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On certain inequalities for φ, ψ, σ and related functions, II

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Abstract: We offer new proofs and refinements of two inequalities from paper [2]. The unitary functions variants are also considered.
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1 Introduction

Let $\varphi(n), \psi(n)$ and $\sigma(n)$ denote the classical arithmetic functions, representing Euler's totient, Dedekind's function, and the sum of divisors functions, respectively. Let $\varphi^*(n)$ and $\sigma^*(n)$ denote the unitary analogues of the functions φ and σ. Is is well-known that these arithmetical functions with the convention $\varphi(1) = \psi(1) = \sigma(1) = \varphi^*(1) = \sigma^*(1) = 1$ are multiplicative, and for prime powers $n = p^a$ (p prime, $a \geq 1$ integer) one has (see [4])

$$\varphi(p^a) = p^a \cdot \left(1 - \frac{1}{p}\right), \ \ \psi(p^a) = p^a \cdot \left(1 + \frac{1}{p}\right), \ \ \sigma(p^a) = \frac{p^{a+1} - 1}{p - 1} \tag{1}$$

and

$$\varphi^*(p^a) = p^a - 1, \ \ \sigma^*(p^a) = p^a + 1. \tag{2}$$

Inspired by a paper by K. T. Atanassov [1] and the author [3], in a recent note S. Dimitrov [2] has proved the following interesting inequalities:

Theorem 1. *For all $n \geq 2$ one has*

$$\varphi^2(n) + \psi^2(n) + \sigma^2(n) \geq 3n^2 + 2n + 3. \tag{3}$$

Theorem 2. *For all $n \geq 2$ one has*

$$\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \geq 3n^2 + 2n - 1. \tag{4}$$

In what follows, we shall prove the following refinements of (3) and (4), as well as a unitary analogue for each of these relations:

Theorem 3. *For all $n \geq 2$ one has*

$$\varphi^2(n) + \psi^2(n) + \sigma^2(n) \geq \varphi^2(n) + 2\psi^2(n) \geq 3n^2 + 2n + 3. \tag{5}$$

Theorem 4. *For all $n \geq 2$ one has*

$$\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \geq \psi^2(n) + 2\varphi(n)\psi(n) \geq 3n^2 + 2n - 1. \tag{6}$$

Theorem 5. *For all $n \geq 2$ one has*

$$(\varphi^*(n))^2 + (\psi(n))^2 + (\sigma^*(n))^2 \geq (\varphi^*(n))^2 + 2(\sigma^*(n))^2 \geq 3n^2 + 2n + 3. \tag{7}$$

Theorem 6. *For all $n \geq 2$ one has*

$$\varphi^*(n)\psi(n) + \varphi^*(n)\sigma^*(n) + \psi(n)\sigma^*(n) \geq 2\varphi^*(n)\sigma^*(n) + (\sigma^*(n)^2) \geq 3n^2 + 2n - 1. \tag{8}$$

2 Proofs of main results

Two of the ingredients of the proofs are the following auxiliary results:

Lemma 1. *Let $x_i > 1$ $(i = 1, 2, \ldots, n)$ be real numbers. Then*

$$\prod_{i=1}^{n}(x_i - 1)^2 + 2\prod_{i=1}^{n}(x_i + 1)^2 \geq 3\prod_{i=1}^{n} x_i^2 + 2\prod_{i=1}^{n} x_i + 3. \tag{9}$$

Lemma 2. *Let $x_i > 1$ $(i = 1, 2, \ldots, n)$ be real numbers. Then*

$$\prod_{i=1}^{n}(x_i + 1)^2 + 2\prod_{i=1}^{n}(x_i - 1)^2 \geq 3\prod_{i=1}^{n} x_i^2 + 2\prod_{i=1}^{n} x_i - 1. \tag{10}$$

