Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2024, Volume 30, Number 3, 602–612
DOI: 10.7546/nntdm.2024.30.3.602-612

Insulated primes

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Dedicated to an amazing teacher, Ms. Sneh Lata.

Received: 23 June 2023 **Revised:** 27 May 2024
Accepted: 1 October 2024 **Online First:** 24 October 2024

Abstract: The degree of insulation of a prime p is defined as the largest interval around p within which no other prime exists. A prime p is classified as insulated if its degree of insulation is greater than that of its neighbouring primes. This leads to the emergence of a new sequence, known as the insulated primes, which starts with $7, 13, 23, 37, 53, 67, 89, 103, 113, 131, 139, 157, 173, 181, 193, 211, 233, 277, 293$, and so on. This paper explores several properties and intriguing relationships concerning the degree of insulation, and includes a brief heuristic study of the insulated primes. Finally, the reader is left with a captivating open problem.
Keywords: Special prime sequences, Prime gaps.
2020 Mathematics Subject Classification: 11A41, 11K31.

1 Introduction

Prime numbers (denoted by $\mathbb{P}$) and their several special sub-sequences have continuously fascinated both young enthusiasts and experienced researchers [7]. During the covid-19 pandemic, we reported two new sequences A339270 and A339148 to OEIS, namely degree of insulation and insulated primes, respectively.

Definition 1.1. *The degree of insulation* $D : \mathbb{P} \to \mathbb{N}$ *of a prime* p *is defined as* $D(p) \triangleq \max X_p$*, where the set* $X_p = \{m \in \mathbb{N} : \pi(p - m) = \pi(p + m) - 1\}$ *and* $\pi(x)$ *is the prime counting function. As a convention, fix* $D(2) = 0$.

Since $D(p_n)$ can be interpreted as the largest spread around p_n containing only the prime p_n, any procedure to evaluate $D(p_n)$ will either compute the prime counting function $\pi(x)$ or determine the surrounding primes (p_{n-1}, p_{n+1}). The plot of $D(p)$ values for primes less than 1000 is shown in Figure 1.

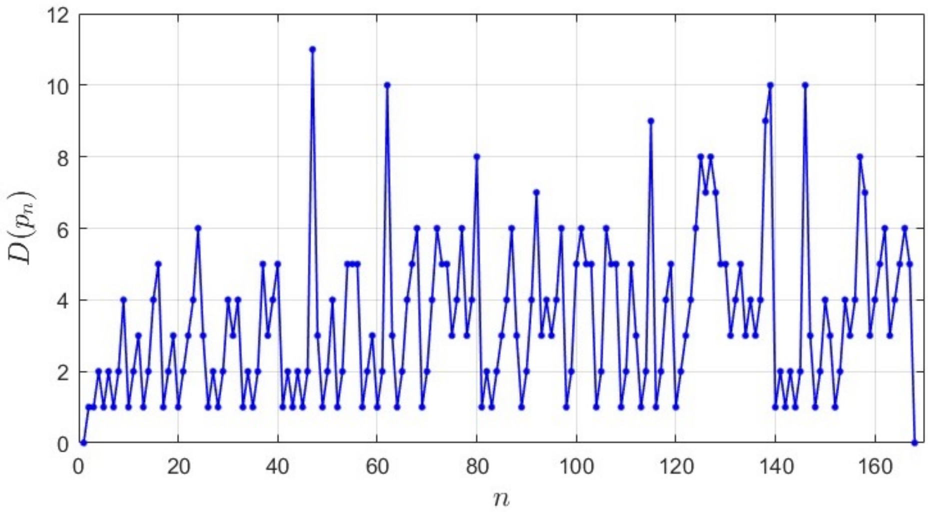

Figure 1. Plot of $D(p)$ for primes less than 1000.

