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A partial recurrence Fibonacci link

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Abstract: The purpose of this note is to develop a conjecture for a Fibonacci number generating function in terms of the elements of a second-order two parameter partial recurrence relation which arose in an operations research problem on Poisson distributed lead time in inventory control.

Keywords: Fibonacci numbers, Lucas numbers, Operations research, Partial recurrence relations.
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1 Introduction

The Fibonacci and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci and Lucas sequences have been applied in various fields such as Algebraic Coding Theory, Architecture, Engineering, Biomathematics, Computer Science, and so on.

For $n \in \mathbb{N}$, Fibonacci numbers F_n and Lucas numbers L_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n \text{, with } F_0 = 0 \text{ and } F_1 = 1,$$
$$L_{n+2} = L_{n+1} + L_n \text{, with } L_0 = 2 \text{ and } L_1 = 1.$$

For F_n and L_n the Binet formulas are given by the following relations, respectively,

$$F_n = \frac{\varphi^n - \omega^n}{\varphi - \omega} \text{ and } L_n = \varphi^n + \omega^n$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\omega = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $r^2 - r - 1 = 0$. Here the number φ is the known Golden ratio.

Many kinds of generalizations of the Fibonacci sequence have been presented in the literature (see for details in [1, 3, 5, 11]). Stanton [10] in a paper on Poisson distributed lead time in inventory control [4] utilized the following second order two parameter partial recurrence relation in his key j-th order equation

$$a_{j,m} = \begin{cases} 0, & j < m \\ m a_{j-1,m} + a_{j-1,m-1}, & j \geq m \end{cases} \tag{1.1}$$

with initial conditions $a_{1,1} = 1$. From this he obtained Table 1 which we shall use initially. With different initial conditions we obtain Table 2, which we shall use for the interested reader to extend, and then to generalize as it has many connections with the Sloane sequences [9].

Table 1. Initial term $a_{1,1} = 1$.

j \ m	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0
3	1	3	1	0	0	0	0	0	0	0
4	1	7	6	1	0	0	0	0	0	0
5	1	15	25	10	1	0	0	0	0	0
6	1	31	90	65	15	1	0	0	0	0
7	1	63	301	350	140	21	1	0	0	0
8	1	127	966	1701	1050	266	28	1	0	0
9	1	255	3025	7770	6951	2646	462	36	1	0
10	1	511	9330	34105	42525	22827	5880	750	45	1
…	…	…	…	…	…	…	…	…	…	…
A…	000012	001348	000392	003519	000481	000770	000771	049434	049447	—

The purpose of this paper is to link the sequences $\{a_{j,m}\}$ with the sequence of Fibonacci numbers; that is, the aim to prove that for the Fibonacci numbers $\{F_n\}$

$$\sum_{k=1}^{n} F_k k^m = \sum_{r=0}^{m-1} (-1)^r \Delta^r \left(n^m\right) F_{n+2r+1} \tag{1.2}$$

where

$$\Delta f(n) = f(n+1) - f(n) \tag{1.3}$$

and

$$\Delta^r \left(n^m\right) = \sum_{i=0}^{m-1} \binom{m}{i} r! a_{m-i,r} n^i. \tag{1.4}$$

Collectively, (1.2) with (1.3) and (1.4) are, in effect, a generalization of the results in [7] and [8], which, in turn, extended [2] and [6] respectively. The coefficients of $\Delta^r\left(n^m\right)$ for $r = 1, 2, \ldots, 10$, are tabulated in Brousseau [2] and summarized in our appendices.

2 Preliminary results

The formulas that are used include with slight adaptations from [2] and marginal modifications from standard usage include

$$\Delta\left(n^2 F_n\right) = (n+1)^2 F_{n+1} - n^2 F_n$$

and its nominal inverse

$$n^2 F_n = \Delta^{-1}\left[n^2 F_{n-1} + (2n+1) F_{n+1}\right] + C,$$

so that

$$\Delta\left(n^m F_{n+r+1}\right) = n^m F_{n+r} + \Delta\left(n^m\right) F_{n+r+2} \tag{2.1}$$

and

$$\Delta^{-1}\left(n^m F_{n+r}\right) = \sum_{t=0}^{m} (-1)^t \Delta^t \left(n^m\right) F_{n+r+2t+1} + C \tag{2.2}$$

with

$$n^m F_{n+r} = \sum_{k=1}^{n} k^m F_{k+r} - \sum_{k=1}^{n-1} k^m F_{k+r} \tag{2.3}$$

and

$$\Delta^{-1}\left(n^m F_{n+r}\right) = \sum_{k=1}^{n-1} k^m F_{k+r}, \tag{2.4}$$

where the arbitrary constant C may involve Fibonacci numbers (independent of n).

3 Main results

We conjecture from Table 1 that

$$a_{j,m} = \frac{1}{m!}\sum_{r=0}^{m}(-1)^r \binom{m}{r}(m-r)^j . \tag{3.1}$$

Proof. The right-hand side of (1.1) is

$$\begin{aligned} ma_{j-1,m} + a_{j-1,m-1} &= \frac{1}{(m-1)!}\sum_{r=0}^{m}(-1)^r \binom{m}{r}(m-r)^{j-1} + \frac{1}{(m-1)!}\sum_{r=0}^{m-1}(-1)^r \binom{m-1}{r}(m-r-1)^{j-1} \\ &= \frac{1}{(m-1)!}\sum_{r=0}^{m-1}(-1)^r \left\{\binom{m-1}{r}+\binom{m-1}{r-1}\right\}(m-r)^{j-1} \\ &= \frac{1}{m!}\sum_{r=0}^{m}(-1)^r \binom{m}{r}(m-r)^j \end{aligned}$$

as suggested.

