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On positive sequences of reals whose block sequence has an asymptotic distribution function

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Abstract: In this paper we study the properties of the unbounded sequence $0 < y_1 \leq y_2 \leq y_3 \leq \cdots$ of positive reals having asymptotic distribution function of the form x^λ. As a consequence, we immediately get information on the asymptotic behavior of the power means of order $r > 0$ of function values of some arithmetic functions, e.g., the first n prime numbers or the values of the prime counting function.
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1 Introduction

Let $0 < y_1 \leq y_2 \leq y_3 \leq \cdots$ be an unbounded sequence of positive real numbers. This sequence we shortly denote by Y. The following sequence y_m/y_n, $n = 1, 2, \ldots, m = 1, 2, \ldots, n$ is called *the ratio block sequence* of the sequence Y. It is formed by the blocks $Y_1, Y_2, \dots, Y_n, \dots$, where

$$Y_n = \left(\frac{y_1}{y_n}, \frac{y_2}{y_n}, \dots, \frac{y_n}{y_n}\right), \quad n = 1, 2, \dots$$

is called the *n-th block*. This kind of block sequences derived from strictly increasing sequence of positive integers was introduced by O. Strauch and J. T. Tóth [14] and they studied the set of their distribution functions. For further results on this topic, see [2], Chapter 1.8.23 in [13], expository paper [12] and the references therein.

If one allows the Y sequence to not exclusively consist of integers, a significant portion of the previously known statements will remain unchanged or only slightly modified. We will show that a substantial portion of the results on the asymptotic behaviour of arithmetic mean (geometric mean, power mean) of the sequence Y are strongly associated with the property that the distribution function of the corresponding block sequence is of the form x^λ.

We note that in this paper we employ the notations Y and Y_n for the sequences under examination, as opposed to the previous papers where X and X_n were used. This distinction is made to illustrate that the scenario is not the same when we focus exclusively on sequences of positive integers.

2 Definitions

- If r is a non-zero real number, and $y_1, y_2, \dots, y_n$, are positive real numbers, then the generalized mean or power mean with exponent r of these positive real numbers is

$$M_r(y_1, y_2, \dots, y_n) = \left(\frac{1}{n}\sum_{i=1}^{n} y_i^r\right)^{\frac{1}{r}}.$$

  For $r = 0$ we set it equal to the geometric mean (which is the limit of means with exponents approaching zero).

  The following basic definitions are from the paper of O. Strauch [12].

- For each $n \in \mathbf{N}$ consider the *step distribution function*

$$F(Y_n, x) = \frac{\#\{i \le n; \frac{y_i}{y_n} < x\}}{n},$$

  of the sequence $0 < y_1 \le y_2 \le y_3 \le \cdots$ for $x \in [0, 1)$, and for $x = 1$, we define $F(Y_n, 1) = 1$.

  Using

$$\frac{y_i}{y_m} < x \iff \frac{y_i}{y_n} < x\frac{y_m}{y_n}$$

  from the definition above of $F(Y_n, x)$, it directly follows that

$$F(Y_m, x) = \frac{n}{m}F\left(Y_n, x\frac{y_m}{y_n}\right) \le \frac{n}{m}F(Y_n, x) \tag{1}$$

  for every $m \le n$ and $x \in [0, 1]$.

- A non-decreasing function $g : [0,1] \to [0,1]$, $g(0) = 0$, $g(1) = 1$ is called *distribution function*. We shall identify any two distribution functions coinciding at common points of continuity.

- A distribution function $g(x)$ is a distribution function of the sequence of blocks Y_n, $n = 1, 2, \dots$, if there exists an increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that

$$\lim_{k\to\infty} F(Y_{n_k}, x) = g(x)$$

  almost everywhere on $[0,1]$. This is equivalent to the weak convergence, i.e., the preceding limit holds for every point $x \in [0,1]$ of continuity of $g(x)$.

