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Lower bounds on expressions dependent on functions $\varphi(n)$, $\psi(n)$ and $\sigma(n)$, II

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Abstract: In this paper we establish lower bounds on several expressions dependent on functions $\varphi(n)$, $\psi(n)$ and $\sigma(n)$.
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1 Notations and formulas

The letter p with or without subscript will always denote prime number. Let $n > 1$ be a positive integer with prime factorization

$$n = p_1^{a_1} \cdots p_k^{a_k} .$$

The function $\Omega(n)$ counts the total number of prime factors of n honoring their multiplicity. We have

$$\Omega(n) = \sum_{i=1}^{k} a_i \quad \text{and} \quad \Omega(1) = 0 .$$

We denote by $\varphi(n)$ the Euler totient function which is defined as the number of positive integers not greater than n that are coprime to n. We have

$$\varphi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i - 1) \quad \text{and} \quad \varphi(1) = 1\,.$$

We define the Dedekind function $\psi(n)$ by the formula

$$\psi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i + 1) \quad \text{and} \quad \psi(1) = 1\,.$$

The function $\sigma(n)$ denotes the sum of the positive divisors of n. We have

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1} \quad \text{and} \quad \sigma(1) = 1\,.$$

2 Introduction and statement of the results

In 2013 Atanassov [2] proved that for every natural number $n \geq 2$ the lower bound

$$\varphi(n)\psi(n)\sigma(n) \geq n^3 + n^2 - n - 1$$

holds. Afterwards Sándor [4] improved Atanassov's result proving that for all $n \geq 1$ one has the inequalities

$$\varphi(n)\psi(n)\sigma(n) \geq \varphi^*(n)(\sigma^*(n))^2 \geq n^3 + n^2 - n - 1\,,$$

where $\psi^*(n)$ and $\sigma^*(n)$ are the unitary analogues of the functions $\psi(n)$ and $\sigma(n)$. We refer to [6, 7] for definitions, properties and references. Very recently the author [3] showed that

$$\varphi^2(n) + \psi^2(n) + \sigma^2(n) \geq 3n^2 + 2n + 3\,,$$
$$\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \geq 3n^2 + 2n - 1$$

for every natural number $n \geq 2$. As a continuation of these studies, we establish the following five theorems.

Theorem 1. *For every natural number $n \geq 2$ the lower bound*

$$\varphi^3(n) + \psi^3(n) + \sigma^3(n) \geq 3n^3 + 3n^2 + 9n + 1 \tag{1}$$

holds.

Theorem 2. *For every natural number $n \geq 2$ the lower bound*

$$\varphi^4(n) + \psi^4(n) + \sigma^4(n) \geq 3n^4 + 4n^3 + 18n^2 + 4n + 3 \tag{2}$$

holds.

Theorem 3. *For every natural number $n \geq 2$ the lower bound*

$$\varphi^2(n)\psi^2(n) + \varphi^2(n)\sigma^2(n) + \sigma^2(n)\psi^2(n) \geq 3n^4 + 4n^3 + 2n^2 + 4n + 3 \tag{3}$$

holds.

Theorem 4. *For every natural number $n \geq 2$ the lower bound*

$$\varphi^2(n)\big(\psi(n)+\sigma(n)\big)+\psi^2(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^2(n)\big(\varphi(n)+\psi(n)\big)\geq 6n^3+6n^2+2n+2 \quad (4)$$

holds.

Theorem 5. *For every natural number $n \geq 2$ the lower bound*

$$\varphi^3(n)\big(\psi(n)+\sigma(n)\big)+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)\geq 6n^4+8n^3+12n^2+8n-2 \quad (5)$$

holds.

3 Lemmas

Lemma 1. *For every natural number $n \geq 2$ the lower bound*

$$\varphi(n)+\psi(n)\geq 2n$$

holds.