Consider the prime triplet $(p_{n-1}, p_n, p_{n+1}) = (19, 23, 29)$, then $D(23)$ is calculated as follows:

$$\pi(23-1) \stackrel{?}{=} \pi(23+1) - 1 \Rightarrow 8 \stackrel{?}{=} 9 - 1 \Rightarrow 8 = 8$$
$$\pi(23-2) \stackrel{?}{=} \pi(23+2) - 1 \Rightarrow 8 \stackrel{?}{=} 9 - 1 \Rightarrow 8 = 8$$
$$\pi(23-3) \stackrel{?}{=} \pi(23+3) - 1 \Rightarrow 8 \stackrel{?}{=} 9 - 1 \Rightarrow 8 = 8$$
$$\pi(23-4) \stackrel{?}{=} \pi(23+4) - 1 \Rightarrow 8 \stackrel{?}{=} 9 - 1 \Rightarrow 8 = 8$$
$$\pi(23-5) \stackrel{?}{=} \pi(23+5) - 1 \Rightarrow 7 \stackrel{?}{=} 9 - 1 \Rightarrow 7 \neq 8$$

which gives $D(23) = 4$. This process highlights two key results: (a) if $\alpha \notin X_p$ then $(\alpha + r) \notin X_p$ for all $r \geq 0$, and (b) if $\alpha \notin X_p$ then $D(p) < \alpha$ for prime p. On similar lines, as illustrated above, one can evaluate $D(19) = 2$ and $D(29) = 1$. Observing that $D(23) > D(19)$ and $D(23) > D(29)$ gives rise to the concept of insulated primes which is formally defined below.

Definition 1.2. *The* n*-th prime* p_n *is said to be insulated if and only if*

$$D(p_n) > \max\{D(p_{n-1}), D(p_{n+1})\}.$$

Figure 2 shows the plot of n-th insulated prime i_n versus n. Some quick observations regarding insulated primes* are: (a) primes just adjacent to an insulated prime can never be insulated, and (b) i_n seems to obey a linear-like fit.

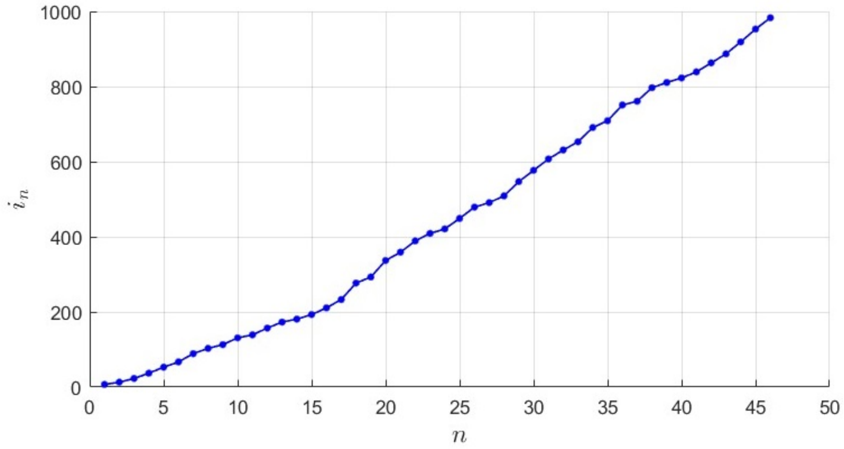

Figure 2. Plot of i_n versus n for primes less than 1000.

With the motivation laid out, the subsequent sections are devoted to its investigation. The remainder of the paper presents analytic/heuristic analysis of $D(p)$ and insulated primes.

2 Main analysis and results

Proposition 2.1. *Primes just adjacent to an insulated prime can never be insulated.*

Proof. Let p_n be an insulated prime, then $D(p_{n-1}) < D(p_n)$ and $D(p_{n+1}) < D(p_n)$ by definition. For p_{n-1} to be an insulated prime, the conditions $D(p_{n-2}) < D(p_{n-1})$ and $D(p_n) < D(p_{n-1})$ must hold. Clearly, the latter condition contradicts the condition for insulation of p_n, therefore, p_{n-1} cannot be an insulated prime. Likewise, for p_{n+1} to be an insulated prime, the conditions $D(p_n) < D(p_{n+1})$ and $D(p_{n+2}) < D(p_{n+1})$ must hold; however, $D(p_n) < D(p_{n+1})$ poses a contradiction, therefore, p_{n+1} also cannot be an insulated prime. □

