Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2024, Volume 30, Number 3, 463–478
DOI: 10.7546/nntdm.2024.30.3.463-478

Euler sine product and the continued fraction of π

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Received: 12 September 2023 **Revised:** 25 August 2024
Accepted: 27 August 2024 **Online First:** 28 August 2024

Abstract: The Euler sine product and the continued fraction of π are discussed in this article. Some of the infinite series for cotangent and its derivative are obtained by implementing the concept of Euler sine product and some of the standard series are derived as the immediate consequence of the main results. Furthermore, the continued fraction for odd powers of π similar to the expression of π derived by Brouncker is presented in this article. Meanwhile, an expression relating the Basel's constant and the cotangent function is obtained as follows:

$$\frac{\coth r}{2} - \frac{1}{2r} = \sum_{n\in\mathbb{N}} \frac{2^{2n}}{2(2n)!} B_{2n} r^{2n-1}.$$

Keywords: Euler product, Continued fraction, Basel constants, Series.
2020 Mathematics Subject Classification: 11A41, 11A55, 11A67.

1 Introduction

Euler's sine product formula is typically discussed in the context of complex numbers, but it can also be applied in the field of real numbers. When the formula is applied to real numbers, it provides a useful representation of the product of two sine functions in terms of exponential functions. For $x, y \in \mathbb{R}$, the formula can be written as:

$$2 \sin x \sin y = \cos (x - y) - \cos (x + y).$$

This formula is useful in various areas of mathematics and engineering, including trigonometry, differential equations, signal processing, etc. For example, in trigonometry, Euler's sine product can be used to evaluate products of sine functions, which are important in many applications, such as in the solution of boundary-value problems in differential equations. In signal processing, Euler's sine product can be used to represent signals in terms of their frequency components, which are then used for processing and analysis. Overall, while the formula is most commonly discussed in the context of complex numbers, it has important applications in the field of real numbers, as well. Oscar [12] mentions that Euler conjectured

$$\frac{\sin x}{x} = \prod_{n \in \mathbb{N}} \left(1 - \frac{x^2}{n^2 \pi^2} \right),$$

which was obtained by factorizing $\sin x$ into a product over its roots as one would a polynomial. Further, Holst [4] proved the infinite sine product of Euler using the Gamma function and the elementary probability theory. Sandifer [14] also described Euler's sine product and gave some Euler's earlier results that led up to some interesting discoveries.

An expression of a number as the sum of an integer and a quotient, the denominator of which is the sum of an integer and a quotient, and so on is called a continued fraction. In general, the continued fraction for some $n \in \mathbb{N}$ is given by

$$n = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdots}}},$$

where a_i and b_i are integers. The continued fraction is called simple continued fraction if all b_i are equal to 1 and all the a_i are positive integers. The unusual patterns of the continued fractions have always fascinated mathematicians and created an interest for them to continue their discoveries. Among such fascinated mathematicians, Lord William Brouncker was the one who gave a beautiful formula for π in terms of a continued fraction:

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cdots}}}}. \tag{1}$$

In the year 1982, Dutka [3] proved Brouncker's continued fraction where he made use of various methods to obtain the Brouncker's continued fraction. Furthermore, Lange [10] also derived a continued fraction for π as

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \cdots}}}}.$$

Osler [13] mentioned some of the formulas for π in terms of continued fractions and also discussed various techniques to derive them.

In 1735, after solving the Basel's problem, Euler started to generalize the Reimann Zeta function for $2n$ and obtained the following expression [15]

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n}. \tag{2}$$

Ayoub [1] has briefed the work of Euler on the Reimann–zeta function. Debnath [2] talks about Euler's contribution to the mathematical world and has provided a generalized proof for Equation (2).

