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Evaluation of certain families of log-cosine integrals using hypergeometric function approach and applications

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Abstract: In this paper, we provide the analytical solutions of the families of certain definite integrals: $\int_0^\pi x^m\{\ln(2\cos\frac{x}{2})\}^n dx$ $(m \in \mathbb{N}_0$ and $n \in \mathbb{N})$, in terms of multiple hypergeometric functions of Kampé de Fériet having the arguments ± 1 and Riemann zeta functions. As applications, we obtain some mixed summation formulas (19), (35) and (46) involving generalized hypergeometric functions ${}_3F_2$, ${}_5F_4$ and ${}_7F_6$ having the arguments ± 1 and other (possibly) new summation formulas (38) and (40) for multiple hypergeometric functions of Kampé de Fériet having the arguments ± 1 also mixed relations (36) and (47) involving Riemann zeta functions.

Keywords: Hypergeometric functions, Log-cosine integrals, Riemann zeta function, Kampé de Fériet multiple hypergeometric functions.
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1 Introduction and Preliminaries

The main aim of this work is to provide the analytical solutions of following families of log-cosine integrals:

- $\int_0^\pi \{\ln(2\cos\frac{x}{2})\}^n dx;\ n \in \{1,2,3,4,5\}$
- $\int_0^\pi x\{\ln(2\cos\frac{x}{2})\}^n dx;\ n \in \{1,2,3,4\}$
- $\int_0^\pi x^2\{\ln(2\cos\frac{x}{2})\}^n dx;\ n \in \{1,2,3\}$
- $\int_0^\pi x^3\{\ln(2\cos\frac{x}{2})\}^n dx;\ n \in \{1,2\}$
- $\int_0^\pi x^4\{\ln(2\cos\frac{x}{2})\}^n dx;\ n = 1.$

in terms of multiple hypergeometric functions of Kampé de Fériet and Riemann zeta functions.

For the sake of convenience, we shall use the following standard notations and other results: $\mathbb{N} := \{1,2,3,\ldots\}; \mathbb{N}_0 := \mathbb{N}\bigcup\{0\}; \mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\ldots\}$. The symbols $\mathbb{C},\mathbb{R},\mathbb{N},\mathbb{Z},\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.

For the definitions of Pochhammer symbol, generalized hypergeometric series (or function) ${}_pF_q$ $(p,\ q \in \mathbb{N}_0)$, which is a natural generalization of the Gaussian hypergeometric series ${}_2F_1$ and convergence conditions, we refer the beautiful monographs [15, 16, 18, 19].

• Multiple hypergeometric functions of Kampé de Fériet

For $(a_A) = a_1, a_2, .., a_A$ and $\left(b^{(i)}_{B^{(i)}}\right) = b^{(i)}_1, b^{(i)}_2, b^{(i)}_3, \ldots,\ b^{(i)}_{B^{(i)}}$, we consider the multiple hypergeometric series [17]:

$$F^{A:\ B^{(1)};\cdots;B^{(n)}}_{C:\ D^{(1)};\cdots;D^{(n)}}\left[\begin{array}{ll}(a_A):\left(b^{(1)}_{B^{(1)}}\right);\cdots;\left(b^{(n)}_{B^{(n)}}\right); & \\ & x_1,\ldots,x_n \\ (c_C):\left(d^{(1)}_{D^{(1)}}\right);\cdots;\left(d^{(n)}_{D^{(n)}}\right); & \end{array}\right]$$

$$= \sum_{m_1,m_2,\ldots,m_n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{m_1+m_2+\cdots+m_n} \prod_{j=1}^{B^{(1)}}\left(b^{(1)}_j\right)_{m_1}\cdots\prod_{j=1}^{B^{(n)}}\left(b^{(n)}_j\right)_{m_n}}{\prod_{j=1}^{C}(c_j)_{m_1+m_2+\cdots+m_n} \prod_{j=1}^{D^{(1)}}\left(d^{(1)}_j\right)_{m_1}\cdots\prod_{j=1}^{D^{(n)}}\left(d^{(n)}_j\right)_{m_n}} \prod_{i=1}^{n}\frac{x_i^{m_i}}{m_i!}, \tag{1}$$

which unifies and extends the four Lauricella series $F^{(n)}_A, F^{(n)}_B, F^{(n)}_C$ and $F^{(n)}_D$ in n variables. In fact, as already observed in the literature [18, pp. 37–38], the multiple hypergeometric series (1) is a special case of the generalized Lauricella series in several variables, which was introduced by Srivastava and Daoust in 1969.

Suppose $\Delta_k \equiv 1 + C + D^{(k)} - A - B^{(k)}\ (k = 1,\ldots,n)$.

For the convergence [20, p. 1127, Eq.(4.3)–Eq.(4.5)] see also [21] of the multiple hypergeometric series (1), we have

$$(i) \quad \Delta_k > 0; \ |x_1| < \infty, \ |x_2| < \infty, \ldots, |x_n| < \infty, \tag{2}$$

$$(ii) \quad \Delta_k = 0; \ A > C \ \text{ and } \ |x_1|^{\frac{1}{(A-C)}} + \cdots + |x_n|^{\frac{1}{(A-C)}} < 1, \tag{3}$$

$$(iii) \quad \Delta_k = 0; \ \ A \leq C \ \text{ and } \ \max\{|x_1|, \ldots, |x_n|\} < 1. \tag{4}$$

In order to get the clear idea about the absolutely and conditionally convergence of (1), we summarize some results [11, pp. 113–114, Th.(4)–Th.(6)] as follows.