Proposition 2.2. *For prime p, if $\alpha \notin X_p$ then $D(p) < \alpha$.*

Proof. Since $\pi(x)$ is an increasing function, so for every $r \geq 0$, we have $\pi(p+\alpha+r) \geq \pi(p+\alpha)$ and $\pi(p-\alpha) \geq \pi(p-\alpha-r)$ which combines to give

$$\pi(p+\alpha+r) - \pi(p-\alpha-r) \geq \pi(p+\alpha) - \pi(p-\alpha).$$

As X_p contains all the possible candidates for being $D(p)$ and given $\alpha \notin X_p$, then $\pi(p+\alpha) - \pi(p-\alpha) > 1$ since there exists at least one prime $p \in [p-\alpha, p+\alpha]$. Therefore, $\pi(p+\alpha+r) - \pi(p-\alpha-r) \geq \pi(p+\alpha) - \pi(p-\alpha) > 1$ implies $\pi(p+\alpha+r) - \pi(p-\alpha-r) \neq 1$. That is, $\alpha + r$ is also not a possible candidate for $D(p)$, thus, $D(p)$ must be less than α. □

* Since $D(p)$ essentially insulates p from neighboring primes, the name "insulated primes" is given. Initially, "isolated" was intended to be used, but it is already taken (OEIS: A023188).

Theorem 2.1. *We have* $D(p_k) = \min\{p_{k+1} - p_k - 1, p_k - p_{k-1}\}$.

Proof. Let $\lfloor . \rfloor_{\mathbb{P}}$ and $\lceil . \rceil_{\mathbb{P}}$ be the prime floor and prime ceiling functions [14]. Then, the degree of insulation can be equivalently expressed as $D(p) = \max\{m \in \mathbb{N} : \lfloor p+m \rfloor_{\mathbb{P}} = \lceil p-m \rceil_{\mathbb{P}}\}$. This shows $D(p)$ is the largest m such that there is no prime except p from $p-m+1$ to $p+m$. Hence, $D(p) = \min\{\lceil p+1 \rceil_{\mathbb{P}} - p - 1, p - \lfloor p-1 \rfloor_{\mathbb{P}}\}$. ☐

Despite suspicion of a direct connection between degree of insulation and gap between primes, note that unlike gaps, the value of $D(p)$ can be odd as well.

Corollary 2.1. *For a prime* $p > 2$*, if* $D(p)$ *is odd then* $p + (D(p) + 1)$ *is prime, else if* $D(p)$ *is even then* $p - D(p)$ *is prime.*

Proof. From Theorem 2.1, $D(p_k) = \min\{p_{k+1} - p_k - 1, p_k - p_{k-1}\}$. Since the difference of two primes is always even, so, if $D(p_k)$ is odd then $D(p_k) = p_{k+1} - p_k - 1 \Rightarrow p_{k+1} = p_k + D(p_k) + 1$, else if $D(p_k)$ is even then $D(p_k) = p_k - p_{k-1} \Rightarrow p_{k-1} = p_k - D(p_k)$. ☐

Proposition 2.3. *For primes* $p_n \geq 5$*, if* $D(p_n) = 1$ *then* $D(p_{n+1}) = 2$.

Proof. Due to Corollary 2.1, $D(p_n) = 1$ implies $p_n + 2$ is the next prime p_{n+1}. In view of Proposition 2.2, only $m \in \{1, 2\}$ need to be checked within X_p for evaluating $D(p_{n+1})$. To find $D(p_{n+1})$, we have two scenarios based on whether $p_{n+1} + 2$ is prime or not. We can conclude that it is not possible for $p_{n+1} + 2$ to be a prime since it would imply the existence of the prime triplet $(p_n, p_n + 2, p_n + 4)$ which contradicts a known fact that every prime number (greater than 3) is congruent to ± 1 modulo 6. Under the remaining case that $p_{n+1} + 2$ is not a prime, we have

$$\begin{aligned}
&\pi(p_{n+1} - 1) \stackrel{?}{=} \pi(p_{n+1} + 1) - 1 \Rightarrow n \stackrel{?}{=} (n+1) - 1 \Rightarrow n = n \\
&\pi(p_{n+1} - 2) \stackrel{?}{=} \pi(p_{n+1} + 2) - 1 \Rightarrow n \stackrel{?}{=} (n+1) - 1 \Rightarrow n = n \\
&\pi(p_{n+1} - 3) \stackrel{?}{=} \pi(p_{n+1} + 3) - 1 \Rightarrow n - 1 \stackrel{?}{=} (n+1) - 1 \Rightarrow n - 1 \neq n,
\end{aligned}$$

which shows $D(p_{n+1}) = 2$. ☐

The next two results provide bounds on the degree of insulation. Following on the lines as shown below, interested reader may explore further to obtain other bounds.