Mathematicians say that series are the heart of real analysis, it has many real-world applications which were only possible due to many great works of mathematicians in history. The most valuable contribution in the field of series was given by Jacobi and Ramanujan. Nimbran [11] has derived a new product expansion for $\sin n\pi$ and has also derived many algebraic irrational–free infinite products for π. Varadarajan [16] has proved some of Euler's important series. Many new series of Euler's works have been discussed by Kim's [5]. Kim [8] has also shown the value of moments of Poisson random variable associated with degenerate special numbers. There are also some work by Kim [7] which suggest the probabilistic extension of Bernoulli polynomials and Euler polynomials. There are also some new classes of sequences related to fully degenerate Bernoulli numbers and polynomials discussed by Kim [6]. From those sequences, he derive some formulae for the degenerate Bernoulli and Euler polynomials. There is also extended or fully degenerate Bernoulli polynomials and numbers, which are a degenerate version of Bernoulli polynomials and numbers and arise naturally from the Volkenborn integral of the degenerate exponential functions studied by Kim [9].

2 Main results

Lemma 1. *For any $x \in \mathbb{R}$,*

$$\frac{1}{2x^2} - \frac{\cot x}{2x} = \frac{1}{(\pi - x)(\pi + x)} + \frac{1}{(2\pi - x)(2\pi + x)} + \frac{1}{(3\pi - x)(3\pi + x)} + \cdots \tag{3}$$

Proof.

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\cdots$$

$$\ln\sin x = \ln(x) + \ln\left(1-\frac{x^2}{\pi^2}\right) + \ln\left(1-\frac{x^2}{4\pi^2}\right) + \ln\left(1-\frac{x^2}{9\pi^2}\right) + \cdots \tag{4}$$

Differentiating Equation (4) with respect to x, we get:

$$\cot x = \frac{1}{x} + \left(\frac{1}{1-\frac{x^2}{\pi^2}}\right)\left(\frac{-2x}{\pi^2}\right) + \left(\frac{1}{1-\frac{x^2}{4\pi^2}}\right)\left(\frac{-2x}{\pi^2}\right) + \left(\frac{1}{1-\frac{x^2}{9\pi^2}}\right)\left(\frac{-2x}{16\pi^2}\right) + \cdots$$

$$\cot x = \frac{1}{x} - \frac{2x}{\pi^2-x^2} - \frac{2x}{4\pi^2-x^2} - \frac{2x}{9\pi^2-x^2} - \frac{2x}{16\pi^2-x^2} - \cdots$$

$$\frac{\cot x}{2x} - \frac{1}{2x^2} = -\left[\frac{1}{\pi^2-x^2} + \frac{1}{4\pi^2-x^2} + \frac{1}{9\pi^2-x^2} + \frac{1}{16\pi^2-x^2} + \cdots\right]$$

$$\frac{1}{2x^2} - \frac{\cot x}{2x} = \frac{1}{(\pi-x)(\pi+x)} + \frac{1}{(2\pi-x)(2\pi+x)} + \frac{1}{(3\pi-x)(3\pi+x)} + \cdots \tag{5}$$

Hence the proof. □

Lemma 2. *The alternating series for $\frac{6-\pi}{4}$ is*

$$\frac{6-\pi}{4} = 1 - \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} - \cdots .$$

Proof. Upon substituting $x = \frac{\pi}{2}$ in Equation (5), we get

$$\frac{2}{\pi^2} = \frac{2^2}{\pi\cdot 3\pi} + \frac{2^2}{3\pi\cdot 5\pi} + \frac{2^2}{5\pi\cdot 7\pi} + \cdots$$

$$\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots \tag{6}$$

Again by substituting $x = \frac{\pi}{4}$ in Equation (5) we get,

$$\frac{8}{\pi^2} - \frac{\cot\frac{\pi}{4}}{\frac{\pi}{2}} = \frac{4^2}{\pi^2(3\cdot 5)} + \frac{4^2}{\pi^2(7\cdot 9)} + \frac{4^2}{\pi^2(11\cdot 13)} + \cdots$$

$$\frac{1}{2} - \frac{\pi}{8} = \frac{1}{3\cdot 5} + \frac{1}{7\cdot 9} + \frac{1}{11\cdot 13} + \cdots \tag{7}$$