Remark 1. *There is equality in* (9) *or* (10) *only for $n = 1$.*

Proof. We will prove only Lemma 1, the proof of Lemma 2 being similar. It is clear that, for $n = 1$ there is equality in (9), as $(x-1)^2 + 2(x+1)^2 = 3x^2 + 2x + 3$. Let us assume that (9) holds true for $n \geq 2$. It is easy to verify by direct computation that (9) holds true for $n = 2$, with strict inequality. Now assume that (9) is valid for $n \geq 2$, and we will prove that it holds for $n+1$, too. The inequality for $n+1$ can be written as

$$\begin{aligned} &\prod_{i=1}^{n}(x_i - 1)^2 \cdot (x_{n+1} - 1)^2 + 2 \cdot \prod_{i=1}^{n}(x_i + 1)^2 \cdot (x_{n+1} + 1)^2 \\ &\geq 3\prod_{i=1}^{n} x_i^2 \cdot x_{n+1}^2 + 2\prod_{i=1}^{n} x_i \cdot x_{n+1} + 3. \end{aligned} \tag{11}$$

The left side of (11) can be written as

$$\begin{aligned} &\prod_{i=1}^{n}(x_i - 1)^2 \cdot (x_{n+1}^2 - 2x_{n+1} + 1) + 2\prod_{i=1}^{n}(x_i + 1)^2 \cdot (x_{n+1}^2 + 2x_{n+1} + 1) \\ &= \left[\prod_{i=1}^{n}(x_i - 1)^2 + 2\prod_{i=1}^{n}(x_i + 1)^2\right] \cdot x_{n+1}^2 \\ &\quad + 2x_{n+1} \cdot \left[2\prod_{i=1}^{n}(x_i + 1)^2 - \prod_{i=1}^{n}(x_i - 1)^2\right] \\ &\quad + \prod_{i=1}^{n}(x_i - 1)^2 + 2\prod_{i=1}^{n}(x_i + 1)^2 \\ &> 3\prod_{i=1}^{n} x_i^2 \cdot x_{n+1}^2 + 2\prod_{i=1}^{n} x_i \cdot x_{n+1}^2 + 3x_{n+1}^2 \end{aligned}$$

by the induction assumption, and the trivial facts

$$\begin{aligned} &2\prod_{i=1}^{n}(x_i + 1)^2 - \prod_{i=1}^{n}(x_i - 1)^2 > 0, \\ &\prod_{I=1}^{n}(x_i - 1)^2 + 2\prod_{i=1}^{n}(x_i - 1)^2 > 0. \end{aligned}$$

□

Now, by $x_{n+1} > 1$ we get that (9) holds true also for $n+1$ in place of n. By mathematical induction, inequality (9) follows for all $n \geq 1$, with equality only for $n = 1$.

Lemma 3. *If* p_i $(i = 1, 2, \ldots, n)$ *are distinct primes and* a_i $(i = 1, 2, \ldots, n) \geq 1$ *integers, then*

$$\frac{2}{p_1 \cdots p_r} + \frac{3}{p_1^2 \cdots p_r^2} \geq \frac{2}{p_1^{a_1} \cdots p_r^{a_r}} + \frac{3}{p_1^{2a_1} \ldots p_t^{2a_r}} \tag{12}$$

$$\frac{2}{p_1 \cdots p_r} - \frac{1}{p_1^2 \cdots p_r^2} \geq \frac{2}{p_1^{a_1} \cdots p_r^{a_r}} - \frac{1}{p_1^{2a_1} \ldots p_t^{2a_r}}. \tag{13}$$

Proof. (12) follows by $p_1^{a_1} \cdots p_r^{a_r} \geq p_1 \cdots p_r$ and $p_1^{2a_1} \cdots p_r^{2a_r} \geq p_1^2 \cdots p_r^2$. For the proof of (13), write the inequality in the form

$$\frac{2}{p_1 \cdots p_r} - \frac{2}{p_1^{a_1} \cdots p_r^{a_r}} \geq \frac{1}{p_1^2 \cdots p_r^2} - \frac{1}{p_1^{2a_1} \cdots p_r^{2a_r}},$$

or

$$\frac{2\cdot\left(p_1^{a_1-1}\cdots p_r^{a_r-1}-1\right)}{p_1^{a_1}\cdots p_r^{a_r}}\geq\frac{p_1^{2a_1-2}\cdots p_r^{2a_r-2}-1}{p_1^{2a_1}\cdots p_r^{2a_r}}.$$