Proposition 2.4. *For* $k \geq 6$*, we have*

$$D(p_k) \leq k - 1 + \log((k-1)\log(k-1)) + k \log\left(\frac{k \log k}{(k-1)\log(k-1)}\right). \tag{1}$$

Proof. We know the inequality:

$$n(\log(n \log n) - 1) < p_n < n \log(n \log n), \tag{2}$$

where the left-hand side (due to Dusart [6]) holds for $n \geq 2$ and the right-hand side (due to Rosser [18]) holds for $n \geq 6$. Then, due to Theorem 2.1, we have

$$
\begin{aligned}
D(p_k) &\le p_k - p_{k-1} \\
&\le k - 1 + \log((k-1)\log(k-1)) + k\log\left(\frac{k\log k}{(k-1)\log(k-1)}\right)
\end{aligned}
$$

where second line uses (2) to arrive at the final result. □

Proposition 2.5. *There exists a constant θ such that $D(p_k) < p_k^\theta - 1$ for sufficiently large k.*

Proof. Hoheisel [10] showed that there exists a constant $\theta < 1$ such that

$$\pi(x + x^\theta) - \pi(x) \sim \frac{x^\theta}{\log(x)} \quad \text{as } x \to \infty, \tag{3}$$

hence, showing $p_{n+1} - p_n < p_n^\theta$ for large n. Along with Theorem 2.1, we have $D(p_k) \le p_{k+1} - p_k - 1 < p_k^\theta - 1$. □

The constant θ has been extensively studied (for instance, see [1]). Hoheisel obtained the possible value $\frac{32999}{33000}$, which was subsequently improved to $\frac{249}{250}$ by Heilbronn [11]. Thereafter, its value has been substantially reduced [8, 9, 12, 19]. In 2001, Baker, Harman and Pintz [2] obtained that θ may be taken to 0.525 which is the best known unconditional result. Under the assumption that the Riemann hypothesis is true, much better results are known (see [3–5, 13, 17]).

Theorem 2.2. *The definition of insulation of a prime p_k is equivalent to*

$$\max\{g_{k+1}, \min\{g_{k-1}, g_{k-2} + 1\}\} < g_k < g_{k-1} + \max\{0, g_k + 1 - g_{k+1}\}, \tag{4}$$

where $g_k = p_{k+1} - p_k$ is the gap between consecutive primes.

Proof. The prime p_k is insulated if and only if $D(p_k) > \max\{D(p_{k-1}), D(p_{k+1})\}$. It can be equivalently written as

$$
\begin{aligned}
&D(p_k) - \max\{D(p_{k-1}), D(p_{k+1})\} > 0 \\
\Leftrightarrow &\min\{D(p_k) - D(p_{k-1}), D(p_k) - D(p_{k+1})\} > 0 \\
\Leftrightarrow &\min\{\min\{g_k - 1, g_{k-1}\} - D(p_{k-1}), \min\{g_k - 1, g_{k-1}\} - D(p_{k+1})\} > 0 \\
\Leftrightarrow &\min\{g_k - 1 - D(p_{k-1}), g_{k-1} - D(p_{k-1}), g_k - 1 - D(p_{k+1}), g_{k-1} - D(p_{k-1})\} > 0,
\end{aligned}
$$

where $D(p_k) = \min\{g_k - 1, g_{k-1}\}$ using Theorem 2.1. The resulting inequality requires that the minimum of the four entries should be positive, which happens if every entry is positive, that is,

$$
\begin{cases}
\max\{g_k - g_{k-1}, g_k - 1 - g_{k-2}\} > 0 \\
\max\{1, g_{k-1} - g_{k-2}\} > 0 \\
\min\{1, g_{k+1} - g_k\} < 0 \\
\min\{g_k - g_{k-1}, g_{k+1} - g_{k-1} - 1\} < 0
\end{cases} \tag{5}
$$

