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A new quadra polynomial sequence

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Abstract: In this paper, we define a new quadra polynomial sequence by using Özkoç numbers as the coefficients. Then, we derive some properties for this polynomial sequence by the help of Fibonacci and Pell polynomials. Additionally, we attempt to define the companion matrix of this polynomial sequence.
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1 Introduction

The integer sequences and the polynomial sequences corresponds to these integer sequences were studied by many mathematicians. Some of these are Fibonacci polynomials, Pell polynomials and the most general version Horadam polynomials [1, 2, 5].

Now, we will give some properties for the Pell and Fibonacci polynomials which are used in this paper.

Definition 1.1. [1] *For $n \geq 0$, the recurrence relation of Pell polynomials $P_n(x)$ is*

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x),\ P_0(x) = 0,\ P_1(x) = 1. \tag{1}$$

In [1], the generating function for this polynomials is obtained as

$$P_n(x) = \frac{t}{1-2xt-t^2} \tag{2}$$

and the sum of the first n terms of Pell polynomials is

$$\sum_{i=1}^{n} P_i(x) = \frac{P_{n+1}(x)+P_n(x)-1}{2x}. \tag{3}$$

Definition 1.2. [5] *For $n \geq 0$, the recurrence relation of Fibonacci polynomials $F_n(x)$ is*

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ F_0(x) = 0,\ F_1(x) = 1. \tag{4}$$

In [5], the generating function for Fibonacci polynomials is given as

$$F_n(x) = \frac{t}{1-xt-t^2} \tag{5}$$

and the sum of the first n terms of Fibonacci polynomials is

$$\sum_{i=1}^{n} F_i(x) = \frac{F_{n+1}(x)+F_n(x)-1}{x}. \tag{6}$$

Moreover, some sequences defined with the fourth order recurrence relations [7, 8]. In [7], Tasci defined quadrapell numbers with the fourth order recurrence relation

$$D_n = D_{n-2} + 2D_{n-3} + D_{n-4},\ n \geq 4,$$

where $D_0 = D_1 = D_2 = 1$ and $D_3 = 2$ are the initial values. After that, Kızılateş et al. gave some properties and matrix sequences of these numbers [4].

Inspiring the definition of quadrapell numbers, some quadra sequences defined in [3,6]. In [6], Özkoç numbers introduced recursively by

$$W_n = 3W_{n-1} - 3W_{n-3} - W_{n-4},\ n \geq 4,$$

where $W_0 = W_1 = 0, W_2 = 1$ and $W_3 = 3$. The most important property of this sequence is that the characteristic equation of the sequence consists of the roots of the characteristic equations of Fibonacci and Pell numbers.

2 Main results

Definition 2.1. *For $n \geq 4$, the quadra polynomial sequence $W_n(x)$ is defined by the recursive formula*

$$W_n(x) = \begin{cases} 0, & \textit{if } n = 0, \\ 0, & \textit{if } n = 1, \\ x, & \textit{if } n = 2, \\ 3x^2, & \textit{if } n = 3, \\ 3xW_{n-1}(x) - (2x^2-2)W_{n-2}(x) - 3xW_{n-3}(x) - W_{n-4}(x), & \textit{if } n \geq 4. \end{cases}$$

In Table 1, the first few elements of this polynomial sequence $W_n(x)$ can be seen.

Table 1. The first few elements of $W_n(x)$

$\boldsymbol{n}$	$\boldsymbol{W_n(x)}$
0	0
1	0
2	x
3	$3x^2$
4	$7x^3+2x$
5	$15x^4+9x^2$
6	$31x^5+28x^3+3x$
7	$63x^6+75x^4+18x^2$
8	$127x^7+186x^5+70x^3+4x.$

The characteristic equation of $W_n(x)$ is

$$v^4-3xv^3+(2x^2-2)v^2+3xv+1=0,$$

and the roots of this equation are

$$\begin{aligned}\Phi(x)&=x+\sqrt{x^2+1},\quad \Psi(x)=x-\sqrt{x^2+1}\\ \alpha(x)&=\frac{x+\sqrt{x^2+4}}{2},\quad \beta(x)=\frac{x-\sqrt{x^2+4}}{2},\end{aligned} \tag{7}$$

where $\Phi(x)$ and $\Psi(x)$ satisfy the characteristic equation of Pell polynomials and $\alpha(x)$ and $\beta(x)$ satisfy characteristic equation of Fibonacci polynomials.

As an observation, it is seen in Table 2 that $W_i(1)=W(i)$ for $i=1,2,3,\ldots,n$, where $W(i)$ is the i-th Özkoç number.

