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# Towards a new generalized Simson's identity

**A. G. Shannon[1,2], H. M. Srivastava[3,4] and József Sàndor[5]**

[1] Warrane College, University of New South Wales, Sydney NSW 2033, Australia
e-mails: tshannon@warrane.unsw.edu.au, tshannon38@gmail.com

[2] Australian Institute of Technology and Commerce, Sydney NSW 2000, Australia

[3] Department of Mathematics and Statistics, University of Victoria
Victoria, British Columbia V8W 3R4, Canada
e-mails: harimsri@math.uvic.ca, harimsri@uvic.ca

[4] Department of Medical Research, China Medical University Hospital,
China Medical University, Taichung 40402, Taiwan

[5] Department of Mathematics, Babeş-Bolyai University, Cluj-Napoca 400347, Romania
e-mails: jjsandor@hotmail.com, jsandor@math.ubbcluj.ro

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**Abstract:** This paper is an attempt to develop an elegant and simple generalization of what is usually called Simson's Identity, with variations named after Cassini, Catalan and Gelin-Cesàro. It can shed a new light on Simson's identity, and possibly how to extend it to some reciprocals of these identities and how to generalize it to arbitrary order with some conjectures.
**Keywords:** Fibonacci and Lucas numbers, Recurrence relations, Riemann Zeta Function, Simson's Identity, Kronecker delta.
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# 1 Introduction

Variations of the identities of Simson, Cassini, Catalan and Vajda have provided opportunities for extensions and generalizations [10]. In its simplest Fibonacci form, the Simson identity can be expressed as

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n . \tag{1.1}$$

More specifically, the Catalan identity is usually [16, 20] expressed as

$$F_{n-r}F_{n+r} - F_n^2 = (-1)^{n-r+1} F_r^2 ,$$

and a generalization of (1.1), the Gelin-Cesàro identity [5, 15]

$$F_{n-2}F_{n+2}F_{n-1}F_{n+1} - F_n^4 = -1,$$

with a variation by Mangon [14]

$$F_{n-1}F_{n-2} - F_nF_{n-3} = (-1)^n ,$$

and the related Vajda identity [20]

$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^n F_iF_j .$$

Knuth [10] and others have restated (1.1) neatly in its determinant form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n \tag{1.2}$$

and proceeded to generalize this format by induction to an elegant arbitrary order, with examples from a variety of well-known sequences. This, for the third-order Tribonacci sequences $\{T_n\}$, his generalization was, in effect,

$$\begin{vmatrix} T_{n+2} & T_{n+1} & T_n \\ T_{n+1} & T_n & T_{n-1} \\ T_n & T_{n-1} & T_{n-2} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \tag{1.3}$$

which is a neat extension of (1.1) and (1.2); but, the left-hand side of (1.3) when expanded is

$$\begin{vmatrix} T_{n+2} & T_{n+1} & T_n \\ T_{n+1} & T_n & T_{n-1} \\ T_n & T_{n-1} & T_{n-2} \end{vmatrix} = T_{n+2}\left(T_nT_{n-2} - T_{n-1}^2\right) - T_{n+1}\left(T_{n+1}T_{n-2} - T_n^2\right) + T_n\left(T_{n+1}T_{n-2} - T_n^2\right).$$

This raises the question which we seek to answer: whether there can be a succinct and elegant generalization to arbitrary order, with connections to other recursive sequences, which includes a format such as either of the following for arbitrary order *r*,

$$F_{n+1}^{(r)}F_{n-1}^{(r)}-\left(F_{n}^{(r)}\right)^{2} \quad \text{or} \quad F_{n+r-1}^{(r)}F_{n+r-3}^{(r)}-\left(F_{n+r-2}^{(r)}\right)^{2},$$

both of which would have the format of the left-hand side of (1.1) when $r = 2$, such as conjectured in (3.4) and (3.5) below, rather than the longer determinantal expansion when $r = 3$ as above, which obviously becomes less simple as $r$ increases.