Proof. See [1], Lemma 1. □

Lemma 2. *For every natural number $n \geq 2$ the lower bound*

$$\varphi(n)+\sigma(n)\geq 2n$$

holds.

Proof. See [5], Remark 5. □

4 Proof of Theorem 1

Consider several cases.

Case 1. $\Omega(n)=1$. Bearing in mind that n is a prime number we write

$$\varphi^3(n)+\psi^3(n)+\sigma^3(n)=(n-1)^3+2(n+1)^3=3n^3+3n^2+9n+1\,.$$

Case 2. $\Omega(n)=2$, $n=pq$, where p and q are distinct primes. Then

$$\begin{aligned}
&\varphi^3(n)+\psi^3(n)+\sigma^3(n)\\
&=(p-1)^3(q-1)^3+2(p+1)^3(q+1)^3\\
&=3p^3q^3+27p^2q^2+27pq+3+3p^3q^2+3p^2q^3+9p^3q+9pq^3\\
&\quad+p^3+9p^2q+9pq^2+q^3+9p^2+9q^2+3p+3q\\
&>3p^3q^3+3p^2q^2+9pq+1\\
&=3n^3+3n^2+9n+1\,.
\end{aligned}$$

Case 3. $\Omega(n)=2$, $n=p^2$, where p is a prime. Then

$$\begin{aligned}
\varphi^3(n)+\psi^3(n)+\sigma^3(n)&=p^3(p-1)^3+p^3(p+1)^3+(p^2+p+1)^3\\
&=3p^6+3p^5+12p^4+7p^3+6p^2+3p+1\\
&>3p^6+3p^4+9p^2+1\\
&=3n^3+3n^2+9n+1\,.
\end{aligned}$$

Now we assume that (1) is true for every natural number n with $\Omega(n) = m$ for some natural number $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

Case A. $p \nmid n$. Using that

$$\varphi(n) < n, \tag{6}$$

we obtain

$$\begin{aligned}
\varphi^3(np) + \psi^3(np) + \sigma^3(np) &= \varphi^3(n)(p-1)^3 + \psi^3(n)(p+1)^3 + \sigma^3(n)(p+1)^3 \\
&= (p+1)^3\big[\varphi^3(n) + \psi^3(n) + \sigma^3(n)\big] - (6p^2+2)\varphi^3(n) \\
&\geq (p+1)^3(3n^3 + 3n^2 + 9n + 1) - (6p^2+2)n^3 \\
&= 3n^3p^3 + 9n^2p^2 + 27np + 1 + p^3(3n^2 + 9n + 1) \\
&\quad + 3p^2(n^3 + 9n + 1) + 3p(3n^3 + 3n^2 + 1) + n^3 + 3n^2 + 9n \\
&> 3n^3p^3 + 3n^2p^2 + 9np + 1\,.
\end{aligned}$$

Case B. $p \,|\, n$. Using that

$$\varphi(np) = p\varphi(n)\,, \quad \psi(np) = p\psi(n)\,, \quad \sigma(np) > p\sigma(n), \tag{7}$$

we get

$$\begin{aligned}
\varphi^3(np) + \psi^3(np) + \sigma^3(np) &> p^3\big[\varphi^3(n) + \psi^3(n) + \sigma^3(n)\big] \\
&\geq p^3(3n^3 + 3n^2 + 9n + 1) \\
&= 3n^3p^3 + 3n^2p^2 + 9np + 3n^2p^2(p-1) + 9np(p^2-1) + p^3 \\
&> 3n^3p^3 + 3n^2p^2 + 9np + 1\,.
\end{aligned}$$

This completes the proof of Theorem 1. □

5 Proof of Theorem 2

Consider several cases.