since

$$
\begin{cases}
g_k - 1 - D(p_{k-1}) = g_k - 1 - \min\{g_{k-1} - 1, g_{k-2}\} = \max\{g_k - g_{k-1}, g_k - 1 - g_{k-2}\} \\
g_{k-1} - D(p_{k-1}) = g_{k-1} - \min\{g_{k-1} - 1, g_{k-2}\} = \max\{1, g_{k-1} - g_{k-2}\} \\
g_k - 1 - D(p_{k+1}) = g_k - 1 - \min\{g_{k+1} - 1, g_k\} = -\min\{1, g_{k+1} - g_k\} \\
g_{k-1} - D(p_{k+1}) = g_{k-1} - \min\{g_{k+1} - 1, g_k\} = -\min\{g_k - g_{k-1}, g_{k+1} - g_{k-1} - 1\}
\end{cases}.
$$

In (5), notice that the second inequality $\max\{1, g_{k-1} - g_{k-2}\} > 0$ is trivially true due to the presence of 1 which makes the condition independent of the value of $g_{k-1} - g_{k-2}$. The third condition $\min\{1, g_{k+1} - g_k\} < 0$ will be true if and only if $g_{k+1} - g_k < 0$. The first and fourth inequalities can be expressed as $g_k - g_{k-1} + \max\{0, g_{k-1} - 1 - g_{k-2}\} > 0$ and $g_k - g_{k-1} + \min\{0, g_{k+1} - 1 - g_k\} < 0$ respectively, which can be combined into a single condition:

$$\min\{0, g_{k-2} + 1 - g_{k-1}\} < g_k - g_{k-1} < \max\{0, g_k + 1 - g_{k+1}\}.$$

Thus, (5) is reduced to

$$\begin{cases} g_{k+1} < g_k, \\ g_{k-1} + \min\{0, g_{k-2} + 1 - g_{k-1}\} < g_k < g_{k-1} + \max\{0, g_k + 1 - g_{k+1}\}, \end{cases}$$

which can be further combined into a single condition to obtain the final result. □

In order to carry out a formal study, let us first define:

$$f_k(x) \triangleq \frac{\#\{p \in \mathbb{P} : D(p) = k \text{ and } p \leq x\}}{\#\{p \in \mathbb{P} : p \leq x\}} \tag{6}$$

which is the fraction of primes with $D(p) = k$ over all primes below x. Figure 4 depicts that the actual values approximately lie on the Gaussian curve:

$$f_k(x) \approx \frac{1}{\sqrt{2\pi}\sigma(x)} \exp\left(-\frac{1}{2}\left(\frac{k-1}{\sigma(x)}\right)^2\right) \tag{7}$$

for the parameter σ dependent on x. Figure 3 is a scatter plot which is evidently dense for smaller values of $D(p)$. This phenomenon is nicely expressed in Figure 4 which is the plot of true values of f_k along with the Gaussian fit predicted in (8).