# 2 Some preliminary notation

$r$ is the order of the recurrence relation which, with its initial terms, defines a recursive sequence [17]. Let $\{x_n^{(s)}\}$ symbolize an "$r$-related sequence of order $s$", which satisfies the $s$-th order recurrence relation with yet to be initial terms

$$\left.\begin{aligned} x_n^{(s)} &= \sum_{i=1}^{s}(-1)^i Q_{r,i}x_{n-i}^{(s)}. && n > 0,\\ x_n^{(s)} &= -1, && n = 0,\\ x_n^{(s)} &= 0, && n < 0, \end{aligned}\right| \tag{2.1}$$

in which the $Q_{r,i}$ are integer functions of $\alpha_{r,i}$ (Equation (3.5)), and $s = \binom{r}{2}$, with $f_2(x)$ as its auxiliary equation, in which

$$f_2(x) = \prod_{\substack{i,j=1\\ i<j}}^{r}\left(x-\alpha_{r,i}\alpha_{r,j}\right) \tag{2.2}$$

and $\alpha_{r,i}$ are the roots, assumed distinct, of the other auxiliary equation for the sequence of generalized Fibonacci numbers of arbitrary order $r$, $\{u_n^{(r)}\}$, represented by

$$f_1(x)=\prod_{j=1}^{r}\left(x-\alpha_{r,j}\right). \tag{2.3}$$

When $r = 2$, $s = 1$, and $x_n^{(1)} = -x_{n-1}^{(1)} = -1$, and for the fundamental sequence of Lucas [10], Simson's identity then takes the form

$$\left(u_n^{(2)}\right)^2-\left(u_{n-1}^{(2)}\right)\left(u_{n+1}^{(2)}\right) = x_n^{(1)}, \tag{2.4}$$

which is what we seek to generalize. Table 1 is the incentive for our conjecture. We are seeking relations with other recurrence relations, if they exist, in a consistent manner. To put the discussion in a broader context, we utilize $r$ "basic" sequences of order $r$, $\{U_{s,n}^{(r)}\}$, $s = 1, 2, \ldots, r$, by the recurrence relation

$$U_{s,n}^{(r)} = \sum_{j=1}^{r}(-1)^{j+1}P_{r,j}U_{s,n-j}^{(r)},\ n > r, \tag{2.5}$$

with initial terms defined by the Kronecker delta: $U_{s,n}^{(r)} = \delta_{s,n}, n = 1, 2, \ldots, r$, and where the $P_{r,j}$ are arbitrary integers [16]. The adjective "basic" is used by analogy with the corresponding third order sequences of Bell [2]. To correspond with the second order "primordial" sequence of Lucas, we define $U_{o,n}^{(r)} = v_{n-1}^{(r)}$, which also satisfies (2.1) but has initial terms given by

$$U_{o,n}^{(r)} = \begin{cases} 0, & n < 1, \\ \sum_{j=1}^{r} \alpha_{r,j}^{n-1}, & 1 \le n \le r, \end{cases} \tag{2.6}$$

where the $\alpha_{r,j}$ are the distinct roots of $f_1(x)$ in (1.3). In the literature, generally only one basic sequence is mentioned, namely the fundamental one, but Gootherts [7] has shown a need for two basic second order sequences as well as the primordial sequence [12].

One of the basic sequences is labelled "fundamental" by analogy with Lucas' "fundamental" sequence, $U_{2,n}^{(2)}$. The sequence of arbitrary order, *r*, is labelled $\{U_{r,n+r}^{(r)}\}$. Since, this sequence is used frequently, we let $U_{r,n+r}^{(r)} = \{u_n^{(r)}\}$ for notational convenience. The fundamental nature of the sequence $\{u_n^{(r)}\}$ was illustrated by d'Ocagne (Dickson [5]) who effectively established that any element $\{w_n^{(r)}\}$ of the set $\Omega = \Omega(P_{r,1}, P_{r,2}, \ldots, P_{r,r})$ of all sequences which satisfy (2.1) can be expressed in terms of the fundamental sequence and the initial terms of $\{w_n^{(r)}\}$ [8]:

$$w_n^{(r)} = \sum_{j=0}^{r-1}\sum_{k=j}^{r-1} (-1)^{k-j} P_{r,k-j} u_{n-k}^{(r)} w_j^{(r)}, \; n \ge 0, P_{r,0} = 1. \tag{2.7}$$

