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On the set of $\underline{\mathrm{Set}}(n)$'s

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Abstract: The set of $\underline{\mathrm{Set}}(n)$'s for natural numbers n is constructed. For this set it is proved that it is a commutative semi-group. The conditions for which it is a monoid are given.
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1 Introduction

For a fixed natural number $n \geq 2$, having the canonical form

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where $k, \alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ are natural numbers and $p_1 < p_2 < \cdots < p_k$ are different prime numbers, in [1], we defined the set:

$$\underline{\mathrm{Set}}(n) = \left\{ m \mid m = \prod_{i=1}^{k} p_i^{\beta_i} \ \ \& \ \ \delta(n) \leq \beta_i \leq \Delta(n) \right\},$$

where

$$\delta(n) = \min(\alpha_1, \dots, \alpha_k),$$
$$\Delta(n) = \max(\alpha_1, \dots, \alpha_k).$$

Other authors (see, e.g. [5]) denote the functions δ and Δ by h and H, respectively.

In the present paper, for the natural numbers n, the set of $\underline{\text{Set}}(n)$s is constructed and for this set it is proved that it is a commutative monoid.

2 Main results

Let everywhere below the natural numbers n and m have the canonical forms

$$m = \prod_{i=1}^{k+l} p_i^{\alpha_i} \geq 2,$$
$$n = \prod_{i=k+1}^{k+l+u} p_i^{\beta_i} \geq 2,$$

where $k, l, u \geq 0$ are natural numbers and obviously, $(m, n) = 1$ is and only if $l = 0$.

Therefore, they generate the sets

$$\underline{\text{Set}}(m) = \left\{ a \mid a = \prod_{i=1}^{k+l} p_i^{\gamma_i} \ \& \ \delta(m) \leq \gamma_i \leq \Delta(m) \right\},$$

$$\underline{\text{Set}}(n) = \left\{ b \mid b = \prod_{i=k+1}^{k+l+u} p_i^{\varepsilon_i} \ \& \ \delta(n) \leq \varepsilon_i \leq \Delta(m) \right\},$$

where

$$\delta(m) = \min(\alpha_1, \dots, \alpha_{k+l}),$$
$$\Delta(m) = \max(\alpha_1, \dots, \alpha_{k+l}).$$
$$\delta(n) = \min(\beta_{k+1}, \dots, \beta_{k+l+u}),$$
$$\Delta(n) = \max(\beta_{k+1}, \dots, \beta_{k+l+u}).$$

Obviously,

$$mn = \prod_{i=1}^{k+l} p_i^{\alpha_i} \cdot \prod_{i=k+1}^{k+l+u} p_i^{\beta_i} = \prod_{i=1}^{k} p_i^{\alpha_i} \cdot \prod_{i=k+1}^{k+l} p_i^{\alpha_i+\beta_i} \cdot \prod_{i=k+l+1}^{k+l+u} p_i^{\beta_i}.$$

The following question arises: *What is the relation between sets* $\underline{\text{Set}}(m), \underline{\text{Set}}(n)$ *and* $\underline{\text{Set}}(mn)$ *that can be constructed for the natural number* mn*?*

The latter set must have the form:

$$\underline{\text{Set}}(mn) = \left\{ c \mid c = \prod_{i=1}^{k+l+u} p_i^{\zeta_i} \ \& \ \delta(mn) \leq \zeta_i \leq \Delta(mn) \right\}. \quad (2)$$

Let us define the operation $*$ by

$$\underline{\mathrm{Set}}(m) * \underline{\mathrm{Set}}(n) = \underline{\mathrm{Set}}(mn). \tag{3}$$

Let

$$\Sigma = \{\underline{\mathrm{Set}}(n) \mid n \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of the natural numbers.

For the set Σ we will prove the following theorem.

Theorem 1. $\langle \Sigma, * \rangle$ *is a commutative semi-group.*

Proof. First, we will show that the operation $*$, defined by (3) really gives the set in the right side of (2). Let $\underline{\mathrm{Set}}(m), \underline{\mathrm{Set}}(n) \in \Sigma$. Then

$$\begin{aligned}
\underline{\mathrm{Set}}(m) * \underline{\mathrm{Set}}(n) = & \left\{ a \mid a = \prod_{i=1}^{k+l} p_i^{\gamma_i} \ \ \& \ \ \delta(m) \leq \gamma_i \leq \Delta(m) \right\} \\
& * \left\{ b \mid b = \prod_{i=k+1}^{k+l+u} p_i^{\varepsilon_i} \ \ \& \ \ \delta(n) \leq \varepsilon_i \leq \Delta(m) \right\} \\
= & \left\{ c \mid c = \prod_{i=1}^{k+l} p_i^{\gamma_i} \cdot \prod_{i=k+1}^{k+l+u} p_i^{\varepsilon_i} \ \ \& \ \ \delta(m) \leq \gamma_i \leq \Delta(m) \ \ \& \ \ \delta(n) \leq \varepsilon_i \leq \Delta(m) \right\}
\end{aligned}$$