Case 1. $\Omega(n) = 1$. Taking into account that n is a prime number we have

$$\varphi^4(n) + \psi^4(n) + \sigma^4(n) = (n-1)^4 + 2(n+1)^4 = 3n^4 + 4n^3 + 18n^2 + 4n + 3\,.$$

Case 2. $\Omega(n) = 2$, $n = pq$, where p and q are distinct primes. Then

$$\begin{aligned}
&\varphi^4(n) + \psi^4(n) + \sigma^4(n) \\
&= (p-1)^4(q-1)^4 + 2(p+1)^4(q+1)^4 \\
&= 3p^4q^4 + 48p^3q^3 + 108p^2q^2 + 48pq + 3 + 4p^4q^3 + 4p^3q^4 + 18p^4q^2 + 18p^2q^4 \\
&\quad + 4p^4q + 24p^3q^2 + 24p^2q^3 + 4pq^4 + 3p^4 + 48p^3q + 48pq^3 + 3q^4 \\
&\quad + 4p^3 + 24p^2q + 24pq^2 + 4q^3 + 18p^2 + 18q^2 + 4p + 4q \\
&> 3p^4q^4 + 4p^3q^3 + 18p^2q^2 + 4pq + 3 \\
&= 3n^4 + 4n^3 + 18n^2 + 4n + 3\,.
\end{aligned}$$

Case 3. $\Omega(n) = 2$, $n = p^2$, where p is a prime. Then

$$\begin{aligned}\varphi^4(n) + \psi^4(n) + \sigma^4(n) &= p^4(p-1)^4 + p^4(p+1)^4 + (p^2+p+1)^4 \\ &= 3p^8 + 4p^7 + 22p^6 + 16p^5 + 21p^4 + 16p^3 + 10p^2 + 4p + 1 \\ &> 3p^8 + 4p^6 + 18p^4 + 4p^2 + 3 \\ &= 3n^4 + 4n^3 + 18n^2 + 4n + 3\,.\end{aligned}$$

Now we assume that (2) is true for every natural number n with $\Omega(n) = m$ for some natural number $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

Case A. $p \nmid n$. Using (6), we find

$$\begin{aligned}\varphi^4(np) + \psi^4(np) + \sigma^4(np) &= \varphi^4(n)(p-1)^4 + \psi^4(n)(p+1)^4 + \sigma^4(n)(p+1)^4 \\ &= (p+1)^4\big[\varphi^4(n) + \psi^4(n) + \sigma^4(n)\big] - (8p^3+8p)\varphi^4 n) \\ &\geq (p+1)^4(3n^4 + 4n^3 + 18n^2 + 4n + 3) - (8p^3+8p)n^4 \\ &= 3n^4p^4 + 16n^3p^3 + 108n^2p^2 + 16np + 3 + 4n^4p^3 + 4n^3p^4 \\ &\quad + 18n^4p^2 + 18n^2p^4 + 4n^4p + 24n^3p^2 + 72n^2p^3 + 4np^4 \\ &\quad + 3n^4 + 16n^3p + 16np^3 + 3p^4 + 4n^3 + 72n^2p + 24np^2 + 12p^3 \\ &\quad + 18n^2 + 18p^2 + 4n + 12p \\ &> 3n^4p^4 + 4n^3p^3 + 18n^2p^2 + 4np + 3\,.\end{aligned}$$

Case B. $p \,|\, n$. By (7) we obtain

$$\begin{aligned}\varphi^4(np) + \psi^4(np) + \sigma^4(np) &> p^4\big[\varphi^4(n) + \psi^4(n) + \sigma^4(n)\big] \\ &\geq p^4(3n^4 + 4n^3 + 18n^2 + 4n + 3) \\ &= 3n^4p^4 + 4n^3p^3 + 18n^2p^2 + 4np + 3 \\ &\quad + 4n^3p^3(p-1) + 18n^2p^2(p^2-1) + 4np(p^3-1) + 3(p^4-1) \\ &> 3n^4p^4 + 4n^3p^3 + 18n^2p^2 + 4np + 3\,.\end{aligned}$$

This completes the proof of Theorem 2. □

6 Proof of Theorem 3

Consider several cases.

