-\frac{1}{m}\right)\frac{\pi}{4}\right]. \tag{47}$$

Simplifying (46) and (47), one obtains that

$$\int_0^{\infty} x^{-1/m} e^{-x} (\sin x + \cos x)\, dx = \sqrt[2m]{2}\,\Gamma\left(1 - \frac{1}{m}\right) \cos\left(\frac{\pi}{4m}\right). \tag{48}$$

Putting the values of (48) and $\int_0^{\infty} x^{-1/m} e^{-x}\, dx$ back into (43), we arrive at

$$\begin{aligned} \mathcal{G}(m) &= m\Gamma\left(1 + \frac{1}{m}\right) \Gamma\left(1 - \frac{1}{m}\right) \left(\sqrt[2m]{2} \cos\left(\frac{\pi}{4m}\right) - 1\right) \\ &= \Gamma\left(\frac{1}{m}\right) \Gamma\left(1 - \frac{1}{m}\right) \left(\sqrt[2m]{2} \cos\left(\frac{\pi}{4m}\right) - 1\right) \end{aligned} \tag{49}$$

Inserting $z = 1/m$ into the reflection formula of the gamma function yields

$$\Gamma\left(\frac{1}{m}\right) \Gamma\left(1 - \frac{1}{m}\right) = \frac{\pi}{\sin\left(\frac{\pi}{m}\right)}.$$

Substituting this value into (49) and simplifying proves (40). □

Remark 3.2. For all $m > 1, 1/m \in (0,1)$. Let $t = 1/m$ in the Proposition 3.4, we obtain

$$\mathcal{G}(1/m) = \mathcal{G}(t) = \pi \operatorname{cosec}(\pi t) \left(2^{t/2} \cos\left(\frac{\pi}{4} t\right) - 1\right). \tag{50}$$

Executing (50) via Mathematica produces the Laurent series around $m = 0$, namely

$$\begin{aligned} \mathcal{G}(t) = \frac{\log(2)}{2} &+ \frac{t}{32}\left(4\log^2(2) - \pi^2\right) + \frac{t^2}{192}\left(4\log^3(2) + 13\pi^2\log(2)\right) \\ &+ \frac{t^3}{6144}\left(16\log^4(2) - 31\pi^4 + 104\pi^2\log^2(2)\right) \\ &+ \frac{t^4}{184320}\left(48\log^5(2) + 520\pi^2\log^3(2) + 1327\pi^4\log(2)\right) + \mathcal{O}(t^5), \end{aligned}$$

where $\mathcal{O}(t)$ denotes the Big O notation. In the above Laurent series, the coefficients of t^p for $p \leq t$ can be obtained by taking the p-th partial derivative of $\mathcal{G}(t)$ and letting $t \to 0$, i.e., $\lim_{t\to 0} \frac{\partial^p}{\partial t^p}\mathcal{G}(t)$.

Remark 3.3. Given Proposition 3.4, Remark 3.2, and the coefficients of the above Laurent series, it is evident that the following logarithmic integrals are valid:

$$\int_0^1 \frac{x}{1+x^2} \log\left(\frac{x}{1-x}\right) dx = \frac{\pi^2}{32} - \frac{\log^2(2)}{8}, \tag{51}$$

$$\int_0^1 \frac{x}{1+x^2} \log^2\left(\frac{x}{1-x}\right) dx = \frac{13\pi^2}{96}\log(2) + \frac{\log^3(2)}{24}, \tag{52}$$

$$\int_0^1 \frac{x}{1+x^2} \log^3\left(\frac{x}{1-x}\right) dx = \frac{31\pi^4}{1024} - \frac{\log^4(2)}{64} - \frac{13\pi^2}{128}\log^2(2). \tag{53}$$

Now, we are ready for our main results and their corresponding proofs.