Remark 1.1. *Let $\Delta_k = 0 \ (k = 1, \ldots, n)$, $A = C$ and $|x_1| = \cdots = |x_n| = 1$. Then the series (1)*

(i) converges absolutely if and only if

$$\rho_k = Re\left(\sum_{j=1}^{A} a_j + \sum_{j=1}^{B^{(k)}} b_j^{(k)} - \sum_{j=1}^{C} c_j - \sum_{j=1}^{D^{(k)}} d_j^{(k)}\right) < 0; \ \ (k = 1, \ldots, n) \tag{5}$$

and

$$\sigma = Re\left\{\sum_{j=1}^{A} a_j + \sum_{k=1}^{n}\left(\sum_{j=1}^{B^{(k)}} b_j^{(k)}\right) - \sum_{j=1}^{C} c_j - \sum_{k=1}^{n}\left(\sum_{j=1}^{D^{(k)}} d_j^{(k)}\right)\right\} < 0; \tag{6}$$

(ii) converges conditionally when $x_k \neq 1 \ \ (k = 1, \ldots, n)$, if

$$\rho_k < 1 \ (k = 1, \ldots, n) \ \text{ and } \ \sigma < n;$$

(iii) diverges if at least one of the following $n + 1$ conditions does not hold true:

$$\rho_k \leqq 1 \ (k = 1, \ldots, n) \ \text{ and } \ \sigma < n.$$

Remark 1.2. *Let $\Delta_k = 0 \ (k = 1, \ldots, n)$, $A < C$ and $|x_1| = \cdots = |x_n| = 1$. Then the series (1)*

(i) converges absolutely if and only if $\rho_k < 0 \ (k = 1, \ldots, n)$,

(ii) converges conditionally when $x_k \neq 1 \ (k = 1, \ldots, n)$, if $\rho_k < 1 \ (k = 1, \ldots, n)$, ρ_k being defined by (5).

Remark 1.3. *Let $\Delta_k = 0 \ (k = 1, \ldots, n)$ and $A > C$. Then the series (1), converges absolutely when*

$$|x_1|^{\frac{1}{(A-C)}} + \cdots + |x_n|^{\frac{1}{(A-C)}} = 1 \quad (x_k \neq 0; k = 1, \ldots, n),$$

if

$$\sigma + A - C < 1,$$

where σ is defined by (6).

The following results will be needed in our present investigation:

$$\ln(1 + e^{ix}) = \ln(2\cos\frac{x}{2}) + i\frac{x}{2}. \tag{7}$$

$$\ln(1 + z) = z \ \ {}_2F_1\left[\begin{array}{c} 1, \ 1; \\ \qquad -z \\ 2\,; \end{array}\right]; |z| \leq 1 \text{ and } z \neq -1. \tag{8}$$

$$\ln(2) = {}_2F_1\left[\begin{matrix} 1,\ 1; \\ \\ 2\ ; \end{matrix}\ -1\right]. \tag{9}$$

A definite integral of Borwein and Borwein [3, p. 1192] (see also [6, p. 780]; [10, p. 206, Eq.(1)] and [13]) was given in the year 1995.

$$\int_0^{\pi} x^2\{\ln(2\cos\frac{x}{2})\}^2 dx = \frac{11\pi^5}{180}. \tag{10}$$

A definite integral of Choi and Srivastava [6, p. 780] was given in the year 2011.

$$\int_0^{\pi} x^2\{\ln(2\cos\frac{x}{2})\}^3 dx = -\pi^3\,\zeta(3) - 6\pi\,\zeta(5). \tag{11}$$

The Riemann zeta function is defined by [6, p. 767, Eq.(1.1)]:

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{1-2^{-s}}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; s \neq 1). \end{cases} \tag{12}$$

General identity: If $s = 2, 3, 4, 5, \ldots,$ (see [1]), then

$$\zeta(s) = {}_{s+1}F_s\left[\begin{matrix} \overbrace{1, 1, \ldots, 1, 1;}^{s+1} \\ \\ \underbrace{2, 2, \ldots, 2;}_{s} \end{matrix}\ 1\right]. \tag{13}$$

Another general identity: If $s = 2, 3, 4, 5, \ldots,$ (see [1]), then

$${}_{s+1}F_s\left[\begin{matrix} \overbrace{1, 1, \ldots, 1, 1;}^{s+1} \\ \\ \underbrace{2, 2, \ldots, 2;}_{s} \end{matrix}\ -1\right] = (1 - 2^{1-s})\zeta(s). \tag{14}$$

Some special values of Riemann zeta functions are given below (see [16, 22]):

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(3) = \frac{\pi^3}{25.79436},\ \ \zeta(4) = \frac{\pi^4}{90},\ \zeta(5) = \frac{\pi^5}{295.1215},\ \zeta(6) = \frac{\pi^6}{945},$$
$$\zeta(7) = \frac{\pi^7}{2995.286},\ \zeta(8) = \frac{\pi^8}{9450},\ \zeta(9) = \frac{\pi^9}{29749.35},\ \ \zeta(10) = \frac{\pi^{10}}{93555},\ldots$$

Some summations of Baboo [1, p. 169, Eq. (9.6.5) and Eq. (9.6.2)]:

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} \mathbf{2:}\ \ 1,1\ ;\ 1,1\ ; \\ \\ \mathbf{3:}\ \ \ 2\ \ ;\ \ 2\ \ ; \end{matrix}\ 1,1\right] = 4\zeta(3) = 4\,{}_4F_3\left[\begin{matrix} 1,1,1,1; \\ \\ 2,2,2; \end{matrix}\ 1\right]. \tag{15}$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} \mathbf{2:}\ \ 1,1\ ;\ 1,1\ ; \\ \\ \mathbf{3:}\ \ \ 2\ \ ;\ \ 2\ \ ; \end{matrix}\ -1,-1\right] = \frac{\zeta(3)}{2} = \frac{2}{3}\,{}_4F_3\left[\begin{matrix} 1,1,1,1; \\ \\ 2,2,2; \end{matrix}\ -1\right]. \tag{16}$$

Generalization of successive integration by parts:

$$\int [U(x)T(x)]dx = +(U)\left(\int T dx\right) - \left(\frac{dU}{dx}\right)\left(\iint T dx dx\right) + \left(\frac{d^2U}{dx^2}\right)\left(\iiint T dx dx dx\right) - \cdots \tag{17}$$

The present article is organized as follows. Motivated by the work collected in beautiful monographs [2, 4, 5, 7–10, 12, 14, 22, 23], in Sections 2 to 6, we mention our main findings as theorems and related applications. In Section 7, we obtain the solutions of some auxiliary integrals: $\int_0^\pi x^p\{\ln(1+e^{ix})\}^q dx = \int_0^\pi x^p\{\ln(2\cos\frac{x}{2}) + \frac{ix}{2}\}^q dx$ ($p \in \mathbb{N}_0$ and $q \in \mathbb{N}$), separate real, imaginary parts and classify (i.e. arrange in systematic way) the obtained results in Sections 2 to 6. We also obtain some mixed summation formulas involving generalized hypergeometric functions ${}_3F_2$, ${}_5F_4$, ${}_7F_6$, other new summation formulas of multiple hypergeometric functions of Kampé de Fériet and mixed relations involving Riemann zeta functions.