Subtracting Equation (6) from Equation (7), we get

$$\frac{\pi}{8} = \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{11\cdot 13} + \frac{1}{13\cdot 15} + \cdots \tag{8}$$

Subtracting Equation (7) from Equation (8), we get

$$\frac{\pi}{4} - \frac{1}{2} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$$

$$\frac{6-\pi}{4} = 1 - \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} - \cdots \tag{9}$$

Hence the theorem. □

Theorem 1. *The term* $\frac{4}{6-\pi}$ *can be expressed as the following continued fraction:*

$$\frac{4}{6-\pi} = 1 + \cfrac{1}{1\cdot 2 + \cfrac{(1\cdot 3)^2}{3\cdot 4 + \cfrac{(3\cdot 5)^2}{5\cdot 4 + \cfrac{(5\cdot 7)^2}{7\cdot 4 + \cdots}}}}.$$

Proof. The term $\frac{4}{6-\pi}$ can be expressed as the reciprocal of $\frac{6-\pi}{4}$ whose alternating series is determined in Lemma 2. Upon adding -1 we get,

$$\frac{4}{6-\pi} - 1 = \frac{1}{\frac{6-\pi}{4}} - 1 = \frac{1 - \frac{6-\pi}{4}}{\frac{6-\pi}{4}}. \tag{10}$$

Incorporating Lemma 2 here we get:

$$\frac{1 - \frac{6-\pi}{4}}{\frac{6-\pi}{4}} = \frac{1 - 1 + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots}{1 - \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} - \cdots}$$

Upon adding and subtracting $-\frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \frac{2}{9\cdot 11} - \cdots$ in the denominator we get:

$$= \frac{\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots}{\frac{2}{1\cdot 3} + \left[\begin{matrix}\left[-\frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \cdots\right] + \\ \left[\frac{2}{3\cdot 5} - \frac{2}{5\cdot 7} + \frac{2}{7\cdot 9 - \cdots}\right]\end{matrix}\right] + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \cdots}$$

$$= \frac{\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots}{2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots\right] + \frac{3}{3\cdot 5} - \frac{3}{5\cdot 7} + \cdots}$$

$$= \cfrac{1}{2 + (1\cdot 3)^2 \cfrac{\frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} - \cdots}{1 - \frac{3}{3\cdot 5} + \frac{3}{5\cdot 7} - \frac{3}{7\cdot 9} - \cdots}}$$

$$= \cfrac{1}{2+\cfrac{(1\cdot 3)^2}{3\cdot 4+(3\cdot 5)^2\cfrac{\left[\frac{1}{5\cdot 7}-\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}-\frac{1}{9\cdot 11}+\cdots\right]}{1-\frac{3\cdot 5}{5\cdot 7}+\frac{3\cdot 5}{7\cdot 9}-\frac{3\cdot 5}{9\cdot 11}+\cdots}}}.$$

Continuing this process, we arrive at the following continued fraction

$$\frac{4}{6-\pi}-1=\cfrac{1}{1\cdot 2+\cfrac{(1\cdot 3)^2}{3\cdot 4+\cfrac{(3\cdot 5)^2}{5\cdot 4+\cfrac{(5\cdot 7)^2}{\cdots}}}}$$

By rearranging the terms we arrive at the result. □

Theorem 2. *For any $n \in \mathbb{N}$, $\forall\, k \geq 1$ and $x \in \mathbb{R}$, we have*

$$\frac{1}{2xk}\frac{d}{dx}\left(\frac{1}{2(k-1)x}\frac{d}{dx}\left(\cdots\left(\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{2x^2}-\frac{\cot x}{2x}\right)\right)\right)\right)=\sum_{n\in\mathbb{N}}\left[\frac{1}{(n\pi-x)(n\pi+x)}\right]^{k+1}. \quad (11)$$