Case 1. $\Omega(n) = 1$. Having in mind that n is a prime number we deduce

$$\varphi^2(n)\psi^2(n) + \varphi^2(n)\sigma^2(n) + \sigma^2(n)\psi^2(n) = 2(n^2-1)^2 + (n+1)^4 = 3n^4 + 4n^3 + 2n^2 + 4n + 3\,.$$

Case 2. $\Omega(n) = 2$, $n = pq$, where p and q are distinct primes. Then

$$\begin{aligned}
&\varphi^2(n)\psi^2(n) + \varphi^2(n)\sigma^2(n) + \sigma^2(n)\psi^2(n) \\
&= 2(p^2-1)^2(q^2-1)^2 + (p+1)^4(q+1)^4 \\
&= 3p^4q^4 + 16p^3q^3 + 44p^2q^2 + 16pq + 3 + 4p^4q^3 + 4p^3q^4 + 2p^4q^2 + 2p^2q^4 \\
&\quad + 4p^4q + 24p^3q^2 + 24p^2q^3 + 4pq^4 + 3p^4 + 16p^3q + 16pq^3 + 3q^4 \\
&\quad + 4p^3 + 24p^2q + 24pq^2 + 4q^3 + 2p^2 + 2q^2 + 4p + 4q \\
&> 3p^4q^4 + 4p^3q^3 + 2p^2q^2 + 4pq + 3 \\
&= 3n^4 + 4n^3 + 2n^2 + 4n + 3\,.
\end{aligned}$$

Case 3. $\Omega(n) = 2$, $n = p^2$, where p is a prime. Then

$$\begin{aligned}
\varphi^2(n)\psi^2(n) + \varphi^2(n)\sigma^2(n) + \sigma^2(n)\psi^2(n) &= p^4(p^2-1)^2 + p^2(p^3-1)^2 + (p^2+p)^2(p^2+p+1)^2 \\
&= 3p^8 + 4p^7 + 6p^6 + 8p^5 + 9p^4 + 4p^3 + 2p^2 \\
&> 3p^8 + 4p^6 + 2p^4 + 4p^2 + 3 \\
&> 3n^4 + 4n^3 + 2n^2 + 4n + 3\,.
\end{aligned}$$

Let us assume that (3) is true for every natural number n with $\Omega(n) = m$ for some natural number $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

Case A. $p \nmid n$. Using that

$$\psi(n) \geq n+1\,, \quad \sigma(n) \geq n+1, \tag{8}$$

we derive

$$\begin{aligned}
&\varphi^2(np)\psi^2(np) + \varphi^2(np)\sigma^2(np) + \sigma^2(np)\psi^2(np) \\
&= \varphi^2(n)\psi(n)^2(p^2-1)^2 + \varphi^2(n)\sigma^2(n)(p^2-1)^2 + \psi^2(n)\sigma^2(n)(p+1)^4 \\
&= (p^2-1)^2\big[\varphi^2(n)\psi^2(n) + \varphi^2(n)\sigma^2(n) + \sigma^2(n)\psi^2(n)\big] + 4p(p+1)^2\psi(n)\sigma(n) \\
&\geq (p^2-1)^2(3n^4 + 4n^3 + 2n^2 + 4n + 3) + 4p(p+1)^2(n+1)^4 \\
&= 3n^4p^4 + 16n^3p^3 + 44n^2p^2 + 16np + 3 + 4n^4p^3 + 4n^3p^4 + 2n^4p^2 + 2n^2p^4 \\
&\quad + 4n^4p + 24n^3p^2 + 24n^2p^3 + 4np^4 + 3n^4 + 16n^3p + 16np^3 + 3p^4 \\
&\quad + 4n^3 + 24n^2p + 24np^2 + 4p^3 + 2n^2 + 2p^2 + 4n + 4p \\
&> 3n^4p^4 + 4n^3p^3 + 2n^2p^2 + 4np + 3\,.
\end{aligned}$$