2 Theorems related with $\int_0^\pi\{\ln(2\cos\frac{x}{2})\}^n dx$; $n \in \{1, 2, 3, 4, 5\}$

$$\int_0^\pi \{\ln(2\cos\frac{x}{2})\} dx = 0. \tag{18}$$

$$\frac{\pi^2}{4} = {}_3F_2\left[\begin{array}{c} 1,1,1; \\ \\ 2,2; \end{array} 1\right] + {}_3F_2\left[\begin{array}{c} 1,1,1; \\ \\ 2,2; \end{array} -1\right]. \tag{19}$$

$$\int_0^\pi \{\ln(2\cos\frac{x}{2})\}^2 dx = \frac{\pi^3}{12}. \tag{20}$$

$$\int_0^\pi \{\ln(2\cos\frac{x}{2})\}^3 dx = -\frac{3\pi}{2}\, {}_4F_3\left[\begin{array}{c} 1,1,1,1; \\ \\ 2,2,2; \end{array} 1\right]. \tag{21}$$

Special case of the integral (21):

$$\int_0^\pi \{\ln(2\cos\frac{x}{2})\}^3 dx = -\frac{3\pi}{2}\,\zeta(3). \tag{22}$$

$$\int_0^\pi \{\ln(2\cos\frac{x}{2})\}^4 dx = \frac{\pi^5}{16} + \frac{3\pi}{4}\, F^{2:2;2}_{2:1;1}\left[\begin{array}{c} 2,2\mathbf{:}1,1;1,1; \\ \\ 3,3\mathbf{:}\ 2\ ;\ 2\ ; \end{array} 1,1\right]. \tag{23}$$

Special case of the integral (23):

$$\int_0^\pi \{\ln(2\cos\frac{x}{2})\}^4 dx = \frac{19\pi^5}{240}. \tag{24}$$

$$\int_0^{\pi}\{\ln(2\cos\frac{x}{2})\}^5dx=\frac{75\pi}{2}\,{}_6F_5\left[\begin{array}{c}1,1,1,1,1,1;\\ \\ 2,2,2,2,2;\end{array}\;1\right]-\frac{25\pi^3}{4}\,{}_4F_3\left[\begin{array}{c}1,1,1,1;\\ \\ 2,2,2;\end{array}\;1\right]-$$

$$-\frac{5\pi}{9}\;F^{2:2;2;2}_{2:1;1;1}\left[\begin{array}{c}3,3\!:\!1,1;1,1;1,1;\\ \\ 4,4\!:2\;;\;2\;;\;2\;;\end{array}\;1,1,1\right]. \tag{25}$$

Special case of the integral (25):

$$\int_0^{\pi}\{\ln(2\cos\frac{x}{2})\}^5dx=-\frac{45\pi}{2}\zeta(5)-\frac{5\pi^3}{4}\,\zeta(3). \tag{26}$$

3 Theorems related with $\int_0^{\pi}x\{\ln(2\cos\frac{x}{2})\}^n dx$; $n\in\{1,2,3,4\}$

$$\int_0^{\pi}x\{\ln(2\cos\frac{x}{2})\}dx=-\frac{1}{2}\;F^{1:2;2}_{1:1;1}\left[\begin{array}{c}2\!:\!1,1;1,1;\\ \\ 3\!:2\;;\;2\;;\end{array}\;1,1\right]+\frac{1}{2}\;F^{1:2;2}_{1:1;1}\left[\begin{array}{c}2\!:\!1,1;1,1;\\ \\ 3\!:2\;;\;2\;;\end{array}\;-1,-1\right]. \tag{27}$$

Special case of the integral (27):

$$\int_0^{\pi}x\{\ln(2\cos\frac{x}{2})\}dx=-\frac{7}{4}\zeta(3). \tag{28}$$

$$\int_0^{\pi}x\{\ln(2\cos\frac{x}{2})\}^2dx=\frac{\pi^4}{48}+\frac{2}{9}\;F^{1:2;2;2}_{1:1;1;1}\left[\begin{array}{c}3\!:\;\;1,1;\;1,1;\;1,1;\\ \\ 4\!:\;\;\;2;\;\;2;\;\;2;\end{array}\;1,1,1\right]+$$

$$+\frac{2}{9}\;F^{1:2;2;2}_{1:1;1;1}\left[\begin{array}{c}3\!:\;\;1,1\;;\;1,1\;;\;1,1\;;\\ \\ 4\!:\;\;2\;;\;\;2\;;\;\;2\;;\end{array}\;-1,-1,-1\right]. \tag{29}$$

$$\int_0^{\pi}x\{\ln(2\cos\frac{x}{2})\}^3dx=\frac{1}{16}\;F^{3:2;2}_{3:1;1}\left[\begin{array}{c}2,2,2\!:1,1\;;\;1,1\;;\\ \\ 3,3,3\!:2\;;\;\;2\;;\end{array}\;1,1\right]-$$

$$-\frac{1}{16}\;F^{3:2;2}_{3:1;1}\left[\begin{array}{c}2,2,2\!:1,1\;;\;1,1\;;\\ \\ 3,3,3\!:2\;;\;\;2\;;\end{array}\;-1,-1\right]-\frac{\pi^2}{8}\;F^{1:2;2}_{1:1;1}\left[\begin{array}{c}2\!:1,1\;;\;1,1\;;\\ \\ 3\!:2\;;\;\;2\;;\end{array}\;1,1\right]-$$

$$-\frac{1}{8}F^{1:2;2;2;2}_{1:1;1;1;1}\left[\begin{array}{c}4\!:\!1,1;1,1;1,1;1,1;\\ \\ 5\!:2\;;\;2\;;\;2\;;\;2\;;\end{array}\;1,1,1,1\right]+\frac{1}{8}F^{1:2;2;2;2}_{1:1;1;1;1}\left[\begin{array}{c}4\!:\!1,1;1,1;1,1;1,1;\\ \\ 5\!:2\;;\;2\;;\;2\;;\;2;\end{array}\;-1,-1,-1,-1\right]. \tag{30}$$