# 3 Simson's identity

We define sequences

$$x_n^{(2)} = -x_{n-1}^{(2)} = \mp 1 \tag{3.1}$$

and

$$x_n^{(3)} = -x_{n-1}^{(3)} - x_{n-2}^{(3)} + x_{n-3}^{(3)}, \; n \ge 3, \tag{3.2}$$

with initial terms −1, 1, 0, so that the first few terms of $\{x_n^{(3)}\}$ are

$$\{x_n^{(3)}\} \equiv \{-1, 1, 0, -2, 3, -1, -4, 8, -5, -7, 20, 18, -9, \ldots\}. \tag{3.3}$$

which is effectively $\{ x_n^{(3)} \}$ = A057597 of Sloane [19]. For example, when $r = 2$,

$$x_n^{(2)} = -x_{n-1}^{(2)} = -1,$$

and for the fundamental sequence of Lucas [12], Simson's identity then takes the form

$$\left(u_n^{(2)}\right)^2 - \left(u_{n-1}^{(2)}\right)\left(u_{n+1}^{(2)}\right) = x_n^{(2)}, \tag{3.4}$$

For the Tribonacci sequence $\{w_n^{(3)}\} \equiv \{1, 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \ldots\}$ (A000073) of Sloane [19]), we have Table 1 which seems to support the conjecture that

$$u_{n-1}^{(3)} u_{n+1}^{(3)} - \left(u_n^{(3)}\right)^2 = x_n^{(3)}. \tag{3.5}$$

Table 1. Simson conjecture for Tribonacci numbers

| $n$ | $w_n^{(3)}$ | $\left(w_n^{(3)}\right)^2$ | $w_{n-1}^{(3)}w_{n+1}^{(3)}$ | $w_{n-1}^{(3)}w_{n+1}^{(3)}-\left(w_n^{(3)}\right)^2$ | $x_n^{(3)}$ |
|---|---|---|---|---|---|
| **0** | 1 | 1 | — | — | **–1** |
| **1** | 0 | 0 | 0 | 0 | **1** |
| **2** | 0 | 0 | 0 | 0 | **0** |
| **3** | 1 | 1 | 0 | **–1** | **–2** |
| **4** | 1 | 1 | 2 | **1** | **3** |
| **5** | 2 | 4 | 4 | **0** | –1 |
| **6** | 4 | 16 | 14 | **–2** | **–4** |
| **7** | 7 | 49 | 52 | **3** | **8** |
| **8** | 13 | 169 | 168 | **–1** | **–5** |
| **9** | 24 | 576 | 572 | **–4** | **–7** |
| **10** | 44 | 1936 | 1944 | **8** | **20** |
| **11** | 81 | 6561 | 6556 | **–5** | –18 |
| **12** | 149 | 22201 | 22194 | **–7** | –9 |
| **13** | 274 | 75096 | 75076 | **–20** | 47 |

These equations do not tell us much about the specific terms of $\{x_n^{(s)}\}$, but it seems that we can relate them to the initial terms of $\{u_n^{(r)}\}$, such as,

$$\left(u_0^{(r)}\right)^2 - u_{-1}^{(r)}u_1^{(r)} = 1 = x_0^{(s)};$$

$$\left(u_1^{(r)}\right)^2 - u_0^{(r)}u_2^{(r)} = \sum_{i<m}\alpha_{r,i}\alpha_{r,m} = x_1^{(s)};$$

$$\left(u_2^{(r)}\right)^2 - u_1^{(r)}u_3^{(r)} = \sum_{\sum\lambda=4}\prod_{i=1}^{r}\alpha_{r,i}^{\lambda_i} = x_2^{(s)}.$$