Proof. By differentiating Equation (5) once with respect to x we get the case for $k=1$ and it is observed that the result holds true. Let $m<k$, then we arrive at the following expression $\forall\, m \geq 1$:

$$\frac{1}{2xm}\frac{d}{dx}\left(\frac{1}{2(m-1)x}\frac{d}{dx}\left(\cdots\left(\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{2x^2}-\frac{\cot x}{2x}\right)\right)\right)\right)=\sum_{n\in\mathbb{N}}\left[\frac{1}{(n\pi-x)(n\pi+x)}\right]^{m+1}. \quad (12)$$

Replacing m by $m+1, \forall m>1$ on the left-hand side of Equation (12), we get

$$\frac{1}{2x(m+1)}\frac{d}{dx}\left(\frac{1}{2mx}\frac{d}{dx}\left(\cdots\left(\frac{1}{4x}\frac{d}{dx}\left(\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{2x^2}-\frac{\cot x}{2x}\right)\right)\right)\right)\right). \quad (13)$$

Incorporating Equation (12) in Equation (13) we arrive at

$$\frac{1}{2x(m+1)}\frac{d}{dx}\left(\sum_{n\in\mathbb{N}}\left[\frac{1}{(n\pi-x)(n\pi+x)}\right]^{m+1}\right). \quad (14)$$

Upon simplifying the above equation, we arrive at

$$\sum_{n\in\mathbb{N}}\left[\frac{1}{(n\pi-x)(n\pi+x)}\right]^{m+2}. \quad (15)$$

Thus by induction, we prove that $\forall\, k \geq 1$

$$\frac{1}{2xk}\frac{d}{dx}\left(\frac{1}{2(k-1)x}\frac{d}{dx}\left(\cdots\left(\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{2x^2}-\frac{\cot x}{2x}\right)\right)\right)\right)=\sum_{n\in\mathbb{N}}\left[\frac{1}{(n\pi-x)(n\pi+x)}\right]^{k+1}.$$

The convergence of Equation (11) is shown in Figure 1. □

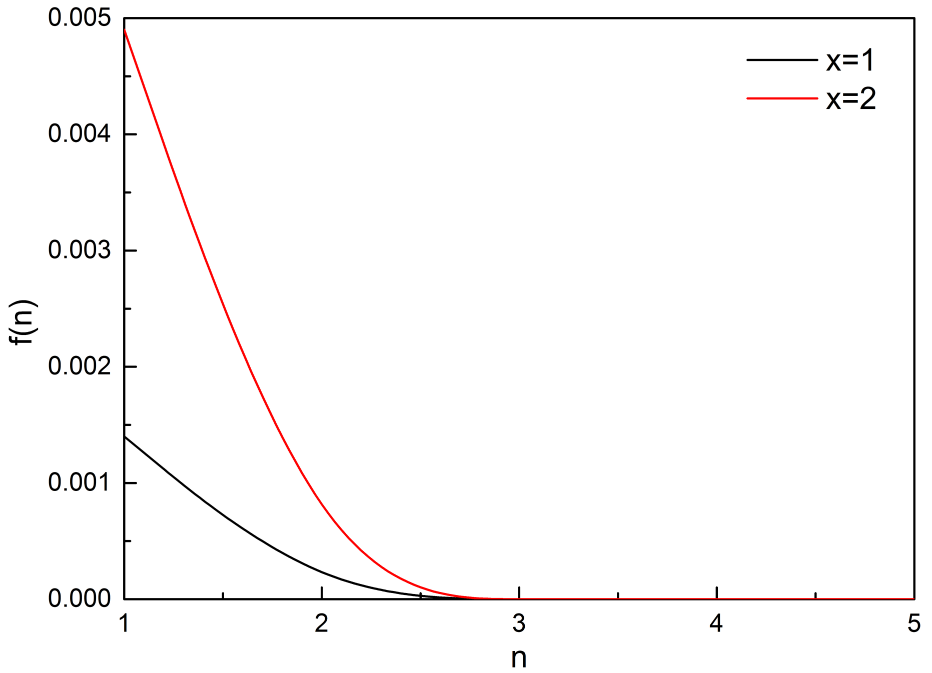

Figure 1. The graph showing the convergence of $\sum_{n\in\mathbb{N}}\left[\frac{1}{(n\pi-x)(n\pi+x)}\right]^{k+1}$