Special case of the integral (30):

$$\int_0^{\pi} x\{\ln(2\cos\frac{x}{2})\}^3 dx = \frac{1}{16}\, F^{3:2;2}_{3:1;1}\left[\begin{array}{c} 2,2,2{:}1,1;1,1; \\ \qquad\qquad 1,1 \\ 3,3,3{:}\ 2\ ;\ 2; \end{array}\right] - \frac{1}{16}\, F^{3:2;2}_{3:1;1}\left[\begin{array}{c} 2,2,2{:}1,1;1,1; \\ \qquad\qquad -1,-1 \\ 3,3,3{:}\ 2\ ;\ 2; \end{array}\right] -$$

$$-\frac{\pi^2}{2}\,\zeta(3) - \frac{1}{8}\, F^{1:2;2;2;2}_{1:1;1;1;1}\left[\begin{array}{c} 4{:}1,1;1,1;1,1;1,1; \\ \qquad\qquad 1,1,1,1 \\ 5{:}\ 2\ ;\ 2\ ;\ 2\ ;\ 2\ ; \end{array}\right] +$$

$$+\frac{1}{8}\, F^{1:2;2;2;2}_{1:1;1;1;1}\left[\begin{array}{c} 4{:}1,1;1,1;1,1;1,1; \\ \qquad\qquad -1,-1,-1,-1 \\ 5{:}\ 2\ ;\ 2\ ;\ 2\ ;\ 2\ ; \end{array}\right]. \tag{31}$$

$$\int_0^{\pi} x\{\ln(2\cos\frac{x}{2})\}^4 dx = \frac{7\pi^6}{1440} + \frac{\pi^2}{9}\, F^{1:2;2;2}_{1:1;1;1}\left[\begin{array}{c} 3{:}1,1;1,1;1,1; \\ \qquad\qquad 1,1,1 \\ 4{:}\ 2\ ;\ 2\ ;\ 2\ ; \end{array}\right] -$$

$$-\frac{2}{81} F^{3:2;2;2}_{3:1;1;1}\left[\begin{array}{c} 3,3,3{:}1,1;1,1;1,1; \\ \qquad\qquad 1,1,1 \\ 4,4,4{:}\ 2\ ;\ 2\ ;\ 2; \end{array}\right] - \frac{2}{81} F^{3:2;2;2}_{3:1;1;1}\left[\begin{array}{c} 3,3,3{:}1,1;1,1;1,1; \\ \qquad\qquad -1,-1,-1 \\ 4,4,4{:}\ 2\ ;\ 2\ ;\ 2; \end{array}\right] +$$

$$+\frac{2}{25}\, F^{1:2;2;2;2;2}_{1:1;1;1;1;1}\left[\begin{array}{c} 5{:}1,1;1,1;1,1;1,1;1,1; \\ \qquad\qquad 1,1,1,1,1 \\ 6{:}\ 2\ ;\ 2\ ;\ 2\ ;\ 2\ ;\ 2\ ; \end{array}\right] +$$

$$+\frac{2}{25}\, F^{1:2;2;2;2;2}_{1:1;1;1;1;1}\left[\begin{array}{c} 5{:}1,1;1,1;1,1;1,1;1,1; \\ \qquad\qquad -1,-1,-1,-1,-1 \\ 6{:}\ 2\ ;\ 2\ ;\ 2\ ;\ 2\ ;\ 2\ ; \end{array}\right]. \tag{32}$$

4 Theorems related with $\int_0^{\pi} x^2\{\ln(2\cos\frac{x}{2})\}^n dx; n\in\{1,2,3\}$

$$\int_0^{\pi} x^2\{\ln(2\cos\frac{x}{2})\} dx = -2\pi\ {}_4F_3\left[\begin{array}{c} 1,1,1,1; \\ \qquad 1 \\ 2,2,2; \end{array}\right]. \tag{33}$$

Special case of the integral (33):

$$\int_0^{\pi} x^2\{\ln(2\cos\frac{x}{2})\} dx = -2\pi\,\zeta(3). \tag{34}$$

$$\frac{\pi^4}{8} = \pi^2\ {}_3F_2\left[\begin{array}{c} 1,1,1\ ; \\ \qquad 1 \\ 2,2\ \ ; \end{array}\right] - 2\ {}_5F_4\left[\begin{array}{c} 1,1,1,1,1\ ; \\ \qquad 1 \\ 2,2,2,2; \end{array}\right] - 2\ {}_5F_4\left[\begin{array}{c} 1,1,1,1,1\ ; \\ \qquad -1 \\ 2,2,2,2\ ; \end{array}\right]. \tag{35}$$

Special case of the result (35):

$$\frac{\pi^4}{8} = \pi^2\,\zeta(2) - \frac{15}{4}\,\zeta(4). \tag{36}$$

$$\int_0^{\pi} x^2\{\ln(2\cos\frac{x}{2})\}^2 dx = \frac{\pi^5}{20} + \frac{\pi}{2} F^{2:2;2}_{2:1;1}\left[\begin{array}{c} 2,2{:}1,1;1,1; \\ \quad\quad\quad\quad 1,1 \\ 3,3{:}\,2\,;\,2\,; \end{array}\right]. \tag{37}$$

Special case of the integral (37):

$$F^{2:2;2}_{2:1;1}\left[\begin{array}{c} 2,2{:}1,1;1,1; \\ \quad\quad\quad\quad 1,1 \\ 3,3{:}\,2\,;\,2\,; \end{array}\right] = \frac{\pi^4}{45}. \tag{38}$$

$$\int_0^{\pi} x^2\{\ln(2\cos\frac{x}{2})\}^3 dx = 18\pi\, {}_6F_5\left[\begin{array}{c} 1,1,1,1,1,1; \\ \quad\quad\quad 1 \\ 2,2,2,2,2; \end{array}\right] - 3\pi^3\, {}_4F_3\left[\begin{array}{c} 1,1,1,1; \\ \quad\quad 1 \\ 2,2,2; \end{array}\right] -$$

$$-\frac{2\pi}{9} F^{2:2;2;2}_{2:1;1;1}\left[\begin{array}{c} 3,3{:}1,1;1,1;1,1; \\ \quad\quad\quad\quad 1,1,1 \\ 4,4{:}\,2\,;\,2\,;\,2\,; \end{array}\right]. \tag{39}$$