- Denote by $G(Y_n)$ the set of all distribution functions of Y_n, $n = 1, 2, \dots$. If $G(Y_n) = \{g(x)\}$ is a singleton, the distribution function $g(x)$ is also called the *asymptotic distribution function* of Y_n.
  Specifically, if $G(Y_n) = \{x\}$, then we say that the sequence of blocks Y_n is uniformly distributed in $[0,1]$.

- We will use the one-step distribution function $c_\alpha(x)$ with the step at $\alpha \in [0,1]$ defined on $[0,1]$ via $c_\alpha(1) = 1$ and for $x < 1$

$$c_\alpha(x) = \begin{cases} 0, & \text{if } x \le \alpha \\ 1, & \text{if } x > \alpha \end{cases}.$$

  In particular, we always have $c_\alpha(0) = 0$.

3 Results

The following theorem determines which distribution functions can be considered as singletons.

Theorem 3.1. *Assume that* $G(Y_n) = \{g\}$. *Then for* $x \in [0,1]$ *either*

(i) $g(x) = c_0(x)$ *or*

(ii) $g(x) = c_1(x)$ *or*

(iii) $g(x) = x^\lambda$ *for some* $\lambda > 0$.

The proof is identical to the proof of Theorem 8.2 in [14], with the exception that when considering the distribution functions of block sequences of positive real numbers, we must also take into account the distribution function $c_1(x)$. We mention as an example the case $y_n = \ln(n+1)$, $n = 1, 2, \dots$ which has singleton $c_1(x)$ and the case $y_n = \sqrt{n}$, $n = 1, 2, \dots$ with $G(Y_n) = x^2$.

In the next two theorems, we give the necessary and sufficient conditions for a block sequence of the sequence Y to have an asymptotic distribution function of the form x^λ.

Theorem 3.2. *Let $\lambda > 0$ be a real number and $0 < y_1 \le y_2 \le y_3 \le \cdots$ be a sequence of positive real numbers. The necessary and sufficient condition for $G(Y_n) = \{x^\lambda\}$ is that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{y_n} = \frac{\lambda}{\lambda + 1}. \tag{2}$$

The proof is the same as for Theorem 1 in [2].

We give an alternative proof of the following Theorem of Filip and Tóth [3].

Theorem 3.3. *Let $\lambda > 0$ be a real number and $0 < y_1 \le y_2 \le y_3 \le \cdots$ be a sequence of positive real numbers. The necessary and sufficient condition for $G(Y_n) = \{x^\lambda\}$ is that for every positive integer k the following limit holds*

$$\lim_{n\to\infty} \frac{y_{kn}}{y_n} = k^{\frac{1}{\lambda}}. \tag{3}$$

Proof. First, we show that the condition (3) is necessary for the block sequence to have an asymptotic distribution function of the form x^λ. We prove this part by contradiction. To the contrary, we suppose that $G(Y_n) = \{x^\lambda\}$ for some $\lambda > 0$ and the limit (3) does not hold for some k. Then there are two possibilities:

1) $\liminf_{n\to\infty} \frac{y_{kn}}{y_n} > k^{\frac{1}{\lambda}}$,

2) $\limsup_{n\to\infty} \frac{y_{kn}}{y_n} < k^{\frac{1}{\lambda}}$.

In the first case, there exists an $\eta > 0$ such that for infinitely many integers n we have

$$\frac{y_n}{y_{kn}} < \frac{1}{k^{\frac{1}{\lambda}}} - \eta. \tag{4}$$

On the other hand, from the condition $G(Y_n) = \{x^\lambda\}$ it follows that

$$\lim_{n\to\infty} F\left(Y_{kn}, \frac{1}{k^{1/\lambda}} - \eta\right) = \lim_{n\to\infty} \frac{\#\{i \le kn; \frac{y_i}{y_{kn}} < \frac{1}{k^{1/\lambda}} - \eta\}}{kn} = \left(\frac{1}{k^{1/\lambda}} - \eta\right)^\lambda < \frac{1}{k},$$

which contradicts the fact that by (4) for infinitely many n

$$\#\left\{i \le kn; \frac{y_i}{y_{kn}} < \frac{1}{k^{1/\lambda}} - \eta\right\} \ge n.$$

The analysis of the second case is similar and left to the reader.