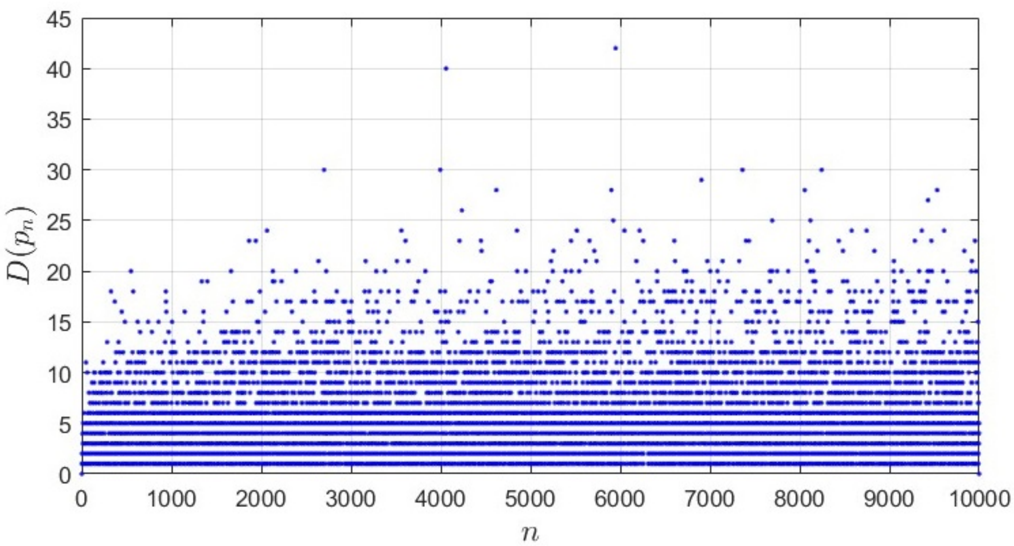

Figure 3. Scatter plot of $D(p_n)$ for $n \leq 10000$.

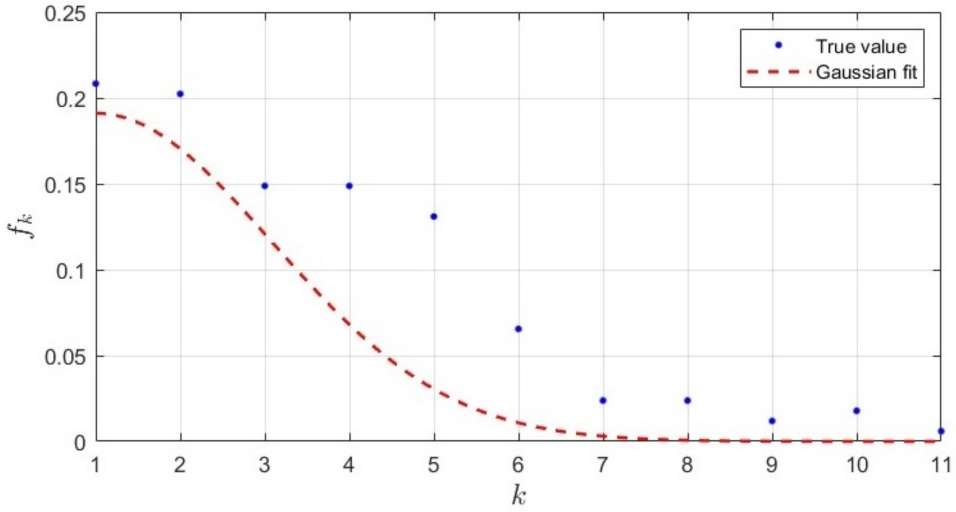

Figure 4. Plot of f_k versus k for primes less than 1000.

Theorem 2.3. *If Eq.* (7) *and the Hardy–Littlewood conjecture are true, then*

$$f_k(x) \sim \frac{2C}{\log x} \exp\left(-4\pi C^2 \left(\frac{k-1}{\log x}\right)^2\right), \tag{8}$$

where $C = \prod_{p \in \mathbb{P}_{>2}} \left(1 - \frac{1}{(p-1)^2}\right) \approx 0.6601618\ldots$ *is a constant.*

Proof. For $k = 1$, we have

$$\frac{\#\{p \in \mathbb{P} : D(p) = 1 \text{ and } p \leq x\}}{\#\{p \in \mathbb{P} : p \leq x\}} \approx \frac{1}{\sqrt{2\pi}\sigma(x)} \exp\left(-\frac{1}{2}\left(\frac{1-1}{\sigma(x)}\right)^2\right) = \frac{1}{\sqrt{2\pi}\sigma(x)},$$

where the left-hand side of the equation can be written as $\frac{\pi_2(x)}{\pi(x)}$ (where $\pi_2(x)$ is the number of twin-primes upto x) since counting the number of primes with $D(p) = 1$ gives the number of twin-prime pairs due to Corollary 2.1. Substituting the asymptotic formula for the prime counting function $\pi(x) \sim \int_2^x \frac{\mathrm{d}t}{\log t} \sim \frac{x}{\log x}$, and the twin-prime counting $\pi_2(x) \sim 2C \int_2^x \frac{\mathrm{d}t}{(\log t)^2} \sim 2C \frac{x}{(\log x)^2}$ (conjectured by Hardy and Littlewood, and heuristically verified in [20]), we obtain

$$\frac{1}{\sigma(x)} \approx \frac{\pi_2(x)}{\pi(x)}\sqrt{2\pi} \sim \frac{2C\frac{x}{(\log x)^2}}{\frac{x}{\log x}}\sqrt{2\pi} = \frac{2C\sqrt{2\pi}}{\log x},$$

where $C \approx 0.6601618\ldots$ is the twin-prime constant. Substituting the expression for $\sigma(x)$ in (7) gives the final result. □