# 4 Related combinatorial properties

We see this in the following because we can also express $\{u_n^{(r)}\}$ in multinomial terms from Macmahon [13], namely, $u_n^{(r)}$ is the product sum of weight *n* of the terms of $P_{r,1}$, and $P_{r,j}$ is the product sum, *j* together of the terms of $P_{r,1}$. By the product sum of weight *n* of the quantities $a_1, a_2, \ldots$, we mean

$$h_n = \sum a_1^n + \sum a_1^{n-1}a_2 + \sum a_1^{n-2}a_2^2 + \cdots,$$

in which each $h_n$ is the sum of a number of symmetric functions each of which is related to a partition of the number *n*. Thus,

$$x_n^{(s)} = u_{2n}^{(r)} - \sum_{n<\lambda\le 2n} \alpha_{r,1}^{\lambda_1}\alpha_{r,2}^{\lambda_2}\cdots$$

$$= \sum_{\Sigma\lambda=2n} \alpha_{r,1}^{\lambda_1}\alpha_{r,2}^{\lambda_2}\cdots - \sum_{n<\lambda\le 2n} \alpha_{r,1}^{\lambda_1}\alpha_{r,2}^{\lambda_2}\cdots$$

and so

$$x_n^{(s)} = \sum_{\Sigma\lambda=2n}\prod_{i=1}^{r}\alpha_{r,i}^{\lambda_i}. \tag{4.1}$$

For example, when $r = 3$, $s = 3$, which seems to work for the cases listed in Table 1,

$$\left.\begin{aligned}
x_1^{(3)} &= \Sigma\alpha_{3,1}\alpha_{3,2}\\
x_2^{(3)} &= \Sigma\alpha_{3,1}^2\alpha_{3,2}^2 + \Sigma\alpha_{3,1}^2\alpha_{3,2}\alpha_{3,3}\\
x_3^{(3)} &= \Sigma\alpha_{3,1}^3\alpha_{3,2}^3 + \Sigma\alpha_{3,1}^3\alpha_{3,2}^2\alpha_{3,3} + \alpha_{3,1}^2\alpha_{3,2}^2\alpha_{3,3}^2
\end{aligned}\right| \tag{4.2}$$

Each term $x_n^{(s)}$ of the *r*-related sequence of order *s* seems to be the product sum of weight *n* of the quantities $\alpha_{r,i}\alpha_{r,j}$ $(i < j)$, such that

$$x_n^{(s)} = \sum_{\Sigma\lambda=2n}\prod_{i=1}^{r}\alpha_{r,i}^{\lambda_i}$$

and

$$u_n^{(r)} = \sum_{\Sigma\lambda=n}\prod_{i=1}^{r}\alpha_{r,i}^{\lambda_i}$$

so that

$$x_n^{(s)} = \sum_{i=1}^{r}\alpha_{r,i}^{r+2n-1} / \prod_{i>j}\left(\alpha_{r,i} - \alpha_{r,j}\right) \tag{4.3}$$

which is expressed entirely in terms of the zeros of $f_1(x)$ rather than $f_2(x)$. Thus, we seem to have the format

$$x_n^{(s)} = \sum_{\Sigma i\mu_i=n}(-1)^{n+\Sigma\mu}\frac{(\Sigma\mu!)}{\mu_1!\ldots\mu_n!}\prod_{i=1}^{r}Q_{r,i}^{\mu_i}, \tag{4.4}$$

the first few terms of which are

$$x_0^{(s)} = 1,$$

$$x_1^{(s)} = Q_{r,1},$$

$$x_2^{(s)} = Q_{r,1}^2 - Q_{r,2},$$

$$x_3^{(s)} = Q_{r,1}^3 - 2Q_{r,1}Q_{r,2} + Q_{r,3},$$

which conform with (4.2) when the terms are simplified, such as when $r = 3$, as follows,

$$
\begin{aligned}
x_0^{(s)} &= 1, \\
x_1^{(s)} &= \sum \alpha_{3,1}\alpha_{3,2} \\
x_2^{(s)} &= \left(\sum \alpha_{3,1}\alpha_{3,2}\right)^2 - \left(\sum \alpha_{3,1}^2\alpha_{3,2}\alpha_{3,3}\right) \\
x_3^{(s)} &= \left(\sum \alpha_{3,1}\alpha_{3,2}\right)^3 + \left(\alpha_{3,1}^2\alpha_{3,2}^2\alpha_{3,3}^2\right) - 2\left(\sum \alpha_{3,1}\alpha_{3,2}\right)\left(\sum \alpha_{3,1}^2\alpha_{3,2}\alpha_{3,3}\right).
\end{aligned}
\tag{4.5}
$$

# 5 Reciprocals

The following theorems harmonize with, and are developed in the style of the family of reciprocal series involving Fibonacci and harmonic numbers [6], which also carries recent developments of the Riemann Zeta function, referred to in the next section.