4 Main results and proofs

Theorem 4.1. *For positive integers, $1 \le n \le 4$, the following relations hold:*

$$\begin{aligned}
\mathcal{P}_{\pm}(1) &= \frac{\log(2)}{2},\\
\mathcal{P}_{\pm}(2) &= \frac{5\pi^2}{96} - \frac{\log^2(2)}{8},\\
\mathcal{P}_{\pm}(3) &= \frac{35}{64}\zeta(3) - \frac{5\pi^2}{192}\log(2) + \frac{\log^3(2)}{48},\\
\mathcal{P}_{\pm}(4) &= \frac{5}{16}\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{343\pi^4}{92160} - \frac{5\pi^2}{768}\log^2(2) + \frac{\log^4(2)}{96}.
\end{aligned}$$

Proof. We will divide the proof into two separate cases. The first case will address the proofs for $n = 1$ and $n = 2$, while the remaining case will cover for $n = 3$ and $n = 4$. In the course of proving the major results, we rely on Newton's binomial formula (see [7, Entry 1.111])

$$(x - y)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{n-k} y^k.$$

First case: $n = 1$ and $n = 2$. By substituting $n = 1$ into Lemma 3.3, one obtains an elementary integral for $\mathcal{P}_{\pm}(1)$, with its value equal to $\lim_{m\to 0} \mathcal{G}(1/m)$. In simpler terms, $\mathcal{P}_{\pm}(1) = \log(2)/2$.

Likewise, for $n = 2$ in Lemma 3.3, we have

$$\begin{aligned}
\mathcal{P}_{\pm}(2) &= -\int_0^1 \frac{x\log(1-x)}{1+x^2}\,dx\\
&= -\int_0^1 \left(\log(x) - \log\left(\frac{x}{1-x}\right)\right)\frac{x}{1+x^2}\,dx\\
&= \int_0^1 \frac{x}{1+x^2}\log\left(\frac{x}{1-x}\right)dx - \int_0^1 \frac{x\log(x)}{1+x^2}\,dx.
\end{aligned}$$

Considering (51), the value of Lemma 3.2 at $m = 1$, and straightforward algebraic operations applied to the last two integrals, we arrive at the proposed result for $\mathcal{P}_{\pm}(2)$.

Second case: $n = 3$ and $n = 4$. Likewise, substituting $n = 3$ into Lemma 3.3 and observing that $\log^2(1-x) = -\log^2(x) + 2\log(x)\log(1-x) + \log^2\left(\frac{x}{1-x}\right)$, we find that

$$\begin{aligned}
\mathcal{P}_{\pm}(3) &= \frac{1}{2}\int_0^1 \frac{x\log^2(1-x)}{1+x^2}\,dx\\
&= \frac{1}{2}\int_0^1 \frac{x\left(2\log(x)\log(1-x) + \log^2\left(\frac{x}{1-x}\right) - \log^2(x)\right)}{1+x^2}\,dx\\
&= \int_0^1 \frac{x\log(x)\log(1-x)}{1+x^2}\,dx - \frac{1}{2}\int_0^1 \frac{x\log^2(x)}{1+x^2}\,dx + \frac{1}{2}\int_0^1 \frac{x}{1+x^2}\log^2\left(\frac{x}{1-x}\right)dx.
\end{aligned}$$

By setting $m = 2$ in Lemma 3.2, we obtain $\frac{3}{16}\zeta(3)$. Substituting the corresponding outputs of the integrals in the last line of (52) and using Proposition 3.1 yields

$$\begin{aligned}
\mathcal{P}_{\pm}(3) &= \frac{41}{64}\zeta(3) - \frac{3\pi^2}{32}\log(2) - \frac{3}{32}\zeta(3) + \frac{1}{2}\left(\frac{13\pi^2}{96}\log(2) + \frac{\log^3(2)}{24}\right)\\
&= \frac{35}{64}\zeta(3) - \frac{5\pi^2}{192}\log(2) + \frac{\log^3(2)}{48}.
\end{aligned}$$

Again for $n = 4$ in Lemma 3.3, and utilizing the identity

$$\log^3(1-x) = \log^3(x) - \log^3\left(\frac{x}{1-x}\right) - 3\log(x)\log(1-x)\log\left(\frac{x}{1-x}\right),$$

we observe

$$\begin{aligned}
\mathcal{P}_{\pm}(4) &= -\frac{1}{6}\int_0^1 \frac{x\log^3(1-x)}{1+x^2}\,dx \\
&= -\frac{1}{6}\int_0^1 \frac{x\left(\log^3(x) - \log^3\left(\frac{x}{1-x}\right) - 3\log(x)\log(1-x)\log\left(\frac{x}{1-x}\right)\right)}{1+x^2}\,dx \\
&= \frac{1}{2}\int_0^1 \frac{x\log^2(x)\log(1-x)}{1+x^2}\,dx - \frac{1}{2}\int_0^1 \frac{x\log(x)\log^2(1-x)}{1+x^2}\,dx \\
&\quad + \frac{1}{6}\int_0^1 \frac{x}{1+x^2}\log^3\left(\frac{x}{1-x}\right)dx - \frac{1}{6}\int_0^1 \frac{x\log^3(x)}{1+x^2}\,dx. \qquad (54)
\end{aligned}$$