Observation 1. *For $n,\ k\in\mathbb{N}$ and $x=\frac{\pi}{2}$, the above theorem gives the following infinite series:*

$$\left(\frac{\pi^2}{4}\right)^{k+1}\frac{1}{2xk}\frac{d}{dx}\left(\cdots\left(\frac{1}{4x}\frac{d}{dx}\left(\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{2x^2}-\frac{\cot x}{2x}\right)\right)\right)\right)=\sum_{n\in\mathbb{N}}\left[\frac{1}{(2n+1)(2n-1)}\right]^{k+1}.$$

Observation 2. *For $n,\ k\in\mathbb{N}$ and $x=\frac{\pi}{4}$, the above theorem gives the following infinite series:*

$$\left(\frac{\pi^2}{16}\right)^{k+1}\frac{1}{2xk}\frac{d}{dx}\left(\cdots\left(\frac{1}{4x}\frac{d}{dx}\left(\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{2x^2}-\frac{\cot x}{2x}\right)\right)\right)\right)=\sum_{n\in\mathbb{N}}\left[\frac{1}{(4n+1)(4n-1)}\right]^{k+1}.$$

Theorem 3. *For any $x\in\mathbb{R}$ and $n\in\mathbb{N}$, the following equation holds true:*

$$\frac{1}{(2n-2)!}\left(\frac{d^{2n-2}}{dx^{2n-2}}\cot x\right)=\sum_{k=0}^{\infty}\left(\frac{1}{k\pi+x}\right)^{2n-1}-\sum_{k\in\mathbb{N}}\left(\frac{1}{k\pi-x}\right)^{2n-1}. \tag{16}$$

Proof. For the case when $n=1$, it can be concluded that the equality holds true as we get $\cot x=\frac{1}{x}-\frac{1}{\pi-x}+\frac{1}{\pi+x}-\frac{1}{2\pi-x}+\frac{1}{2\pi-x}-\cdots$. Thus it can be assumed that for some $m<n$,

$$\frac{1}{(2m-2)!}\left(\frac{d^{2m-2}}{dx^{2m-2}}\cot x\right)=\sum_{k=0}^{\infty}\left(\frac{1}{k\pi+x}\right)^{2m-1}-\sum_{k\in\mathbb{N}}\left(\frac{1}{k\pi-x}\right)^{2m-1} \tag{17}$$

We now proceed with the induction process by considering the $(m+1)^{\text{th}}$ iteration on the left-hand side of Equation (17) as follows:

$$\frac{1}{(2m)!}\left(\frac{d^{2k}}{dx^{2k}}\cot x\right)=\frac{1}{(2m)!}\left(\frac{d^2}{dx^2}\left(\frac{d^{2m-2}}{dx^{2m-2}}\cot x\right)\right) \tag{18}$$

By incorporating Equation (17) into the above equation we arrive at the following expression

$$\frac{(2m-2)!}{(2m)!}\left[\frac{d^2}{dx^2}\left(\sum_{k=0}^{\infty}\left(\frac{1}{k\pi+x}\right)^{2m-1}-\sum_{k\in\mathbb{N}}\left(\frac{1}{k\pi-x}\right)^{2m-1}\right)\right]$$

$$=\frac{(2m-2)!}{(2m)!}\left[(2m)(2m-1)\left(\begin{array}{l}\frac{1}{x^{2k+1}}-\frac{1}{(\pi-x)^{2m+1}}+\frac{1}{(\pi+x)^{2m+1}}-\\ \frac{1}{(2\pi-x)^{2m+1}}+\frac{1}{(2\pi+x)^{2m+1}}-\cdots\end{array}\right)\right]$$

$$=\frac{1}{x^{2m+1}}-\frac{1}{(\pi-x)^{2m+1}}+\frac{1}{(\pi+x)^{2m+1}}-\frac{1}{(2\pi-x)^{2m+1}}+\frac{1}{(2\pi+x)^{2m+1}}+\cdots$$