Case B. $p \mid n$. From (7) we establish

$$\begin{aligned}
&\varphi^2(np)\psi^2(np) + \varphi^2(np)\sigma^2(np) + \sigma^2(np)\psi^2(np) \\
&> p^4\big[\varphi^2(n)\psi^2(n) + \varphi^2(n)\sigma^2(n) + \sigma^2(n)\psi^2(n)\big] \\
&\geq p^4(3n^4 + 4n^3 + 2n^2 + 4n + 3) \\
&= 3n^4p^4 + 4n^3p^3 + 2n^2p^2 + 4np + 3 + 4n^3p^3(p-1) \\
&\quad + 2n^2p^2(p^2-1) + 4np(p^3-1) + 3(p^4-1) \\
&> 3n^4p^4 + 4n^3p^3 + 2n^2p^2 + 4np + 3\,.
\end{aligned}$$

This completes the proof of Theorem 3. □

7 Proof of Theorem 4

Consider several cases.

Case 1. $\Omega(n) = 1$. Bearing in mind that n is a prime number we write

$$\begin{aligned}
&\varphi^2(n)\big(\psi(n)+\sigma(n)\big)+\psi^2(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^2(n)\big(\varphi(n)+\psi(n)\big)\\
&=2(n-1)^2(n+1)+2(n+1)^2(n-1)+2(n+1)^3\\
&=6n^3+6n^2+2n+2\,.
\end{aligned}$$

Case 2. $\Omega(n) = 2$, $n = pq$, where p and q are distinct primes. Then

$$\begin{aligned}
&\varphi^2(n)\big(\psi(n)+\sigma(n)\big)+\psi^2(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^2(n)\big(\varphi(n)+\psi(n)\big)\\
&=2(p-1)^2(q-1)^2(p+1)(q+1)+2(p+1)^2(q+1)^2(p-1)(q-1)+2(p+1)^3(q+1)^3\\
&=6p^3q^3+22p^2q^2+22pq+6+6p^3q^2+6p^2q^3+2p^3q+2pq^3\\
&\quad+2p^3+18p^2q+18pq^2+2q^3+2p^2+2q^2+6p+6q\\
&>6p^3q^3+6p^2q^2+2pq+2\\
&=6n^3+6n^2+2n+2\,.
\end{aligned}$$

Case 3. $\Omega(n) = 2$, $n = p^2$, where p is a prime. Then

$$\begin{aligned}
&\varphi^2(n)\big(\psi(n)+\sigma(n)\big)+\psi^2(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^2(n)\big(\varphi(n)+\psi(n)\big)\\
&=p^3(p-1)^2(p+1)+p^3(p+1)^2(p-1)+p^2(p-1)^2(p^2+p+1)\\
&\quad+p^2(p+1)^2(p^2+p+1)+p(p-1)(p^2+p+1)^2+p(p+1)(p^2+p+1)^2\\
&=6p^6+6p^5+8p^4+6p^3+4p^2\\
&>6p^6+6p^4+2p^2+2\\
&=6n^3+6n^2+2n+2\,.
\end{aligned}$$

Now we assume that (4) is true for every natural number n with $\Omega(n) = m$ for some natural number $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