Table 2. Özkoç numbers and polynomials

$\boldsymbol{n}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
$\boldsymbol{W_n(x)}$	0	0	x	$3x^2$	$7x^3+2x$	$15x^4+9x^2$	$31x^5+28x^3+3x$
$\boldsymbol{W_n}$	0	0	1	3	9	24	62

Theorem 2.1. *For $n\geq 0$, we have*

$$W_n(x)=P_n(x)-F_n(x), \tag{8}$$

where $P_n(x)$ and $F_n(x)$ are the Pell and Fibonacci polynomials, respectively.

Proof. We apply induction method on n. For $n = 0$, it is clear. Assume that the theorem holds for the integers less or equal to $n-1$. Now, we will show that the theorem holds for n.

By using the recurrence relation of Pell and Fibonacci polynomials, we obtain the followings, respectively,

$$\begin{aligned} P_n(x) &= 2xP_{n-1}(x) + P_{n-2}(x) \\ &= 3xP_{n-1}(x) - xP_{n-1}(x) + 2P_{n-2}(x) - P_{n-2}(x) \\ &\quad + xP_{n-3}(x) - xP_{n-3}(x) + P_{n-4}(x) - P_{n-4}(x) \\ &= 3xP_{n-1}(x) - x(P_{n-1}(x) - P_{n-3}(x)) - (P_{n-2}(x) - P_{n-4}(x)) \\ &\quad + 2P_{n-2}(x) - xP_{n-3}(x) - P_{n-4}(x) \\ &= 3xP_{n-1}(x) - (2x^2-2)P_{n-2}(x) - 3xP_{n-3}(x) - P_{n-4}(x) \end{aligned} \tag{9}$$

$$\begin{aligned} F_n(x) &= xF_{n-1}(x) + F_{n-2}(x) \\ &= xF_{n-1}(x) + 2F_{n-2}(x) - F_{n-2}(x) \\ &= xF_{n-1}(x) + 2F_{n-2}(x) - xF_{n-3}(x) - F_{n-4}(x) \\ &= 3xF_{n-1}(x) - 2xF_{n-1}(x) + 2F_{n-2}(x) \\ &\quad - 3xF_{n-3}(x) + 2xF_{n-3}(x) - F_{n-4}(x) \\ &= 3xF_{n-1}(x) - 2x(F_{n-1}(x) - F_{n-3}(x)) + 2F_{n-2}(x) \\ &\quad - 3(F_{n-2}(x) - F_{n-4}(x)) - F_{n-4}(x) \\ &= 3xF_{n-1}(x) - (2x^2-2)F_{n-2}(x) - 3xF_{n-3}(x) - F_{n-4}(x). \end{aligned} \tag{10}$$

By substracting the equation (9) and (10) side by side, we get

$$\begin{aligned} P_n(x) - F_n(x) = 3x(P_{n-1}(x) - F_{n-1}(x)) - (2x^2-2)(P_{n-2}(x) - F_{n-2}(x)) \\ - 3x(P_{n-3}(x) - F_{n-3}(x)) - (P_{n-4}(x) - F_{n-4}(x)). \end{aligned}$$

By using the assumptions of induction and the recurrence relation of $W_n(x)$, we obtain

$$\begin{aligned} P_n(x) - F_n(x) &= 3xW_{n-1}(x) - (2x^2-2)W_{n-2}(x) - 3xW_{n-3}(x) - W_{n-4}(x) \\ &= W_n(x). \end{aligned}$$ □

Theorem 2.2. *The generating function for $W_n(x)$ is*

$$G(t) = \frac{xt^2}{1 - 3xt + (2x^2-2)t^2 + 3xt + t^4}.$$

Proof. The formal power series expansion of the generating function for $W_n(x)$ at $x = 0$ is

$$G(t) = \sum_{n=0}^{\infty} W_n(x)t^n = W_0(x) + W_1(x)t + W_2(x)t^2 + W_3(x)t^3 + \cdots . \tag{11}$$

Then, by multiplying (11) by $-3xt, (2x^2-2)t^2, 3xt^3$ and t^4, respectively, we have

$$\begin{aligned}
-3xtG(t) &= 3xW_0(x)t + 3xW_1(x)t^2 + xW_2(x)t^3 + 3xW_3(x)t^4 + \cdots\,, \\
(2x^2-2)t^2G(t) &= (2x^2-2)W_0(x)t^2 + (2x^2-2)W_1(x)t^3 \\
&\quad + (2x^2-2)W_2(x)t^4 + (2x^2-2)W_3(x)t^5 + \cdots\,, \\
3xt^3G(t) &= 3xW_0(x)t^3 + 3xW_1(x)t^4 + 3xW_2(x)t^5 + 3xW_3(x)t^6 + \cdots\,, \\
t^4G(t) &= W_0(x)t^4 + W_1(x)t^5 + W_2(x)t^6 + W_3(x)t^6 + \cdots\,.
\end{aligned}$$