Thus by induction, it can be concluded that the equality described in Equation (16) holds true. The convergence of $\frac{1}{(2n-2)!}\left(\frac{d^{2n-2}}{dx^{2n-2}}\cot x\right)=\sum_{k=0}^{\infty}\left(\frac{1}{k\pi+x}\right)^{2n-1}-\sum_{k\in\mathbb{N}}\left(\frac{1}{k\pi-x}\right)^{2n-1}$ is shown in Figure 2. □

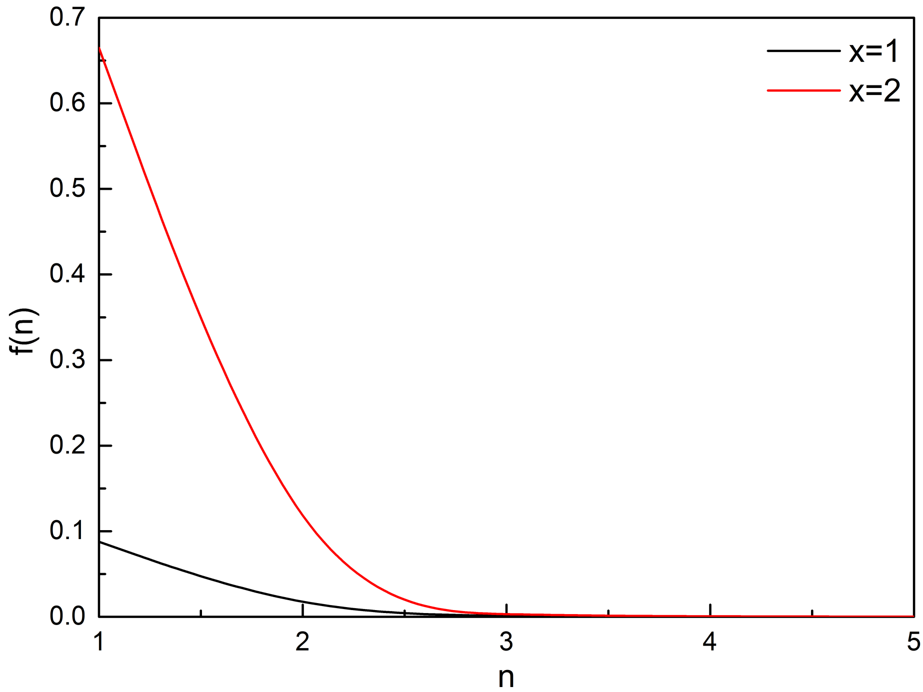

Figure 2. The graph showing the convergence of $\sum_{k=0}^{\infty}\left(\frac{1}{k\pi+x}\right)^{2n-1}-\sum_{k\in\mathbb{N}}\left(\frac{1}{k\pi-x}\right)^{2n-1}$

Observation 3. *The famous infinite $\pi-$series described as $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots$ can be obtained by evaluating Equation* (16) *at $x=\frac{\pi}{4}$ whenever $n=1$.*

Observation 4. *Upon substituting $x=\frac{\pi}{4}$ in Equation* (16) *we arrive at the following alternating series:*

$$\left(\frac{\pi}{4}\right)^{2n-1}\frac{1}{(2n-2)!}\left[\frac{d^{2n-2}}{dx^{2n-2}\cot x}\right]_{x=\frac{\pi}{4}}=1-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\frac{1}{7^{2n-1}}+\frac{1}{9^{2n-1}}-\frac{1}{11^{2n-1}}+\cdots \quad (19)$$

With the help of the previous theorem and its observations, we now head towards arriving at a continued fraction for odd powers of π similar to that of Brouncker's fraction for π given in Equation (1).