Putting $m = 3$ in Lemma 3.2, we have $\int_0^1 \frac{x\log^3(x)}{1+x^2}\,dx = -\frac{7\pi^4}{1920}$. Then, substituting the values of Proposition 3.2, Proposition 3.3, and (53) into (54), and simplying the calculations, proves the announced result. □

5 Proofs of the two integrals

In this section, we present the proofs of the two integrals obtained during the calculation of (28), namely $\int_0^1 \frac{\log^2(1+y^2)}{y}\,dy$ and $\int_0^1 \frac{\left(\tan^{-1}(y)\right)^2}{y}\,dy$.

We show that

$$\int_0^1 \frac{\log^2(1+y^2)}{y}\,dy = \frac{\zeta(3)}{8} \quad \text{and} \quad \int_0^1 \frac{\left(\tan^{-1}(y)\right)^2}{y}\,dy = \frac{\pi G}{2} - \frac{7}{8}\zeta(3).$$

For the former integral, we initiate with the generating function of the harmonic number, we have

$$\begin{aligned}
\int_0^1 \frac{\log^2(1+y^2)}{y}\,dy &= -2\int_0^1 \frac{1}{y}\int_0^{-y^2} \frac{\log(1-x)}{1-x}\,dx\,dy = -\sum_{n=0}^{\infty}(-1)^n\frac{H_n}{(n+1)^2} \\
&= -\sum_{n=0}^{\infty}(-1)^n\frac{H_{n+1}-\frac{1}{n+1}}{(n+1)^2} = \sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n^2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}. \qquad (55)
\end{aligned}$$

Employing the recurrence relation of the harmonic number gives rise to the first sum (55). We employ a well-known identity (see [13, p. 310, Eqn. (4.88)], also see [3, p. 7]), which evaluates to $-\frac{5}{8}\zeta(3)$, for the first sum. The second sum is an elementary series that converges to $-\frac{3}{4}\zeta(3)$. Substituting the respective values into (55) yields the proposed answer.

The second integral is a well-known result, which can be found in [4, p. 3, Eqn. (35)], but we provide an alternative proof.

$$\begin{aligned}
\int_0^1 \frac{\left(\tan^{-1}(x)\right)^2}{x}\,dx &= \int_0^{\frac{\pi}{4}} \frac{y^2}{\sin y\cos y}\,dy = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{y^2}{\sin y}\,dy \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{2}} y\log\left(\tan\left(\frac{y}{2}\right)\right)dy. \qquad (56)
\end{aligned}$$

It is easy to show that $\log\left(\cot\left(\frac{y}{2}\right)\right) = -\log\left(\tan\left(\frac{y}{2}\right)\right)$. Considering this property, we utilize the Fourier series [7, Entry 1.442.2] of

$$\log\left(\tan\left(\frac{y}{2}\right)\right) = -2\sum_{n=1}^{\infty}\frac{\cos\left((2n-1)y\right)}{2n-1}, \quad 0 < y < \pi.$$

Employing it in (56), we get

$$\begin{aligned}
&-\frac{1}{2}\int_0^{\frac{\pi}{2}} y\log\left(\tan\left(\frac{y}{2}\right)\right)\,dy = \sum_{n=1}^{\infty}\frac{1}{2n-1}\int_0^{\frac{\pi}{2}} y\cos\left((2n-1)y\right)\,dy \\
&= \sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{\pi}{2(2n-1)}\sin\left((2n-1)\frac{\pi}{2}\right) - \frac{1}{2n-1}\int_0^{\frac{\pi}{2}}\sin\left((2n-1)y\right)\,dy\right) \\
&= \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2} - \sum_{n=1}^{\infty}\frac{1}{(2n-1)^3} = \frac{\pi G}{2} - \frac{7}{8}\zeta(3). \qquad (57)
\end{aligned}$$