**Theorem 1.** (Reciprocal of Simson [3, 10]): *Let* $\mathbb{N}$ ={0, 1, 2, ...} *and* A: $\mathbb{N} \rightarrow \mathbb{N}$ *on application such that* $A(0)=1, A(1)=2,$ *and*

$$A(n-1).A(n+1)-\left(A(n)\right)^2 =(-1)^n \text{ for } n\geq 1, \tag{5.1}$$

*then*

$$A(n+1)=A(n)+A(n-1) \quad \forall n\geq 1. \tag{5.2}$$

*Proof.* Applying (5.1) for $n$ = 1, we get $A(2)=3,$ so (5.2) is true for $n$ = 1. Now assume that (5.2) is valid for $n$, and we shall prove it for $n$ + 1. Applying (5.1) for $n$ and $n \rightarrow n$ +1, one has in turn

$$A(n+1).A(n-1)=\left(A(n)\right)^2+(-1)^n , \tag{5.3}$$

$$A(n+2).A(n)=\left(A(n+1)\right)^2+(-1)^{n+1} . \tag{5.3′}$$

On adding (5.3) and (5.3′), we get

$$A(n+2).A(n)+A(n+1).A(n-1)=\left(A(n+1)\right)^2+\left(A(n)\right)^2$$

so

$$A(n)\left(A(n+2)-A(n)\right)=\text{A(n+1)}\left(A(n+1)-A(n-1)\right). \tag{5.4}$$

By the inductive hypothesis, one then has

$$A(n+1)-A(n-1)=A(n),$$

so by (5.4) we obtain

$$A(n)\left(A(n+2)-A(n)\right)=A(n+1).A(n). \tag{5.5}$$

Now as $A(n)\geq 1 \forall n$ (which follows from (5.1) and also from a simple induction), by (5.5) we get $A(n+2)-A(n)=A(n+1)$, so that (5.2) holds for "$n$ + 1" too. □

**Theorem 2** (Reciprocal of Mangan [14]). *Assume that* A: $\mathbb{N} \to \mathbb{N}$ *satisfies* $A(0) = 1$, $A(1) = 2$, $A(2) = 3$, *and*

$$A(n-1).A(n-2) - A(n).A(n-3) = (-1)^n \text{ for } n \geq 3, \tag{5.6}$$

*then*

$$A(n+1) = A(n) + A(n-1) \forall n \geq 1. \tag{5.7}$$

*Proof.* From (5.6) we get $A(3) = 5$, so (5.7) holds true for $n = 1$ and $n = 2$. Assume that (5.7) holds true for $n - 1$ and $n$. We shall prove that (5.7) also holds for $n + 1$. Thus assume

$$A(n+1) - A(n-1) = A(n)$$

and

$$A(n) - A(n-2) = A(n-1).$$

Applying (5.6) for $n + 1$ and $n + 2$, one has

$$\left.\begin{aligned} A(n).A(n-1) - A(n+1).A(n-2) &= (-1)^{n+1} \\ A(n+1).A(n) - A(n+2).A(n-1) &= (-1)^{n+2}. \end{aligned}\right| \tag{5.8}$$

By adding the two relations of (5.8), we find

$$A(n+1)\left[A(n) - A(n-2)\right] + A(n-1)\left[A(n) - A(n+2)\right] = 0,$$

and reducing with $A(n-1) \geq 1$, we get

$$A(n+1) = A(n+2) - A(n);$$

that is, (5.7) is valid for $n + 1$. □

**Theorem 3** (Reciprocal of Gelin-Cesàro identity [5, 15]). *Assume that* A: $\mathbb{N} \to \mathbb{N}$ *satisfies* $A(0) = 1, A(1) = 2, A(2) = 3, A(3) = 5$, *and if*

$$A(n-2)A(n+2)A(n-1)A(n+1) = \left(A(n)\right)^4 - 1 \textit{ for } n \geq 2, \tag{5.9}$$