To prove the sufficiency assume that (3) holds for any positive integer k. Let $\alpha \in (0, 1)$. We will show that

$$\lim_{n\to\infty} F(Y_n, \alpha) = \lim_{n\to\infty} \frac{\#\{i \le n; \frac{y_i}{y_n} < \alpha\}}{n} = \alpha^\lambda.$$

Fix such an $\varepsilon > 0$ for which $\varepsilon < \min\{\alpha, 1 - \alpha\}$. Choose α_1 and α_2 such that $\alpha - \varepsilon < \alpha_1 < \alpha$ and $\alpha < \alpha_2 < \alpha + \varepsilon$. Furthermore, let α_1^λ and α_2^λ be rational numbers with the same denominator, $\alpha_1^\lambda = \frac{a}{b}$ and $\alpha_2^\lambda = \frac{c}{b}$ for suitable positive integers a, b, c. As a consequence of (3), we have

$$\lim_{m\to\infty} \frac{y_{\alpha_1^\lambda bm}}{y_{bm}} = \lim_{n\to\infty} \frac{y_{am}}{y_{bm}} = \lim_{m\to\infty} \frac{\frac{y_{am}}{y_m}}{\frac{y_{bm}}{y_m}} = \left(\frac{a}{b}\right)^{\frac{1}{\lambda}} = \alpha_1 < \alpha.$$

Then there exists an m_0 such that for every $m \geq m_0$ we have

$$\frac{y_{\alpha_1^\lambda bm}}{y_{bm}} < \alpha.$$

Hence, if $i \leq \alpha_1^\lambda bm$, then $\frac{y_i}{y_{bm}} < \alpha$. This means that

$$\#\left\{i \leq bm; \frac{y_i}{y_{bm}} < \alpha\right\} \geq \alpha_1^\lambda bm.$$

Then

$$\liminf_{m\to\infty} \frac{\#\{i \leq bm; \frac{y_i}{y_{bm}} < \alpha\}}{bm} \geq \liminf_{m\to\infty} \frac{\alpha_1^\lambda bm}{bm} = \alpha_1^\lambda \geq (\alpha - \varepsilon)^\lambda. \tag{5}$$

Similarly, we can show that

$$\limsup_{m\to\infty} \frac{\#\{i \leq bm; \frac{y_i}{y_{bm}} < \alpha\}}{bm} \leq \limsup_{m\to\infty} \frac{\alpha_2^\lambda bm}{bm} = \alpha_2^\lambda \leq (\alpha + \varepsilon)^\lambda. \tag{6}$$

Since (5) and (6) hold for an arbitrary small positive ε, therefore, we have

$$\lim_{m\to\infty} F(Y_{bm}, \alpha) = \lim_{m\to\infty} \frac{\#\{i \leq bm; \frac{y_i}{y_{bm}} < \alpha\}}{bm} = \alpha^\lambda. \tag{7}$$

Taking into account that for arbitrary positive integers n and b there exists a nonnegative integer m for which $bm \leq n < b(m+1)$ together with the inequality (1), we get

$$F(Y_{bm}, \alpha) \leq \frac{n}{bm} F(Y_n, \alpha) \quad \text{and} \quad F(Y_n, \alpha) \leq \frac{b(m+1)}{n} F(Y_{b(m+1)}, \alpha)$$

which yields

$$\frac{bm}{b(m+1)} F(Y_{bm}, \alpha) \leq F(Y_n, \alpha) \leq \frac{b(m+1)}{bm} F(Y_{b(m+1)}, \alpha).$$

Using the sandwich theorem for the terms in the previous inequality, we get that $F(Y_n, \alpha) \to \alpha^\lambda$ for $n \to \infty$ which completes the proof of the theorem. □

Below, we can see the advantage of not exclusively concentrating on sequences consisting solely of positive integers when considering distribution functions.