Plugging (57) back to (56), we arrive to the proposed result. □

6 An open problem

To date, in the mathematical literature, the real values of the polylogarithm expression, $\mathcal{P}_{\pm}(n)$, are explicitly known only up to the fourth order. Even in this paper, we use standard methods and tools to prove the results only for $1 \leq n \leq 4$. By applying the techniques demonstrated in this paper, we can conclude that

$$\begin{aligned}
\Re\left(\mathrm{Li}_n\left(\frac{1\pm i}{2}\right)\right) &= \frac{(-1)^n}{2^n}\left(1-\frac{1}{2^{n-1}}\right)\zeta(n) - \frac{1}{\Gamma(n)}\sum_{k=1}^{n-2}(-1)^k\binom{n-1}{k}\mathcal{B}(n-k-1,k) \\
&\quad + \frac{(-1)^{n-1}}{\Gamma(n)}\lim_{t\to 0}\frac{\partial^{n-1}}{\partial t^{n-1}}\left(\pi\,\mathrm{cosec}(\pi t)\left(-1+2^{-t/2}\cos\left(\frac{\pi}{4}t\right)\right)\right), \qquad (58)
\end{aligned}$$

where $k \leq n$ and $\mathcal{B}(n-k-1,k)$ represents the integral $\mathcal{B}(a,b)$ for $a = n-k-1$ and $b = k$. For positive integers n and k such that $k \leq n$, consider $a = n-k-1$ and $b = k$. By employing Newton's binomial identity for $\log^n\left(\frac{x}{1-x}\right)$ and separating the first and the last terms of the series, we obtain

$$\log^n\left(\frac{x}{1-x}\right) = \log^n(x) + (-1)^n\log^n(1-x) + \sum_{k=1}^{n-1}\binom{n}{k}\log^{n-k}(x)(-\log(1-x))^k. \qquad (59)$$

Next, by multiplying both sides of (59) by the factor $\frac{x}{1+x^2}$, rearranging the terms, and then integrating, we obtain

$$\begin{aligned}
-(-1)^n\int_0^1\frac{x\log^n(1-x)}{1+x^2}\,dx &= \int_0^1\frac{x\log^n(x)}{1+x^2}\,dx - \int_0^1\frac{x}{1+x^2}\log^n\left(\frac{x}{1-x}\right)\,dx \\
&\quad + \sum_{k=1}^{n-1}(-1)^k\binom{n}{k}\int_0^1\frac{x\log^{n-k}(x)\log^k(1-x)}{1+x^2}\,dx. \qquad (60)
\end{aligned}$$

Replacing n with $n-1$, and then applying Lemma 3.2, Lemma 3.3, and Remark 3.2, expression (60) attains the form as highlighted in (58).

For $n=5$, using (58), we have

$$\Re\left(\operatorname{Li}_5\left(\frac{1\pm i}{2}\right)\right)=\frac{557}{1024}\zeta(5)-\frac{\pi^2}{64}\zeta(3)+\frac{\log^5(2)}{3840}+\frac{13\pi^2}{4608}\log^3(2)$$
$$+\frac{247\pi^4}{184320}\log(2)-\int_0^1\frac{x\log(x)\log^2(1-x)}{1+x^2}\log\left(\frac{x^{1/4}}{(1-x)^{1/6}}\right)dx. \quad (61)$$

The integral $\mathcal{B}(a,b)$ has an explicit solution using standard tools only for $(a,b)=(1,1)$, $(2,1)$, and $(1,2)$. The equation would yield an explicit result if we had the integral results for $\mathcal{B}(2,2)$ and $\mathcal{B}(1,3)$. However, there are no elementary solutions for these integrals. These integrals can be evaluated using Mathematica packages [2] developed by Kam Cheong Au, which combine MZV technique with complex analysis. For instance,

$$\mathcal{B}(1,3)=\frac{45}{16}\operatorname{Li}_5\left(\frac{1}{2}\right)-\frac{35\pi^2}{64}\zeta(3)+\frac{13287}{2048}\zeta(5)-\frac{3\log^5(2)}{128}+\frac{\pi^2}{64}\log^3(2)-\frac{23\pi^4}{1024}\log(2).$$