Special case of the integral (39):

$$F^{2:2;2;2}_{2:1;1;1}\left[\begin{array}{c} 3,3{:}1,1;1,1;1,1; \\ \quad\quad\quad\quad 1,1,1 \\ 4,4{:}\,2\,;\,2\,;\,2\,; \end{array}\right] = -9\pi^2\zeta(3) + 108\zeta(5). \tag{40}$$

Remark 4.1. *The right hand-side of the result (37) is an alternative form (evidently different) of the right hand-side of the result of Coffey [10, p. 206, Eq.(1)].*

Remark 4.2. *The summation formula (38) is not available in a paper of Coffey [10].*

5 Theorems related with $\int_0^{\pi} x^3\{\ln(2\cos\frac{x}{2})\}^n dx;\ n \in \{1,2\}$

$$\int_0^{\pi} x^3\{\ln(2\cos\frac{x}{2})\} dx = \frac{1}{4} F^{3:2;2}_{3:1;1}\left[\begin{array}{c} 2,2,2{:}1,1;1,1; \\ \quad\quad\quad\quad 1,1 \\ 3,3,3{:}\,2\,;\,2\,; \end{array}\right] - \frac{\pi^2}{2} F^{1:2;2}_{1:1;1}\left[\begin{array}{c} 2{:}1,1;1,1; \\ \quad\quad\quad 1,1 \\ 3{:}\,2\,;\,2\,; \end{array}\right] -$$

$$-\frac{1}{4} F^{3:2;2}_{3:1;1}\left[\begin{array}{c} 2,2,2{:}1,1;1,1; \\ \quad\quad\quad\quad -1,-1 \\ 3,3,3{:}\,2\,;\,2\,; \end{array}\right]. \tag{41}$$

Special case of the integral (41):

$$\int_0^{\pi} x^3\{\ln(2\cos\frac{x}{2})\} dx = \frac{1}{4} F^{3:2;2}_{3:1;1}\left[\begin{array}{c} 2,2,2{:}\,1,1\,;\,1,1\,; \\ \quad\quad\quad\quad 1,1 \\ 3,3,3{:}\,2\,;\,2\,; \end{array}\right] - 2\pi^2\,\zeta(3) -$$

$$-\frac{1}{4} F^{3:2;2}_{3:1;1}\left[\begin{array}{c} 2,2,2{:}\ \ 1,1\ \ ;\ \ 1,1\ \ ; \\ \quad\quad\quad\quad -1,-1 \\ 3,3,3{:}\ \ \ 2\ \ ;\ \ \ 2\ \ ; \end{array}\right]. \tag{42}$$

$$\int_0^\pi x^3\{\ln(2\cos\frac{x}{2})\}^2dx=\frac{\pi^6}{72}+\frac{2\pi^2}{9}F^{1:2;2;2}_{1:1;1;1}\left[\begin{array}{c}3\text{:}1,1;1,1;1,1;\\ \qquad\qquad\qquad 1,1,1\\ 4\text{:}\;2\;;\;2\;;\;2\;;\end{array}\right]-$$
$$-\frac{4}{81}F^{3:2;2;2}_{3:1;1;1}\left[\begin{array}{c}3,3,3\text{:}1,1;1,1;1,1;\\ \qquad\qquad\qquad 1,1,1\\ 4,4,4\text{:}\;2\;;\;2\;;\;2\;;\end{array}\right]-\frac{4}{81}F^{3:2;2;2}_{3:1;1;1}\left[\begin{array}{c}3,3,3\text{:}1,1;1,1;1,1;\\ \qquad\qquad\qquad -1,-1,-1\\ 4,4,4\text{:}\;2\;;\;2\;;\;2\;;\end{array}\right].\tag{43}$$

6 Theorems related with $\int_0^\pi x^4\{\ln(2\cos\frac{x}{2})\}^n dx; n=1$

$$\int_0^\pi x^4\{\ln(2\cos\frac{x}{2})\}dx=24\pi\;{}_6F_5\left[\begin{array}{c}1,1,1,1,1,1;\\ \qquad\qquad 1\\ 2,2,2,2,2;\end{array}\right]-4\pi^3\;{}_4F_3\left[\begin{array}{c}1,1,1,1;\\ \qquad\qquad 1\\ 2,2,2;\end{array}\right].\tag{44}$$

Special case of the integral (44):

$$\int_0^\pi x^4\{\ln(2\cos\frac{x}{2})\}dx=24\pi\;\zeta(5)-4\pi^3\;\zeta(3).\tag{45}$$

$$\frac{\pi^6}{12}=\pi^4\;{}_3F_2\left[\begin{array}{c}1,1,1\;;\\ \qquad 1\\ 2,2\;\;;\end{array}\right]-12\pi^2\;{}_5F_4\left[\begin{array}{c}1,1,1,1,1\;;\\ \qquad\qquad 1\\ 2,2,2,2;\end{array}\right]+24\;{}_7F_6\left[\begin{array}{c}1,1,1,1,1,1,1;\\ \qquad\qquad 1\\ 2,2,2,2,2,2\;;\end{array}\right]+$$
$$+\;24\;{}_7F_6\left[\begin{array}{c}1,1,1,1,1,1,1;\\ \qquad\qquad -1\\ 2,2,2,2,2,2\;;\end{array}\right].\tag{46}$$

Special case of the result (46):

$$\frac{\pi^6}{12}=\pi^4\zeta(2)-12\pi^2\zeta(4)+\frac{189}{4}\zeta(6).\tag{47}$$

7 Proof of the results given in Section 2 to Section 6

Proof of the results (44), (45), (46) and (47). Consider

$$\int_0^\pi x^4\;\{\ln(1+e^{ix})\}dx.$$

Now using the result (7), we get

$$\int_0^\pi x^4\;\{\ln(1+e^{ix})\}dx=\int_0^\pi x^4\{\ln(2\cos\frac{x}{2})\}dx+i\int_0^\pi\frac{x^5}{2}dx.\tag{48}$$