Reducing with $p_1^{a_1-1}\cdots p_r^{a_1-1}-1$, (when it is not zero) and $p_1^{a_1}\cdots p_r^{a_r}$, we get

$$2p_1^{a_1}\cdots p_r^{a_r}\geq p_1^{a_1-1}\cdots p_r^{a_r-1}+1,$$

or

$$p_1^{a_1-1}\cdots p_r^{a_r-1}\cdot(2p_1\cdots p_r-1)\geq 1,$$

which is true, or $p_i^{a_i-1}\geq 1$ and $2p_1\cdots p_r-1>1$. This proves (12). □

Proofs of Theorems 5 and 6

First remark that relation (7) of Theorem 5 follows by the known inequality $\psi(n)\geq\sigma^*(n)$ (see e.g. [3]). In a similar manner, the first inequality of (8) of Theorem 6 follows by the same inequality. Now, for the proof of second inequality of (7), apply Lemma 1 for $x_i=p_i^{a_i}$ $(i=1,2,\ldots,r)$, where $n=\prod_{i=1}^r p_i^{a_i}$ is the prime factorization of n.

Applying (9) (with "r" in place of "n") and using relation (2), we get the desired inequality of (7). In the same manner, the second inequality of (8) of Theorem 6 follows by Lemma 2, as $(p_i^{a_i}+1)^2=(\sigma^*(p_i^{a_i}))^2$ and $\varphi^*(p_i^{a_i})\sigma^*(p_i^{a_i})=p_i^{2a_i}-1$ by relation (2). □

Proofs of Theorem 3 and 4

The first inequality of relation (5) of Theorem 3 follows by $\sigma(n)\geq\psi(n)$. For the proof of second inequality, remark that this inequality can be written as

$$\left(\frac{\varphi(n)}{n}\right)^2+2\cdot\left(\frac{\psi(n)}{n}\right)^2\geq 3+\frac{2}{n}+\frac{3}{n^2}. \tag{14}$$

Remark that if $n=\prod_{i=1}^r p_i^{a_i}$ is the prime factorization on n, then

$$\frac{\varphi(n)}{n}=\prod_{i=1}^r\frac{\varphi(p_i)}{p_i}\quad\text{and}\quad\frac{\psi(n)}{n}=\prod_{i=1}^r\frac{\psi(p_i)}{p_i}.$$

By relation (12) of Lemma 3, it will be sufficient to prove inequality (14) for $n=p_1p_2\cdots p_r$. Applying now Lemma 1 for $x_i=p_i$ $(i=1,2,\ldots,r)$ and "r" in place of "n", inequality (14) follows. This proves Theorem 3.

The proof of Theorem (4) is similar, by application of inequality (13) of Lemma 3, and using the same methods as in the case of Theorem 3.

As there is equality in Lemma 1 (and Lemma 2) only for $n=1$, and there is equality in Lemma 3 only for $a_1=\cdots=a_r=1$, we get that there is Theorem 3 and 4 only form $n=$ prime, while in case of Theorems 5 and 6 again only for $n=$ prime (as $\psi(n)=\sigma^*(n)$ only for $n=$ squarefree, and by taking into account Lemma 2). □

References

[1] Atanassov, K. (2013). Note on φ, ψ and σ-functions. Part 6. *Notes on Number Theory and Discrete Mathematics*, 19(1), 22–24.

[2] Dimitrov, S. (2023). Lower bounds on expressions dependent on functions $\varphi(n)$, $\psi(n)$ and $\sigma(n)$. *Notes on Number Theory and Discrete Mathematics*, 29(4), 713–716.

[3] Sándor, J. (2014). On certain inequalities for σ, φ, ψ and related functions. *Notes on Number Theory and Discrete Mathematics*, 20(2), 52–60.

[4] Sándor, J., Mitrinović, D. S., & Crstici, B. (2006). *Handbook of Number Theory I*. Springer.