The above analysis also motivates the following thought. Let $\nu(x, g)$ be the number of primes up to x that differ by gap g. In 2014, Zhang [21] proved that there are infinitely many pairs of primes that differ by g for some g less than 7×10^7; a bound which subsequently has been improved to 246 unconditionally (see [15, 16] and the references therein). Now, if $\nu(x, 2) \geq \nu(x, 4) \geq \nu(x, 6) \geq \cdots \geq \nu(x, g)$ for at least up to $g = 246$ then it would directly imply the infinitude of twin-primes. This is a separate topic for a future work.

3 Growth pattern of insulated primes

The most natural method for finding $D(p)$ is its definition, which involves computing $\pi(x)$. Alternatively, one could identify the surrounding primes of a given prime and apply Theorem 2.1, but this approach becomes impractical for extremely large primes. To reduce the number of $\pi(x)$ computations, one can leverage the results established in the previous section and employ bracketing techniques. For example, instead of starting from 1 and performing a linear search, it is more efficient to begin from a suitable initial guess, m_0, and then refine it using methods like the bisection method or genetic algorithm. In the case of very large primes, by applying Proposition 2.5 or a sharper inequality, one can select a good starting point, enabling faster convergence.

Insulated primes can be interpreted as the sequence of local maxima in the plot of degree of insulation. The sequence of insulated primes is $7, 13, 23, 37, 53, 67, 89, 103, 113, 131, 139, 157, 173, 181, 193$, and so on. Using MATLAB command `cftool` for the curve fitting toolbox, a variety of curves (with different settings) were tested which suggested i_n obeys a power law. It was observed that the equation $y = 18.41n^{1.097}$ is an extremely good fit as shown in Figure 5.

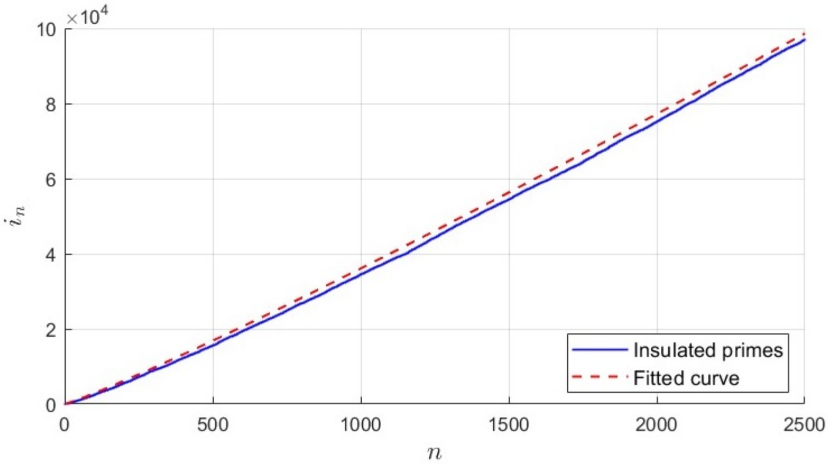

Figure 5. Comparison of i_n plots for primes less than 10^5.

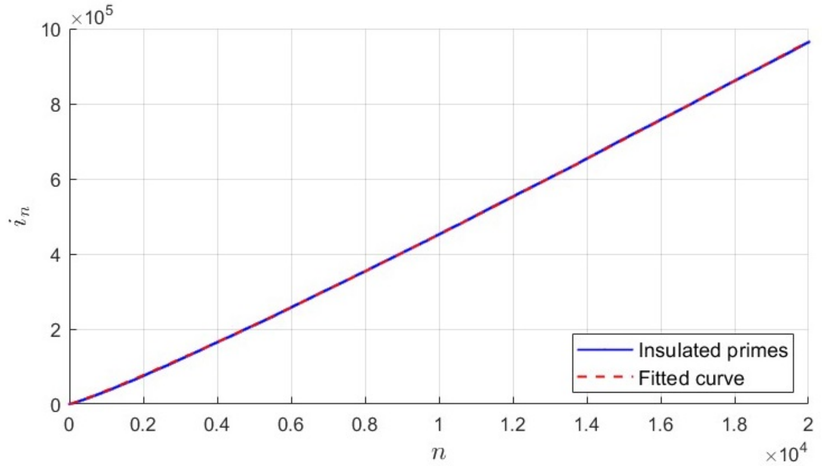

Figure 6. Comparison of i_n plots for primes less than 10^6.