Additionally, the conjectured values due to Kam Cheong Au for $\Re\left(\operatorname{Li}_5(1+i)\right)$ and $\Re\left(\operatorname{Li}_6(1+i)\right)$ can be found in [6]. For instance,

$$\Re\left(\operatorname{Li}_5(1+i)\right)=\frac{5}{32}\operatorname{Li}_5\left(\frac{1}{2}\right)+\frac{2139\zeta(5)}{4096}-\frac{\ln^5(2)}{768}+\frac{\pi^2\ln^3(2)}{288}+\frac{97\pi^4\ln(2)}{18432}.$$

In view of Lemma 3.3, the desired result of $\mathcal{P}_{\pm}(5)$ is equivalent to calculating the integral $\int_0^1\frac{x\log^4(1-x)}{1+x^2}\,dx$, which can easily be evaluated by using Kam Cheong Au's Mathematica package. To extract the real part of $\mathcal{P}_{\pm}(5)$, we use the polylogarithmic inversion formula [8, p. 196, Eqn. (7.38)], namely

$$\operatorname{Li}_n(r,\theta)+(-1)^n\operatorname{Li}_n(1/r,\theta)=-\frac{\log^n(r)}{n!}+2\sum_{m=1}^{\lfloor n/2\rfloor}\frac{\log^{n-2m}(r)}{(n-2m)!}\operatorname{Gl}_{2m}(\theta).$$

Since $\operatorname{Li}_n(r,\theta)=\operatorname{Li}_n(-r,\pi-\theta)$, replacing r with $-r$, putting $n=5,\ \theta=\pi$, and using the values of $\operatorname{Gl}_{2m}(\theta)$ given in [8, p. 202], for $m=1$ and $m=2$, we have

$$\operatorname{Li}_5(r)-\operatorname{Li}_5(1/r)=-\frac{\log^5(-r)}{5!}-2\eta(2)\frac{\log^3(-r)}{3!}-2\eta(4)\log(-r).$$

This particular case can be found in [9, p. 43, Eqn. (1.98)]. Now, putting $r=1+i$, and after some computation and extraction of the real parts, we obtain

$$\Re\left(\operatorname{Li}_5(1+i)\right)-\Re\left(\operatorname{Li}_5\left(\frac{1}{1+i}\right)\right)=\frac{1313\pi^4}{184320}\log(2)-\frac{\log^5(2)}{3840}+\frac{11\pi^2}{4608}\log^3(2).$$

Using the conjectured value of $\Re\left(\operatorname{Li}_5(1+i)\right)$, we find

$$\mathcal{P}_{\pm}(5)=\frac{5}{32}\operatorname{Li}_5\left(\frac{1}{2}\right)+\frac{2139}{4096}\zeta(5)-\frac{343\pi^4}{184320}\log(2)+\frac{5\pi^2}{4608}\log^3(2)-\frac{\log^5(2)}{960},$$

which can be verified by employing the PSLQ-algorithm. Similarly, using the inversion formula and the conjectured value of $\Re\left(\operatorname{Li}_6(1+i)\right)$, we can easily conjecture the value of $\mathcal{P}_{\pm}(6)$.

Furthemore, in view of (61) and $\mathcal{P}_{\pm}(5)$, we can readily conclude that

$$\int_0^1 \frac{x\log(x)\log^2(1-x)}{1+x^2}\log\left(\frac{x^{1/4}}{(1-x)^{1/6}}\right)dx = \frac{89}{4096}\zeta(5) - \frac{5}{32}\operatorname{Li}_5\left(\frac{1}{2}\right) + \frac{59\pi^4}{184320}\log(2) - \frac{\pi^2}{64}\zeta(3) + \frac{\log^5(2)}{768} + \frac{\pi^2}{576}\log^3(2).$$

The above integral is equivalent to computing the value of $\mathcal{B}(2,2)/4 - \mathcal{B}(1,3)/6$. Using the noted value of $\mathcal{B}(1,3)$, it is straightforward to deduce the value of $\mathcal{B}(2,2)$. However, the real challenge lies in evaluating these integrals without the use of Mathematica packages. Furthermore, the question remains whether there exists any generalization for $\mathcal{B}(a,b)$. For brevity, we leave this question for future research.

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