So, we get

$$\begin{aligned}
&(1 - 3xt + (2x^2-2)t^2 + 3xt^3 + t^4)G(t) \\
&\quad = W_0(x) + (W_1(x) - 3xW_0(x))t + (W_2(x) - 3xW_1(x) + (2x^2-2)W_0(x))t^2 \\
&\qquad + (W_3(x) - 3xW_2(x) + (2x^2-2)W_1(x) + 3xW_0(x))t^3 \\
&\qquad + (W_4(x) - 3xW_3(x) + (2x^2-2)W_2(x) + 3xW_1(x) + W_0(x))t^4 + \cdots\,.
\end{aligned}$$

Since, $W_0(x) = W_1(x) = 0, W_2(x) = x, W_3(x) = 3x^2$ and

$$W_n(x) = 3xW_{n-1}(x) - (2x^2-2)W_{n-2}(x) - 3xW_{n-3}(x) - W_{n-4}(x),$$

we have

$$(1 - 3xt + (2x^2-2)t^2 + 3xt^3 + t^4)G(t) = xt^2.$$

Therefore, the desired result is achieved. □

Theorem 2.3. *For $n \geq 0$, the Binet Formula for $W_n(x)$ is*

$$W_n(x) = \frac{\Phi^n(x) - \Psi^n(x)}{\Phi(x) - \Psi(x)} - \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)},$$

where $\Phi(x), \Psi(x), \alpha(x)$, and $\beta(x)$ defined in (7) .

Proof. In Theorem 2.2, we get the generating function for $W_n(x)$ as

$$W_n(x) = \frac{xt^2}{1 - 3xt + (2x^2-2)t^2 + 3xt + t^4}. \tag{12}$$

By using the partial fraction decomposition, (12) can be written as

$$W_n(x) = \frac{t}{1 - 2xt - t^2} - \frac{t}{1 - xt - t^2},$$

where $\frac{t}{1-2xt-t^2}$ is the generating function for the Pell polynomials and $\frac{t}{1-xt-t^2}$ is the generating function for the Fibonacci polynomials given in the equations (2) and (5) , respectively.

Hence, we get

$$W_n(x) = \frac{\Phi^n(x) - \Psi^n(x)}{\Phi(x) - \Psi(x)} - \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)},$$

where $\frac{\Phi^n(x) - \Psi^n(x)}{\Phi(x) - \Psi(x)}$ is the Binet formula for the Pell polynomials and $\frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}$ is the Binet formula for the Fibonacci polynomials obtained in [1] and [5], respectively. □

Theorem 2.4. *The sum of the first n terms of $W_n(x)$ can be evaluated by the following formula*

$$\sum_{i=1}^{n} W_i(x) = \frac{W_{n+1}(x) + W_n(x) - F_{n+1}(x) - F_n(x) + 1}{2x},$$

where $F_n(x)$ is the n-th Fibonacci polynomial.

Proof. By equation (8) , it is obvious that

$$\sum_{i=1}^{n} W_i(x) = \sum_{i=1}^{n} P_i(x) - \sum_{i=1}^{n} F_i(x).$$

If we use equations (3) and (6), then

$$\begin{aligned}\sum_{i=1}^{n} W_i(x) &= \frac{P_{n+1}(x) + P_n(x) - 1}{2x} - \frac{F_{n+1}(x) + F_n(x) - 1}{x} \\ &= \frac{P_{n+1}(x) - F_{n+1}(x) + P_n(x) - F_n(x) - F_{n+1}(x) - F_n(x) + 1}{2x} \\ &= \frac{W_{n+1}(x) + W_n(x) - F_{n+1}(x) - F_n(x) + 1}{2x}.\end{aligned}$$

□

Definition 2.2. *The companion matrix for $W_n(x)$ is*

$$M_x = \begin{bmatrix} 3x & -(2x^2-2) & -3x & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where $\det(M_x) = 1$ and the characteristic equation of M_x is

$$\lambda^4 - 3x\lambda^3 + (2x^2-2)\lambda^2 + 3x\lambda + 1 = 0$$

with the eigenvalues $\lambda_1 = \Psi(x), \lambda_2 = \Phi(x), \lambda_3 = \alpha(x)$, and $\lambda_4 = \beta(x)$.

Notice that, setting $x = 1$ we get the companion matrix of the Özkoç numbers.

3 Conclusion

In this paper, we define a new quadra polynomial sequence by using the Özkoç numbers as the coefficients. Then, we get some properties for this polynomial sequence by the help of Fibonacci and Pell polynomials. Also, the companion matrix of these polynomials defined.

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