We consider the array on the right-hand side of (2.1):

$$\sum_{r=0}^{m-1}(-1)^r F_{n+2r+1}\sum_{i=0}^{m-1}\binom{m}{i} r!a_{m-i,r}n^i$$

and use induction on *n* for which we take the case for *m* = 7, simplified from [2]:

$$\begin{aligned} n=1: \quad F_{r+1} = F_{r+2} - 127F_{r+4} + 1932F_{r+6} - 10206F_{r+8} + 25200F_{r+10} - 31920F_{r+12} \\ +\ 20160F_{r+14} - 5040F_{r+16} \end{aligned}$$

which can be confirmed when *r* = 1, for instance.

The complication in practice is that we have 3 variables: *m, n, r*, so that if we let $A^2(u,v)$ denote the entry in the *v*-th row and the (*u* + 2)-th column of Table A2 in the appendices, where $v \geq 2$ and $u \geq 0$, then

$$\left.\begin{array}{lll} \text{a)} & A^2(0,v) = A^2(0,v-1) + 4\left[A^2(0,v-3)+2\right] & \text{for } v \geq 5. \\ \text{b)} & A^2(u,u+2) = A^2(u-1,u+1) + 2(u+1) & \text{for } u \geq 1. \\ \text{c)} & A^2(u,u+2) = (u+2)(u+1) & \text{for } u \geq 0. \\ \text{d)} & A^2(u,u+3) = (u+3)(u+2)(u+1) & \text{for } u \geq 0. \end{array}\right| \tag{3.2}$$

4 Concluding comments

The interested read might like to try the foregoing with the elements of Table 2, or A111669 as a triangular array in [9].

Table 2. Initial term $a_{1,0} = a_{0,1} = 1$.

$j \setminus m$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0
3	1	2	1	0	0	0	0	0	0	0	0
4	1	3	4	1	0	0	0	0	0	0	0
5	1	4	11	7	1	0	0	0	0	0	0
6	1	5	26	32	11	1	0	0	0	0	0
7	1	6	57	122	76	16	1	0	0	0	0
8	1	7	120	423	426	156	22	1	0	0	0
9	1	8	247	1389	2127	1206	288	29	1	0	0
10	1	9	502	4414	9897	8257	2934	491	37	1	0
11	1	10	1013	13744	94684	210990	172980	49566	787	46	1
A…	000012	000027	000259	249999	—	—	—	—	—	—	—

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Appendices (from Brousseau and Sloane)

Table A1. Coefficients of $\Delta(n^m)$

m	1	n^1	n^2	n^3	n^4	n^5	n^6	n^7	n^8	n^9
1	1									
2	1	2								
3	1	3	3							
4	1	4	6	4						
5	1	5	10	10	5					
6	1	6	15	20	15	6				
7	1	7	21	35	35	21	7			
8	1	8	28	56	70	56	28	8		
9	1	9	36	84	126	126	84	36	9	
10	1	10	45	120	210	252	210	120	45	10
A	000012	000027	000217	000292	000332	000389	000579			

Table A2. Coefficients of $\Delta^2(n^m)$

m	1	n^1	n^2	n^3	n^4	n^5	n^6	n^7	n^8
2	2								
3	6	6							
4	14	24	12						
5	30	70	60	20					
6	62	180	210	120	30				
7	126	434	630	490	210	42			
8	254	1008	1736	1680	980	336	56		
9	510	2286	4536	5208	3780	1764	504	72	
10	1022	5100	11430	15120	13020	7560	2940	720	90
A	000198	052749	000554	---	---				

Table A3. Coefficients of $\Delta^3(n^m)$

m	**1**	n^1	n^2	n^3	n^4	n^5	n^6	n^7
3	6							
4	36	24						
5	150	180	60					
6	540	900	540	120				
7	1806	3780	3150	1260	210			
8	5796	14448	15120	8400	2520	336		
9	18150	52164	65016	45360	18900	4536	504	
10	55980	181500	260820	216720	113400	37800	7560	720

Table A4. Coefficients of $\Delta^4(n^m)$

m	**1**	n^1	n^2	n^3	n^4	n^5	n^6
4	24						
5	240	120					
6	1560	1440	360				
7	8400	10920	5040	840			
8	40824	67200	43680	13440	1680		
9	186480	367416	302400	131040	30240	3024	
10	818520	1864800	1837080	1080000	327600	60480	5040

Table A5. Coefficients of $\Delta^5(n^m)$

m	**1**	n^1	n^2	n^3	n^4	n^5
5	120					
6	1800	720				
7	16800	12600	2520			
8	126000	134400	50400	6720		
9	834120	1134000	604800	151200	15120	
10	5103000	8341200	5670000	2016000	378000	30240

Table A6. Coefficients of $\Delta^6(n^m)$

m	**1**	n^1	n^2	n^3	n^4
6	720				
7	15120	5040			
8	191520	120960	20160		
9	1905120	1723680	544320	60480	
10	16435440	19051200	8618400	1814400	151200

Table A7. Coefficients of $\Delta^7(n^m)$

m	1	n^1	n^2	n^3
7	5040			
8	141120	40320		
9	2328480	1270080	181440	
10	29635200	23284800	6350400	604800

Table A8. Coefficients of $\Delta^8(n^m)$

m	1	n^1	n^2
8	40320		
9	1451520	362880	
10	30240000	14515200	1814400

Table A9. Coefficients of $\Delta^9(n^m)$

m	1	n^1
9	362880	
10	16329600	3628800

Table A10. Coefficients of $\Delta^{10}(n^m)$

m	1
10	3628800