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Note on a quadratic inequality

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Abstract: In this note we obtain a quadratic inequality based on a result of Atanassov but in a more symmetric form. Somewhat surprisingly, well-known properties of Chebyshev polynomials can be used to give a straightforward proof.
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1 Introduction

In [1] Atanassov proved the following elementary quadratic inequality. (See also [2] and [3] for similar results.) The notation here differs slightly from the notation used in [1].

For all real numbers $\{a_1, \ldots, a_n\}$ $(n \geq 2)$,

$$\sum_{k=1}^{n-1} a_{k+1}(a_k - a_{k+1}) \leq \frac{1}{2(n-1)} \sum_{k=1}^{n-1} a_k^2.$$

Separating out the cross and square terms, this becomes

$$\sum_{k=1}^{n-1} a_{k+1}(a_k - a_{k+1}) \leq \frac{1}{2(n-1)} a_1^2 + \frac{2n-1}{2(n-1)} \sum_{k=2}^{n-1} a_k^2 + a_n^2. \tag{1}$$

The right hand side of this inequality is assymetrical as it gives different weights to the terms in the sequence. If we look for a more symmetric expression, then a natural problem arises: can we find constants λ_n such that

$$\sum_{k=1}^{n-1} a_{k+1}\, a_k \le \lambda_n \sum_{k=1}^{n} a_k^2 \tag{2}$$

for all real numbers $\{a_1, \ldots, a_n\}$ $(n \ge 2)$.

The purpose of this note is to find the best constants.

2 Main result

Fix n and for simplicity write $\lambda_n = x$ in (2). We can write the quadratic form obtained from (2) as

$$2x \sum_{k=1}^{n} a_k^2 - 2\sum_{k=1}^{n-1} a_{k+1}a_k = \underline{a}^T\, H_n\, \underline{a} \tag{3}$$

where $\underline{a} = (a_1, \ldots, a_n)^T$ and H_n is the $n \times n$ symmetric matrix

$$H_n = \begin{bmatrix} 2x & -1 & 0 & 0 & \cdots \\ -1 & 2x & -1 & 0 & \cdots \\ 0 & -1 & 2x & -1 & \cdots \\ & & \ddots & & \\ \cdots & 0 & -1 & 2x & -1 \\ \cdots & 0 & 0 & -1 & 2x \end{bmatrix}.$$

Then the inequality (2) is equivalent to finding the smallest value of x such that H_n is positive (semi-definite).

Let $U_n = U_n(x)$ be the determinant of H_n. Expanding the determinant along (say) the first row, we obtain

$$U_n = 2x\, U_{n-1} - U_{n-2} \quad (n \ge 2), \tag{4}$$

where we define $U_0 = 1$.

The recurrence relation (4) is the defining relation of the Chebyshev polynomials of the second kind (see, e.g. [4]) whose solutions are

$$U_n(x) = \frac{\sin(n+1)t}{\sin t} \quad \text{where } x = \cos t. \tag{5}$$

For a fixed n, the zeros of U_n in $[-1, 1]$ are the the values

$$x_k = \cos\left(\frac{k}{n+1}\,\pi\right) \quad k = 1 \cdots n \tag{6}$$

Now for H_n to be positive, semi-definite, the determinants of H_k must all be positive, $(k = 1, \ldots, n)$. Because $\lim_{x\to\infty} U_n(x) = \infty$, this means that x cannot be smaller than the largest zero of the H_k's. From (6) this means that $x \ge \cos\left(\frac{\pi}{n+1}\right)$. So we have our main result.

Theorem 2.1. *Fix $n \geq 2$. Then*

$$\sum_{k=1}^{n-1} a_{k+1}\, a_k \leq \cos^{-1}\left(\frac{\pi}{n+1}\right) \sum_{k=1}^{n} a_k^2$$

for all real numbers $\{a_1, \ldots, a_n\}$ $(n \geq 1)$.

(Here we must obviously choose $\cos^{-1}\left(\frac{\pi}{n+1}\right) \in [0, 1]$.)

Note that the inequality is best possible since it becomes equality precisely when we choose $[a_1, \ldots, a_n]$ to be in the null space of H_n.

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