Also consider,

$$\int_0^\pi x^4\{\ln(1+e^{ix})\}dx.$$

Now using result (8), we obtain

$$\int_0^\pi x^4\{\ln(1+e^{ix})\}dx = \sum_{m=0}^{\infty}\frac{(1)_m(1)_m(-1)^m}{(2)_m\ m!}\int_0^\pi x^4\ \ e^{i(1+m)x}dx$$

$$=\sum_{m=0}^{\infty}\frac{(1)_m(1)_m(-1)^m}{(2)_m\ m!}\times$$

$$\times\left[\left\{\frac{-4\pi^3(-1)^m}{\{1+m\}^2}+\frac{24\pi(-1)^m}{\{1+m\}^4}\right\}+i\left\{\frac{\pi^4(-1)^m}{\{1+m\}}-\frac{12\pi^2(-1)^m}{\{1+m\}^3}+\frac{24(-1)^m}{\{1+m\}^5}+\frac{24}{\{1+m\}^5}\right\}\right]. \quad (49)$$

Now equating real and imaginary parts from equation (48) and (49), we have

$$\int_0^\pi x^4\{\ln(2\cos\frac{x}{2})\}dx = \sum_{m=0}^{\infty}\frac{(1)_m(1)_m(-1)^m}{(2)_m\ m!}\left[\left\{\frac{-4\pi^3(-1)^m}{\{1+m\}^2}+\frac{24\pi(-1)^m}{\{1+m\}^4}\right\}\right]$$

$$=-4\pi^3\sum_{m=0}^{\infty}\frac{(1)_m(1)_m\ \{(1)_m\}^2}{(2)_m\ m!\ \{(2)_m\}^2}+24\pi\sum_{m=0}^{\infty}\frac{(1)_m(1)_m\ \{(1)_m\}^4}{(2)_m\ m!\ \{(2)_m\}^4},$$

and using the definition of generalized hypergeometric function of one variable, we get the required result (44).

Again using the result (13) in the right-hand side of equation (44), we obtain the result (45).

$$\int_0^\pi \frac{x^5}{2}dx = \sum_{m=0}^{\infty}\frac{(1)_m(1)_m(-1)^m}{(2)_m\ m!}\left[\left\{\frac{\pi^4(-1)^m}{\{1+m\}}-\frac{12\pi^2(-1)^m}{\{1+m\}^3}+\frac{24(-1)^m}{\{1+m\}^5}+\frac{24}{\{1+m\}^5}\right\}\right]$$

$$\frac{\pi^6}{12}=\pi^4\sum_{m=0}^{\infty}\frac{(1)_m(1)_m\ \{(1)_m\}}{(2)_m\ m!\ \{(2)_m\}}-12\pi^2\sum_{m=0}^{\infty}\frac{(1)_m(1)_m\ \{(1)_m\}^3}{(2)_m\ m!\ \{(2)_m\}^3}+24\sum_{m=0}^{\infty}\frac{(1)_m(1)_m\ \{(1)_m\}^5}{(2)_m\ m!\ \{(2)_m\}^5}+$$

$$+24\sum_{m=0}^{\infty}\frac{(1)_m(1)_m(-1)^m\ \{(1)_m\}^5}{(2)_m\ m!\ \{(2)_m\}^5},$$

and using the definition of generalized hypergeometric function of one variable, we have the required result (46).

Also using the results (13) and (14) in the right-hand side of equation (46), we obtain a mixed relation (47). □

Proof of the results (33), (34), (35) and (36). Consider,

$$\int_0^\pi x^2\ \{\ln(1+e^{ix})\}dx.$$

Now using the result (7), we get

$$\int_0^\pi x^2\ \{\ln(1+e^{ix})\}dx = \int_0^\pi x^2\{\ln(2\cos\frac{x}{2})\}dx + i\int_0^\pi\frac{x^3}{2}dx. \quad (50)$$

Also consider,

$$\int_0^\pi x^2\{\ln(1+e^{ix})\}dx.$$

Now using result (8), we obtain

$$\int_0^{\pi} x^2\{\ln(1+e^{ix})\}dx = \sum_{m=0}^{\infty}\frac{(1)_m(1)_m(-1)^m}{(2)_m\ m!}\int_0^{\pi} x^2\ \ e^{i(1+m)x}dx$$

$$= \sum_{m=0}^{\infty}\frac{(1)_m(1)_m(-1)^m}{(2)_m\ m!}\times$$

$$\times\left[\left\{\frac{-2\pi(-1)^m}{\{1+m\}^2}\right\}+i\left\{\frac{\pi^2(-1)^m}{\{1+m\}}-\frac{2(-1)^m}{\{1+m\}^3}-\frac{2}{\{1+m\}^3}\right\}\right]. \tag{51}$$

Now equating real and imaginary parts from equation (50) and (51), we have

$$\int_0^{\pi} x^2\{\ln(2\cos\frac{x}{2})\}dx = \sum_{m=0}^{\infty}\frac{(1)_m(1)_m(-1)^m}{(2)_m\ m!}\left[\left\{\frac{-2\pi(-1)^m}{\{1+m\}^2}\right\}\right]$$

$$= -2\pi\sum_{m=0}^{\infty}\frac{(1)_m(1)_m\ \{(1)_m\}^2}{(2)_m\ m!\ \{(2)_m\}^2},$$

and using the definition of generalized hypergeometric function of one variable, we get the required result (33).

Again using the result (13) in the right-hand side of equation (33), we obtain the result (34).

$$\int_0^{\pi}\frac{x^3}{2}dx = \sum_{m=0}^{\infty}\frac{(1)_m(1)_m(-1)^m}{(2)_m\ m!}\left[\left\{\frac{\pi^2(-1)^m}{\{1+m\}}-\frac{2(-1)^m}{\{1+m\}^3}-\frac{2}{\{1+m\}^3}\right\}\right]$$

$$\frac{\pi^4}{8} = \pi^2\sum_{m=0}^{\infty}\frac{(1)_m(1)_m\ \{(1)_m\}}{(2)_m\ m!\ \{(2)_m\}} - 2\sum_{m=0}^{\infty}\frac{(1)_m(1)_m\ \{(1)_m\}^3}{(2)_m\ m!\ \{(2)_m\}^3} - 2\sum_{m=0}^{\infty}\frac{(1)_m(1)_m(-1)^m\ \{(1)_m\}^3}{(2)_m\ m!\ \{(2)_m\}^3},$$

and using the definition of generalized hypergeometric function of one variable, we get the required result (35).