Having in mind that

$$\delta(mn) = \min(\alpha_1, \ldots, \alpha_k, \alpha_{k+1} + \beta_{k+1}, \ldots, \alpha_{k+l} + \beta_{k+l}, \beta_{k+l+1}, \ldots, \beta_{k+l+u}),$$
$$\Delta(mn) = \max(\alpha_1, \ldots, \alpha_k, \alpha_{k+1} + \beta_{k+1}, \ldots, \alpha_{k+l} + \beta_{k+l}, \beta_{k+l+1}, \ldots, \beta_{k+l+u}),$$

if we put

$$\zeta_i = \begin{cases} \gamma_i, & \text{for } i = 1, \ldots, k \\ \gamma_i + \varepsilon_i, & \text{for } i = k+1, \ldots, k+l \\ \varepsilon_i, & \text{for } i = k+l+1, \ldots, k+l+u \end{cases}$$

then we will obtain that

$$\underline{\mathrm{Set}}(m) * \underline{\mathrm{Set}}(n) = \left\{ c \mid c = \prod_{i=1}^{k+l+u} p_i^{\zeta_i} \ \ \& \ \ \delta(mn) \leq \zeta_i \leq \Delta(mn) \right\} = \underline{\mathrm{Set}}(mn).$$

Hence $\underline{\mathrm{Set}}(mn) \in \Sigma$.

Second, in a similar, but essentially longer way, it is checked that for every three natural numbers m, n, r:

$$\underline{\mathrm{Set}}(m) * (\underline{\mathrm{Set}}(n) * \underline{\mathrm{Set}}(r)) = (\underline{\mathrm{Set}}(m) * (\underline{\mathrm{Set}}(n)) * \underline{\mathrm{Set}}(r),$$

i.e., the operation * is associative.

Third, for the natural numbers m and n we see as above that

$$\underline{\text{Set}}(m) * \underline{\text{Set}}(n) = \left\{ c \mid c = \prod_{i=1}^{k+l+u} p_i^{\zeta_i} \;\; \& \;\; \delta(mn) \leq \zeta_i \leq \Delta(mn) \right\}$$

$$= \left\{ c \mid c = \prod_{i=1}^{k+l+u} p_i^{\zeta_i} \;\; \& \;\; \delta(nm) \leq \zeta_i \leq \Delta(nm) \right\}$$

$$= \underline{\text{Set}}(n) * \underline{\text{Set}}(m),$$

i.e., the operation $*$ is commutative.
This proves the Theorem. □

It is easy to see that if we define

$$\underline{\text{Set}}(1) = \{1\},$$

then $\langle \Sigma, *, \underline{\text{Set}}(1) \rangle$ is not a (commutative) monoid, because for $\underline{\text{Set}}(n) * \underline{\text{Set}}(1)$ we will have that $\delta(n.1)$ must be equal to 1 even when $\delta(n) > 1$. Obviously, for each natural number n, $\delta(n) = 1$ if and only if n has at least one divisor with a degree 1. Now, let us define

$$\Sigma^* = \{\underline{\text{Set}}(n) \mid n \in \mathbb{N} \;\; \& \;\; \delta(n) = 1\}.$$

For it the following theorem is valid.

Theorem 2. $\langle \Sigma^*, *, \underline{\text{Set}}(1) \rangle$ *is a commutative monoid.*

Really, now

$$\underline{\text{Set}}(n) * \underline{\text{Set}}(1) = \underline{\text{Set}}(n) = \underline{\text{Set}}(1) * \underline{\text{Set}}(n).$$

Now, we can define

$$(\underline{\text{Set}}(n))^2 = \underline{\text{Set}}(n) * \underline{\text{Set}}(n).$$

Then by induction we can prove that for every two natural numbers $n, s \geq 2$:

$$(\underline{\text{Set}}(n))^s = (\underline{\text{Set}}(n))^{s-1} * \underline{\text{Set}}(n) = \left\{ m \mid m = \prod_{i=1}^{k} p_i^{\beta_i} \;\; \& \;\; s\delta(n) \leq \beta_i \leq s\Delta(n) \right\}.$$

Really, from (3) we obtain as the first step of the induction that

$$(\underline{\text{Set}}(n))^2 = \underline{\text{Set}}(n) * \underline{\text{Set}}(n) = \left\{ m \mid m = \prod_{i=1}^{k} p_i^{\beta_i} \;\; \& \;\; 2\delta(n) \leq \beta_i \leq 2\Delta(n) \right\}.$$

All operators from modal and topological type, defined over $\underline{\text{Set}}(n)$ in [1–4] can be applied over $\underline{\text{Set}}(m) * \underline{\text{Set}}(n)$, too.

3 Conclusion

In the paper, the set Σ was defined and some of its properties have been studied. An **Open Problem** is what other interesting properties Σ has.

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