Case A. $p \nmid n$. Now (8), Lemma 1 and Lemma 2 imply

$$\begin{aligned}
&\varphi^2(np)\big(\psi(np)+\sigma(np)\big)+\psi^2(np)\big(\varphi(np)+\sigma(np)\big)+\sigma^2(np)\big(\varphi(np)+\psi(np)\big)\\
&=\varphi^2(n)\psi(n)(p-1)^2(p+1)+\varphi^2(n)\sigma(n)(p-1)^2(p+1)+\psi^2(n)\varphi(n)(p+1)^2(p-1)\\
&\quad+\psi^2(n)\sigma(n)(p+1)^3+\sigma^2(n)\varphi(n)(p+1)^2(p-1)+\sigma^2(n)\psi(n)(p+1)^3\\
&=(p-1)^2(p+1)\big[\varphi^2(n)\big(\psi(n)+\sigma(n)\big)+\psi^2(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^2(n)\big(\varphi(n)+\psi(n)\big)\big]\\
&\quad+(2p^2-2)\big[\psi^2(n)\varphi(n)+\sigma^2(n)\varphi(n)\big]+(4p^2+4p)\big[\psi^2(n)\sigma(n)+\sigma^2(n)\psi(n)\big]\\
&\geq(p-1)^2(p+1)(6n^3+6n^2+2n+2)+(2p^2+4p+2)\big[\psi^2(n)\sigma(n)+\sigma^2(n)\psi(n)\big]\\
&\quad+(2p^2-2)\big[\psi^2(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^2(n)\big(\varphi(n)+\psi(n)\big)\big]\\
&\geq(6n^3+6n^2+2n+2)(p-1)^2(p+1)+2(n+1)^3(2p^2+4p+2)+4n(n+1)^2(2p^2-2)\\
&=6n^3p^3+22n^2p^2+22np+6+6n^3p^2+6n^2p^3+2n^3p+2np^3+2n^3+18n^2p+18np^2\\
&\quad+2p^3+2n^2+2p^2+6n+6p\\
&>6n^3p^3+6n^2p^2+2np+2\,.
\end{aligned}$$

Case B. $p \mid n$. Using (7) we obtain

$$\begin{aligned}
&\varphi^2(np)\big(\psi(np)+\sigma(np)\big)+\psi^2(np)\big(\varphi(np)+\sigma(np)\big)+\sigma^2(np)\big(\varphi(np)+\psi(np)\big)\\
&> p^3\big[\varphi^2(n)\big(\psi(n)+\sigma(n)\big)+\psi^2(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^2(n)\big(\varphi(n)+\psi(n)\big)\big]\\
&\geq p^3(6n^3+6n^2+2n+2)\\
&= 6n^3p^3+6n^2p^2+2np+2+6n^2p^2(p-1)+2np(p^2-1)+2(p^3-1)\\
&> 6n^3p^3+6n^2p^2+2np+2\,.
\end{aligned}$$

This completes the proof of Theorem 4. □

8 Proof of Theorem 5

Consider several cases.

Case 1. $\Omega(n)=1$. Bearing in mind that n is a prime number, we write

$$\begin{aligned}
&\varphi^3(n)\big(\psi(n)+\sigma(n)\big)+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)\\
&= 2(n-1)^3(n+1)+2(n+1)^3(n-1)+2(n+1)^4\\
&= 6n^4+8n^3+12n^2+8n-2\,.
\end{aligned}$$

Case 2. $\Omega(n)=2$, $n=pq$, where p and q are distinct primes. Then

$$\begin{aligned}
&\varphi^3(n)\big(\psi(n)+\sigma(n)\big)+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)\\
&= 2(p-1)^3(q-1)^3(p+1)(q+1)+2(p+1)^3(q+1)^3(p-1)(q-1)+2(p+1)^4(q+1)^4\\
&= 6p^4q^4+48p^3q^3+72p^2q^2+48pq+6+8p^4q^3+8p^3q^4+12p^4q^2+12p^2q^4\\
&\quad +p^4(8q-2)+48p^3q^2+48p^2q^3+q^4(8p-2)+16p^3q+16pq^3+8p^3+48p^2q+48pq^2\\
&\quad +8q^3+12p^2+12q^2+8p+8q\\
&> 6p^4q^4+8p^3q^3+12p^2q^2+8pq-2\\
&= 6n^4+8n^3+12n^2+8n-2\,.
\end{aligned}$$