*then we also have*

$$A(n+1) = A(n) + A(n-1) \quad \forall n \geq 1. \tag{5.10}$$

*Proof:* From (5.9) we get $A(4) = 8$, so (5.10) holds true for $n = 1$, 2 and 3. Assume that (5.10) holds true for $n - 1$, $n$ and $n + 1$, and we shall prove it true for $n + 2$. Thus, assume

$$A(n) = A(n-1) + A(n-2)$$
$$A(n+1) = A(n) + A(n-1) \qquad (5.11)$$
$$A(n+2) = A(n+1) + A(n),$$

and we wish to prove that

$$A(n+3) = A(n+2) + A(n+1).$$

Using (5.9) for $n$ and $n + 1$, one has

$$A(n-2)A(n+2)A(n-1)A(n+1) = (A(n))^4 - 1$$
$$A(n-1)A(n+3)A(n)A(n+2) = (A(n+1))^4 - 1. \qquad (5.12)$$

By subtraction in (5.12), we get

$$A(n+2)A(n-1)\left[A(n+3)A(n) - A(n+1)A(n-2)\right]$$
$$= (A(n+1))^4 - (A(n))^4$$
$$= [A(n+1) - A(n)].[A(n+1) + A(n)].\left[(A(n+1))^2 + (A(n))^2\right] \qquad (5.13)$$
$$= [A(n-1)].[A(n+2)].\left[(A(n+1))^2 + (A(n))^2\right]$$

from which we can obtain

$$A(n+3)A(n) = A(n+1)A(n-2) + (A(n+1))^2 + (A(n))^2. \qquad (5.14)$$

From (5.11) we obtain

$$A(n-2) = A(n) - A(n-1) = 2A(n) - A(n+1),$$

So, from (5.14), we have

$$A(n+3)A(n) = A(n+1)\left[2A(n) - A(n+1)\right] + (A(n+1))^2 + (A(n))^2$$
$$= A(n+1)\left[2A(n) - A(n+1)\right] + (A(n+1))^2 + \left[A(n+2) - A(n+1)\right]^2$$
$$= 2A(n)A(n+1) - (A(n+1))^2 + (A(n+1))^2 + (A(n+2))^2$$
$$-2A(n+2)A(n+1) + (A(n+1))^2$$
$$= 2A(n)A(n+1) + (A(n+1))^2 + (A(n+2))^2 + 2A(n+2)A(n+1)$$
$$= 2A(n)A(n+1) + \left[A(n+2) - A(n+1)\right]^2$$
$$= 2A(n)A(n+1) + (A(n))^2.$$

By reducing with $A(n)$ up to

$$A(n+3)=2A(n+1)+A(n)$$
$$=A(n+1)+(A(n+1)+A(n))$$
$$=A(n+1)+A(n+2);$$

thus, (5.10) is valid also for $n + 2$. □

# 6 Conclusion

Vajda [16, 21] has supplied other elegant generalizations which are worth extending, such as

$$F_{n+i}F_{n+j}-F_nF_{n+i+j}=(-1)^n F_iF_j\,. \tag{6.1}$$

The story is not over though, because in this vein, van der Poorten [22] proved that (in our notation) the sequence $\{(w_n^{(r)})^i\}$ satisfies a linear recurrence relation of order $\binom{i+r-1}{i}$ with auxiliary polynomial

$$f_i(\lambda_i x)=\prod_{\sum\lambda_n=i}\left(x-\alpha_{r,1}^{\lambda_1}\alpha_{r,2}^{\lambda_2}\cdots\alpha_{r,r}^{\lambda_r}\right), \tag{6.2}$$