The equation performs even better when tested for primes up to one million as shown in Figure 6. We can conclude that $i_n \sim 18.41n^{1.097}$ is heuristically an accurate fit for the specified magnitude of primes. The analysis so far suffices to convince that insulated primes are definitely well-behaved in comparison to primes or other prime subsets, though any concrete result would need to be rigorously proven.

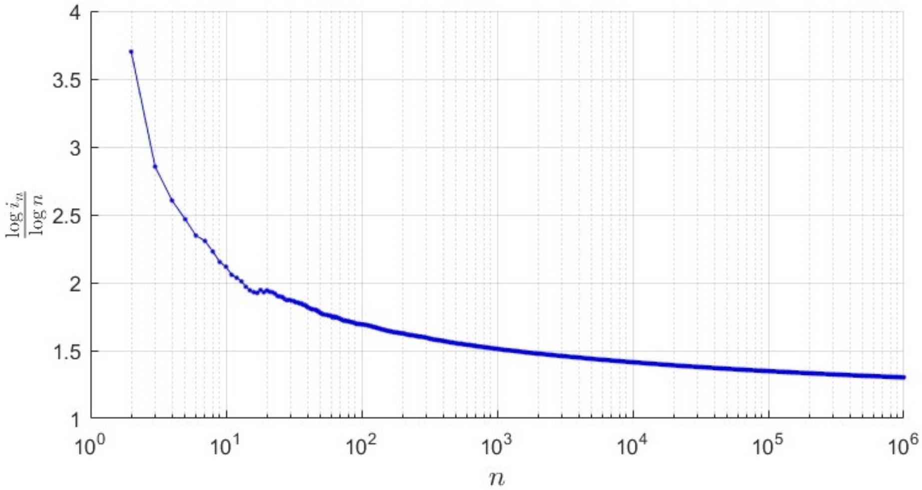

Figure 7. Plot of $\frac{\log i_n}{\log n}$ versus n.

Figure 7 is the plot of $\frac{\log i_n}{\log n}$ for first one million insulated primes. It offers some evidence that the exponent in the power law governing i_n may possibly approach a limiting value as n becomes large. However, a more thorough analytic investigation remains open.

4 Conclusion and Future scope

This paper investigates the properties of a newly defined sequence called the "insulated primes." These primes are characterized by their degree of insulation, a concept that has deep yet intricate connections to the gaps between consecutive primes. Through mathematical analysis, several interesting results are established. Moreover, heuristics suggest that the n-th insulated prime, i_n, follows a power law, indicating that the sequence of insulated primes may exhibit more regular behavior compared to the sequence of all primes.

An extension of this concept leads to the definition of "highly insulated primes." Just as the application of the idea of insulation to the set of primes $\mathbb{P}$ yields the set of insulated primes $\mathbb{I}$, applying the same concept to $\mathbb{I}$ produces the set of highly insulated primes $\mathbb{I}_H$. A prime i_n is considered highly insulated if and only if $D(i_n) > \max\{D(i_{n-1}), D(i_{n+1})\}$. This sequence is documented as A339188 in the OEIS, which begins as 23, 53, 89, 211, 293, and so on. Readers interested in further exploration are encouraged to investigate this sequence.

It is also worth noting that applying the concept of insulation each time reduces the size of the resulting set. This raises a thought-provoking question: could the repeated application of insulation eventually produce a finite set? Therefore, it is worthwhile to study the cardinality of a set formed by iteratively applying insulation.

Availability of code. Mathematica codes are available in OEIS. For implementation in other languages, visit `https://github.com/anuraag-saxena/Insulated-Primes`.

Author Contributions. Both authors contributed equally.

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