Also using the results (13) and (14) in the right-hand side of equation (35), we obtain a mixed relation (36). □

Proof of the results (37), (38), (41) and (42). Consider,

$$\int_0^{\pi} x^2\ \{\ln(1+e^{ix})\}^2 dx.$$

Now using the result (7), we get

$$\int_0^{\pi} x^2\ \{\ln(1+e^{ix})\}^2 dx = \int_0^{\pi} x^2\{\ln(2\cos\frac{x}{2})\}^2 dx - \int_0^{\pi}\frac{x^4}{4}dx + i\int_0^{\pi} x^3\{\ln(2\cos\frac{x}{2})\}dx. \tag{52}$$

Also consider,

$$\int_0^{\pi} x^2\{\ln(1+e^{ix})\}^2 dx.$$

Now using result (8), we obtain

$$\int_0^{\pi} x^2\{\ln(1+e^{ix})\}^2 dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_m(1)_m(1)_n(1)_n(-1)^{m+n}}{(2)_m\ m!(2)_n\ n!}\int_0^{\pi} x^2\ \ e^{i(2+m+n)x}dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n(-1)^{m+n}}{(2)_m\, m!(2)_n\, n!} \times$$

$$\times \left[\left\{\frac{2\pi(-1)^{m+n}}{\{2+(m+n)\}^2}\right\} + i\left\{\frac{2(-1)^{m+n}}{\{2+(m+n)\}^3} - \frac{\pi^2(-1)^{m+n}}{\{2+(m+n)\}} - \frac{2}{\{2+(m+n)\}^3}\right\}\right]. \tag{53}$$

Further equating real and imaginary parts from equations (52) and (53), we have

$$\int_0^{\pi} x^2\{\ln(2\cos\frac{x}{2})\}^2 dx - \int_0^{\pi} \frac{x^4}{4} dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n(-1)^{m+n}}{(2)_m\, m!(2)_n\, n!} \left[\left\{\frac{2\pi(-1)^{m+n}}{\{2+(m+n)\}^2}\right\}\right]$$

$$\int_0^{\pi} x^2\{\ln(2\cos\frac{x}{2})\}^2 dx = \frac{\pi^5}{20} + 2\pi \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n\, \{(2)_{m+n}\}^2}{(2)_m\, m!(2)_n\, n!\, \{2(3)_{m+n}\}^2}$$

$$= \frac{\pi^5}{20} + \frac{\pi}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n\, \{(2)_{m+n}\}^2}{(2)_m\, m!(2)_n\, n!\, \{(3)_{m+n}\}^2},$$

also using the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we get the required result (37).

Again using the value of integral (10) of Borwein and Borwein in left-hand side of equation (37), we obtain a (possibly) new summation formula (38) for double hypergeometric functions of Kampé de Fériet.

$$\int_0^{\pi} x^3\{\ln(2\cos\frac{x}{2})\} dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n(-1)^{m+n}}{(2)_m\, m!(2)_n\, n!} \times$$

$$\times \left[\left\{\frac{2(-1)^{m+n}}{\{2+(m+n)\}^3} - \frac{\pi^2(-1)^{m+n}}{\{2+(m+n)\}} - \frac{2}{\{2+(m+n)\}^3}\right\}\right]$$

$$= 2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n\, \{(2)_{m+n}\}^3}{(2)_m\, m!(2)_n\, n!\, \{2(3)_{m+n}\}^3} - \pi^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n\, \{(2)_{m+n}\}}{(2)_m\, m!(2)_n\, n!\, \{2(3)_{m+n}\}} -$$

$$-2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n(-1)^{m+n}\, \{(2)_{m+n}\}^3}{(2)_m\, m!(2)_n\, n!\, \{2(3)_{m+n}\}^3}$$

$$= \frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n\, \{(2)_{m+n}\}^3}{(2)_m\, m!(2)_n\, n!\, \{(3)_{m+n}\}^3} - \frac{\pi^2}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n\, \{(2)_{m+n}\}}{(2)_m\, m!(2)_n\, n!\, \{(3)_{m+n}\}} -$$

$$-\frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n(-1)^{m+n}\, \{(2)_{m+n}\}^3}{(2)_m\, m!(2)_n\, n!\, \{(3)_{m+n}\}^3},$$

further using the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we get the required result (41).

Also using the summation formula (15) of Baboo in the right-hand side of equation (41), we obtain the result (42). □

Proof of the results (39), (40) and (43). Consider,

$$\int_0^{\pi} x^2\, \{\ln(1+e^{ix})\}^3 dx.$$

Now using the result (7), we get

$$\int_0^\pi x^2\ \{\ln(1+e^{ix})\}^3 dx = \int_0^\pi x^2\{\ln(2\cos\frac{x}{2})\}^3 dx - \frac{3}{4}\int_0^\pi x^4\{\ln(2\cos\frac{x}{2})\}dx +$$

$$+i\left\{\frac{3}{2}\int_0^\pi x^3\{\ln(2\cos\frac{x}{2})\}^2 dx - \int_0^\pi \frac{x^5}{8}dx\right\}. \qquad (54)$$

Also consider,

$$\int_0^\pi x^2\{\ln(1+e^{ix})\}^3 dx.$$

Now using result (8), we obtain

$$\int_0^\pi x^2\{\ln(1+e^{ix})\}^3 dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(1)_m(1)_m(1)_n(1)_n(1)_p(1)_p(-1)^{m+n+p}}{(2)_m\ m!\ (2)_n\ n!\ (2)_p\ p!}\int_0^\pi x^2 e^{i(3+m+n+p)x}dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(1)_m(1)_m(1)_n(1)_n(1)_p(1)_p(-1)^{m+n+p}}{(2)_m\ m!\ (2)_n\ n!\ (2)_p\ p!}\times$$

$$\times\left[\left\{\frac{-2\pi(-1)^{m+n+p}}{\{3+(m+n+p)\}^2}\right\}+i\left\{\frac{\pi^2(-1)^{m+n+p}}{\{3+(m+n+p)\}}-\frac{2(-1)^{m+n+p}}{\{3+(m+n+p)\}^3}-\frac{2}{\{3+(m+n+p)\}^3}\right\}\right]. \qquad (55)$$