the zeros of which are the roots of $f_1(x) = 0$ taken $i$ at a time. If $i = 2$ and $\lambda_1 = \lambda_2 = 1$, then $f_i(\lambda_1 x) = f_2(x)$, and if $i = 1$ and $\lambda_1 = 1$, then $f_i(\lambda_1 x) = f_1(x)$, as in Section 1. The point of mentioning these is for the interested reader to search whether Williams' [23, 24] generalized Lucas numbers can lead to a more elegant generalized Simson's identity. These numbers, $\{L_{s,n}^{(r)}\}$, are defined in effect by

$$L_{s,n}^{(r)}=d^{-s}\sum_{j=1}^{r}\alpha_{r,j}^{n}\varsigma_r^{s(j-1)}\,,\ s=0,1,\dots,r-1, \tag{6.3}$$

in which $\varsigma_m=\exp(2\pi i/m)$, the Riemann Zeta Function $i^2 = -1$, is a modification of Carlitz [4], and $d$ is some real number for $r > 2$; for $r = 2$, $d$ is the difference between the roots, $\alpha_{2,1}$ and $\alpha_{2,2}$, of the auxiliary equation, as usual. Thus, when $r = 2$, Equation 5.2 becomes an ordinary Lucas number, as in Section 2,

$$L_{0,n}^{(2)}=\alpha_{2,1}^{n}+\alpha_{2,2}^{n}=v_n^{(2)}. \tag{6.4}$$

These generalized Lucas numbers, $\{L_{s,n}^{(r)}\}$, are related to $\{H_{s,n}^{(r)}\}$, another arbitrary order recursive sequence defined by Williams [22, 23] in an additive formula

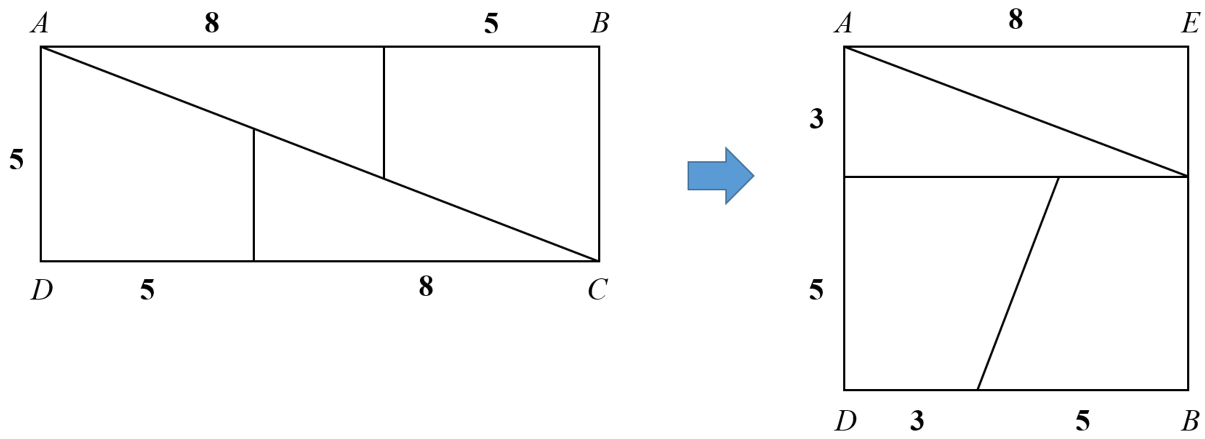
$$H_{s,n+m}^{(r)}=\sum_{h=0}^{r-1}\sum_{j=0}^{r-1}U_{s,h+j}^{(r)}H_{h,n}^{(r)}H_{j,m}^{(r)}, \tag{6.5}$$

which was simplified by Shannon [18]

$$H_{s,m}^{(r)} = \sum_{j=1}^{r} U_{s,j}^{(r)} H_{j,m}^{(r)}, \tag{6.6}$$

of which the main properties (in the sense of current fashions) have yet to be explored.

Finally, the Simson Identity is sometimes presented in disguise as a Fibonacci puzzle of how a rectangle of area 65 $cm^2$ can apparently be transformed into a square of 64 $cm^2$. What is the geometry of any generalized Simson's identity [1, 3, 9]?

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