Corollary 3.1. *Let Y denote the sequence $0 < y_1 \leq y_2 \leq y_3 \leq \cdots$ and Y^c denote the sequence $0 < y_1^c \leq y_2^c \leq y_3^c < \cdots$ for some $c > 0$ and Y_n^c denote the related blocks. If $G(Y_n) = \{x^\lambda\}$, then we have $G(Y_n^c) = \{x^{\frac{\lambda}{c}}\}$.*

Proof. The assertion follows from the definition of the step distribution function,

$$F(Y^c, x) = \frac{\#\big\{i \leq n; \frac{y_i^c}{y_n^c} < x\big\}}{n} = \frac{\#\big\{i \leq n; \frac{y_i}{y_n} < x^{\frac{1}{c}}\big\}}{n} = F\big(Y_n, x^{\frac{1}{c}}\big)$$

and

$$\lim_{n\to\infty} F(Y_n^c, x) = \lim_{n\to\infty} F\big(Y_n, x^{\frac{1}{c}}\big) = \big(x^{\frac{1}{c}}\big)^\lambda.$$ □

In virtue of Theorem 3.2 and Corollary 3.1, we can establish the asymptotic behavior of power mean of the first n terms of the sequences studied.

Corollary 3.2. *Let Y denote the sequence $0 < y_1 \le y_2 \le y_3 \le \cdots$ and $G(Y_n) = \{x^\lambda\}$. Then for any $r > 0$*

$$\lim_{n\to\infty} \frac{M_r(y_1,\dots,y_n)}{y_n} = \lim_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} \frac{y_i^r}{y_n^r}\right)^{\frac{1}{r}} = \left(\frac{\frac{\lambda}{r}}{\frac{\lambda}{r}+1}\right)^{\frac{1}{r}} = \left(\frac{\lambda}{\lambda+r}\right)^{\frac{1}{r}}, \tag{8}$$

further for the sum of powers we get

$$\sum_{i=1}^{n} y_i^r \sim \frac{\lambda}{\lambda+r} n y_n^r. \tag{9}$$

Using (3), we can show that the block sequence of the following number theoretical sequences is uniformly distributed in $[0,1]$ and from (9) we get the related asymptotic results for $\lambda = 1$.

In the case $y_n = p_n$, equation (9) was obtained by [9], see also [7].

Let $c_{n,k}$, $n = 1,2,\dots$ is the sequence of numbers with k prime factors. Equation (9) for the case $y_n = c_{n,k}$ was obtained by Jakimczuk [5]. We note that

$$c_{n,k} \sim \frac{n.(k-1)!\log n}{(\log\log\log n)^{k-1}}.$$

The next asymptotic result seems to be original. Let $\pi_k(x)$ be the number of integers less than or equal to x which can be written as product of k prime factors. It was already known to Landau [8], Section 56, that

$$\pi_k(x) \sim \frac{1}{(k-1)!}\frac{x(\log\log x)^{k-1}}{\log x}.$$

In this case, we get

$$\sum_{i=1}^{n} \pi_k^r(i) \sim \frac{1}{1+r} n\pi_k^r(n).$$

If we consider the sequence $p_n\pi(n)$ (or p_n^2, respectively $\pi(n)^2$), we get the related sums of powers in (9) for $\lambda = \frac{1}{2}$. The details are left to the reader.

There are several papers concerning the geometric mean of the first n primes (see [11], [10], [6], [1]), which in our notation is

$$\lim_{n\to\infty} M_0\left(\frac{p_1}{p_n},\dots,\frac{p_n}{p_n}\right) = \frac{1}{e}. \tag{10}$$

For simplicity, we will denote

$$M_r\left(\frac{p_1}{p_n},\dots,\frac{p_n}{p_n}\right)$$

by $M_r(P_n)$. We get the same limit as above if we consider the limit

$$\lim_{r\to 0^+}\lim_{n\to\infty} M_r(P_n) = \lim_{r\to 0^+}\left(\frac{1}{1+r}\right)^{\frac{1}{r}} = \frac{1}{e},$$