Now equating real and imaginary parts from equations (54) and (55), we have

$$\int_0^\pi x^2\{\ln(2\cos\frac{x}{2})\}^3 dx - \frac{3}{4}\int_0^\pi x^4\{\ln(2\cos\frac{x}{2})\}dx$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(1)_m(1)_m(1)_n(1)_n(1)_p(1)_p(-1)^{m+n+p}}{(2)_m\ m!\ (2)_n\ n!\ (2)_p\ p!}\times\left[\left\{\frac{-2\pi(-1)^{m+n+p}}{\{3+(m+n+p)\}^2}\right\}\right]$$

$$\int_0^\pi x^2\{\ln(2\cos\frac{x}{2})\}^3 dx$$

$$= \frac{3}{4}\int_0^\pi x^4\{\ln(2\cos\frac{x}{2})\}dx - 2\pi\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(1)_m(1)_m(1)_n(1)_n(1)_p(1)_p}{(2)_m\ m!\ (2)_n\ n!\ (2)_p\ p!}\times\frac{\{(3)_{m+n+p}\}^2}{\{3\ (4)_{m+n+p}\}^2},$$

further using the result (44) and also the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we get the required result (39).

Again using the value of integral (11) of Choi and Srivastava in left-hand side of equation (39) and also using the result (13) in right-hand side of equation (39), we get another (possibly) new summation formula (40) for triple hypergeometric functions of Kampé de Fériet.

$$\frac{3}{2}\int_0^\pi x^3\{\ln(2\cos\frac{x}{2})\}^2 dx - \int_0^\pi \frac{x^5}{8}dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(1)_m(1)_m(1)_n(1)_n(1)_p(1)_p(-1)^{m+n+p}}{(2)_m\ m!\ (2)_n\ n!\ (2)_p\ p!}\times$$

$$\times\left[\left\{\frac{\pi^2(-1)^{m+n+p}}{\{3+(m+n+p)\}}-\frac{2(-1)^{m+n+p}}{\{3+(m+n+p)\}^3}-\frac{2}{\{3+(m+n+p)\}^3}\right\}\right]$$

$$\frac{3}{2}\int_0^{\pi} x^3\{\ln(2\cos\frac{x}{2})\}^2 dx = \frac{\pi^6}{48} + \pi^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n(1)_p(1)_p\{(3)_{m+n+p}\}}{(2)_m\ m!\ (2)_n\ n!\ (2)_p\ p!\{3\ (4)_{m+n+p}\}} -$$

$$-2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n(1)_p(1)_p\ \{(3)_{m+n+p}\}^3}{(2)_m\ m!\ (2)_n\ n!\ (2)_p\ p!\ \{3\ (4)_{m+n+p}\}^3} -$$

$$-2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n(1)_p(1)_p(-1)^{m+n+p}\ \{(3)_{m+n+p}\}^3}{(2)_m\ m!\ (2)_n\ n!\ (2)_p\ p!\ \{3\ (4)_{m+n+p}\}^3}$$

$$\int_0^{\pi} x^3\{\ln(2\cos\frac{x}{2})\}^2 dx = \frac{\pi^6}{72} + \frac{2\pi^2}{9} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n(1)_p(1)_p\ \{(3)_{m+n+p}\}}{(2)_m\ m!\ (2)_n\ n!\ (2)_p\ p!\ \{(4)_{m+n+p}\}} -$$

$$-\frac{4}{81}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n(1)_p(1)_p\ \{(3)_{m+n+p}\}^3}{(2)_m\ m!\ (2)_n\ n!\ (2)_p\ p!\ \{(4)_{m+n+p}\}^3} -$$

$$-\frac{4}{81}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n(1)_p(1)_p(-1)^{m+n+p}\ \{(3)_{m+n+p}\}^3}{(2)_m\ m!\ (2)_n\ n!\ (2)_p\ p!\ \{(4)_{m+n+p}\}^3},$$

and using the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we get the required result (43). □

Proof of the results (18) and (19). Consider the auxiliary integral $\int_0^{\pi}\{\ln(1+e^{ix})\}dx$ and use the results (7) and (8). Now equating real and imaginary parts, we get the required results (18) and (19). □

Proof of the results (20), (27) and (28). Consider the auxiliary integral $\int_0^{\pi}\{\ln(1+e^{ix})\}^2dx$ and use the results (7) and (8). Now equating real and imaginary parts, we get the required results (20) and (27).

Again using the summation formulas (15) and (16) of Baboo in the right-hand side of equation (27), we obtain the result (28). □

Proof of the results (21), (22) and (29). Consider the auxiliary integral $\int_0^{\pi}\{\ln(1+e^{ix})\}^3dx$ and use the results (7) and (8). Now equating real and imaginary parts. Using the result (33), we get the required result (21). Again using the result (13) in the right-hand side of equation (21), we obtain the result (22).

Now using the definition (1), we get the required result (29). □

Proof of the results (23), (24), (30) and (31). Consider the auxiliary integral $\int_0^{\pi}\{\ln(1+e^{ix})\}^4dx$ and use the results (7) and (8). Now equating real and imaginary parts. Using the equation (37), we get the required result (23) . Again using our new summation formula (38) in the right-hand side of equation (23), we obtain the result (24).

Now using the result (41) and also using the definition (1), we get the required result (30).

Also using the summation formula (15) of Baboo in the right-hand side of equation (30), we obtain the result (31). □

Proof of the results (25), (26) and (32). Consider the auxiliary integral $\int_0^\pi \{\ln(1+e^{ix})\}^5 dx$ and use the results (7) and (8). Now equating real and imaginary parts. Using the results (39) and (44), we get the required result (25). Again using the result (13) and our another new summation formula (40) in the right-hand side of equation (25), we obtain the result (26). Now using the equation (43) and also using the definition (1), we get the required result (32).

Remark 7.1. *We have also verified the results (19), (35), (36), (38), (40), (46) and (47) numerically by using* Mathematica *software.*

8 Conclusion

Here in this paper, we provide the analytical solutions of the families of certain definite integrals: $\int_0^\pi x^m \{\ln(2\cos\frac{x}{2})\}^n dx$ $(m \in \mathbb{N}_0$ and $n \in \mathbb{N})$, in terms of multiple hypergeometric functions of Kampé de Fériet having the arguments ± 1 and Riemann zeta functions. The results proved in this paper appear to be (possibly) new as well as some known and likely to have useful applications to a wide range of problems of mathematics and statistics.

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