Case 3. $\Omega(n)=2$, $n=p^2$, where p is a prime. Then

$$\begin{aligned}
&\varphi^3(n)\big(\psi(n)+\sigma(n)\big)+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)\\
&= p^4(p-1)^3(p+1)+p^4(p+1)^3(p-1)+p^3(p-1)^3(p^2+p+1)\\
&\quad +p^3(p+1)^3(p^2+p+1)+p(p-1)(p^2+p+1)^3+p(p+1)(p^2+p+1)^3\\
&= 6p^8+8p^7+20p^6+20p^5+16p^4+6p^3+2p^2\\
&> 6p^8+8p^6+12p^4+8p^2-2\\
&= 6n^4+8n^3+12n^2+8n-2\,.
\end{aligned}$$

Now we assume that (5) is true for every natural number n with $\Omega(n)=m$ for some natural number $m\geq 2$. Let p be a prime number. Then $\Omega(np)=\Omega(n)+1$.

Case A. $p \nmid n$. Now (8), Lemma 1 and Lemma 2 yield

$$
\begin{aligned}
&\varphi^3(np)\big(\psi(np)+\sigma(np)\big)+\psi^3(np)\big(\varphi(np)+\sigma(np)\big)+\sigma^3(np)\big(\varphi(np)+\psi(np)\big)\\
&=\varphi^3(n)\psi(n)(p-1)^3(p+1)+\varphi^3(n)\sigma(n)(p-1)^3(p+1)+\psi^3(n)\varphi(n)(p+1)^3(p-1)\\
&\quad+\psi^3(n)\sigma(n)(p+1)^4+\sigma^3(n)\varphi(n)(p+1)^3(p-1)+\sigma^3(n)\psi(n)(p+1)^4\\
&=(p-1)^3(p+1)\big[\varphi^3(n)\big(\psi(n)+\sigma(n)\big)+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)\big]\\
&\quad+(4p^3-4p)\big[\psi^3(n)\varphi(n)+\sigma^3(n)\varphi(n)\big]+(6p^3+6p^2+2p+2)\big[\psi^3(n)\sigma(n)+\sigma^3(n)\psi(n)\big]\\
&\geq(p-1)^3(p+1)(6n^4+8n^3+12n^2+8n-2)\\
&\quad+(4p^3-4p)\big[\psi^3(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)\big]\\
&\quad+(2p^3+6p^2+6p+2)\big[\psi^3(n)\sigma(n)+\sigma^3(n)\psi(n)\big]\\
&\geq(6n^4+8n^3+12n^2+8n-2)(p-1)^3(p+1)\\
&\quad+4n(n+1)^3(4p^3-4p)+2(n+1)^4(2p^3+6p^2+6p+2)\\
&=6n^4p^4+48n^3p^3+72n^2p^2+48np+6+n^4(8p^3-2)+8n^3p^4+12n^4p^2\\
&\quad+p^4(12n^2-2)+8n^4p+48n^3p^2+48n^2p^3+8np^4+16n^3p+16np^3+8n^3+48n^2p\\
&\quad+48np^2+8p^3+12n^2+12p^2+8n+8p\\
&>6n^4p^4+8n^3p^3+12n^2p^2+8np-2\,.
\end{aligned}
$$

Case B. $p \mid n$. Using (7), we deduce

$$
\begin{aligned}
&\varphi^3(np)\big(\psi(np)+\sigma(np)\big)+\psi^3(np)\big(\varphi(np)+\sigma(np)\big)+\sigma^3(np)\big(\varphi(np)+\psi(np)\big)\\
&>p^4\big[\varphi^3(n)\big(\psi(n)+\sigma(n)\big)+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)\big]\\
&\geq p^4(6n^4+8n^3+12n^2+8n-2)\\
&=6n^4p^4+8n^3p^3+12n^2p^2+8np+8n^3p^3(p-1)\\
&\quad+12n^2p^2(p^2-1)+4np(p^3-2)+2p^4(2n-1)\\
&>6n^4p^4+8n^3p^3+12n^2p^2+8np-2\,.
\end{aligned}
$$

This completes the proof of Theorem 5. □

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