(see (8)), but to prove (10), we have to consider the limit

$$\lim_{n\to\infty}\lim_{r\to 0^+} M_r(P_n). \tag{11}$$

In general, the problem of proving or disproving whether under the conditions of Corollary 3.2 we can exchange the limit order

$$\lim_{n\to\infty}\lim_{r\to 0^+} M_r\left(\frac{y_1}{y_n},\ldots,\frac{y_n}{y_n}\right) = \lim_{r\to 0^+}\lim_{n\to\infty} M_r\left(\frac{y_1}{y_n},\ldots,\frac{y_n}{y_n}\right)$$

is open.

We can prove it only in the special case, considering the sequence of prime numbers. Here, we give an alternative proof of (10).

Theorem 3.4. *Let p_n denote the n-th prime number. Then the limit (10) holds.*

Proof. The Moore–Osgood Theorem states that if one of the limits converges pointwise and the other converges uniformly, then we can switch limits. Therefore, to use this theorem to prove (11), we must demonstrate the uniform convergence of the sequence $M_r(P_n)$.

A consequence of a deep result of [4] says that the sum of the r-th powers ($r > -1$) of the primes less than x is asymptotic to $\pi(x^{r+1})$. This implies that (8) holds for the sequence of prime numbers in the case $-1 < r < 0$, too.

Let us consider the increasing function

$$h(x) = \begin{cases} \frac{1}{e}, & \text{if } x = 0 \\ \left(\frac{1}{1+x}\right)^{\frac{1}{x}}, & \text{if } x \neq 0 \end{cases}.$$

Since $h(x)$ is continuous on the set of real numbers, for any $\varepsilon > 0$ there exists an $\eta > 0$ for which

$$f(\eta) - f(-\eta) = f(\eta) - f(0) + f(0) - f(-\eta) < \frac{\varepsilon}{3}.$$

By (8), for the sequence of prime numbers and for $r = \eta$ ($r = -\eta$) we have that for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for all $n > N(\varepsilon)$ we have

$$M_\eta(P_n) < h(\eta) + \frac{\varepsilon}{3} \quad \text{and} \quad M_{-\eta}(P_n) > h(-\eta) - \frac{\varepsilon}{3}.$$

We will use the Cauchy criterion to prove the uniform convergence of the sequence $M_r(P_n)$, $n = 1, 2, \ldots$ in the neighbourhood of $r = 0$.

For a given $\varepsilon > 0$, let $r \in (-\eta, \eta) \setminus \{0\}$. Taking into account that the power mean M_r is increasing in r, for any $n, m > N(\varepsilon)$ we have

$$|M_r(P_n) - M_r(P_m)| < |M_\eta(P_n) - M_{-\eta}P_m| < h(\eta) + \frac{\varepsilon}{3} - \left(h(-\eta) - \frac{\varepsilon}{3}\right) < \varepsilon.$$

Applying the Moore–Osgood Theorem, we can exchange the limit order, so

$$\lim_{n\to\infty} M_0\left(\frac{p_1}{p_n},\ldots,\frac{p_n}{p_n}\right) = \lim_{n\to\infty}\lim_{r\to 0} M_r\left(\frac{p_1}{p_n},\ldots,\frac{p_n}{p_n}\right)$$

$$= \lim_{r\to 0}\lim_{n\to\infty} M_r\left(\frac{p_1}{p_n},\ldots,\frac{p_n}{p_n}\right) = \lim_{r\to 0}\left(\frac{1}{1+r}\right)^{\frac{1}{r}} = \frac{1}{e}.$$

□

4 Conclusion

We have extended the concept of distribution function of the block sequences for the sequences of positive real numbers. Using the necessary and sufficient conditions (2) and (3) for the block sequence of sequence $0 < y_1 \leq y_2 \leq y_3 \leq \cdots$ having asymptotic distribution function of the form x^λ, new asymptotic results for the power means of some number theoretic sequences were derived.

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