Theorem 4. *For any natural number $n \geq 1$ and $x = \frac{\pi}{4}$ the Equation* (19) *will satisfy the following continued fraction:*

$$\pi^{2n-1} = \cfrac{\cfrac{4^{2n-1}(2n-2)!}{\left[\frac{d^{2n-2}}{dx^{2n-2}}\cot x\right]_{x=\frac{\pi}{4}}}}{1^n + \cfrac{1^{2(2n-1)}}{3^n - 1^n + \cfrac{3^{2(2n-1)}}{5^n - 3^n + \cfrac{5^{2(2n-1)}}{7^n - 5^n + \cfrac{7^{2(2n-1)}}{9^n - 7^n + \cdots}}}}}. \tag{20}$$

Proof. From observation (19) we have,

$$\frac{4^{2n-1}(2n-2)!}{\pi^{2n-1}\frac{d^{2n-2}}{dx^{2n-2}}\cot x} - 1 = \frac{1}{\frac{\pi^{2n-1}\frac{d^{2n-2}}{dx^{2n-2}}\cot x}{4^{2n-1}(2n-2)!}} - 1 = \frac{1 - \frac{\pi^{2n-1}\frac{d^{2n-2}}{dx^{2n-2}}\cot x}{4^{2n-1}(2n-2)!}}{\frac{\pi^{2n-1}\frac{d^{2n-2}}{dx^{2n-2}}\cot x}{4^{2n-1}(2n-2)!}} \tag{21}$$

By implementing observation 19 to the above equation we arrive at the following equation:

$$\frac{4^{2n-1}(2n-2)!}{\pi^{2n-1}\frac{d^{2n-2}}{dx^{2n-2}}\cot x} - 1 = \frac{\frac{1}{3^{2n-1}} - \frac{1}{5^{2n-1}} + \frac{1}{7^{2n-1}}\cdots}{1 - \frac{1}{3^{2n-1}} + \frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}}\cdots}$$

Upon adding and subtracting $-\frac{3^{2n-1}-1}{5^{2n-1}} + \frac{3^{2n-1}-1}{7^{2n-1}} - \frac{3^{2n-1}-1}{9^{2n-1}} + \frac{3^{2n-1}-1}{11^{2n-1}} - \cdots$ in the denominator of the right-hand side of (21) we get:

$$= \frac{\frac{1}{3^{2n-1}} - \frac{1}{5^{2n-1}} + \frac{1}{7^{2n-1}}\cdots}{\frac{3^{2n-1}-1}{3^{2n-1}} + \left[\begin{array}{c}\left[-\frac{3^{2n-1}-1}{5^{2n-1}} + \frac{3^{2n-1}-1}{7^{2n-1}} + \cdots\right] + \\ \left[\frac{3^{2n-1}-1}{5^{2n-1}} - \frac{3^{2n-1}-1}{7^{2n-1}} - \cdots\right]\end{array}\right] + \frac{1}{5^{2n-1}} + \cdots}$$

$$= \cfrac{1}{3^{2n-1} + \cfrac{3^{2n-1}\left(\frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}}\frac{1}{9^{2n-1}} - \frac{1}{11^{2n-1}} + \cdots\right)}{\frac{1}{3^{2n-1}} - \frac{1}{5^{2n-1}} + \frac{1}{7^{2n-1}} - \frac{1}{9^{2n-1}} + \cdots}}$$

$$= \cfrac{1}{3^{2n-1} - 1 + \cfrac{3^{2(2n-1)}\left(\frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{11^{2n-1}} + \cdots\right)}{1 - \frac{3^{2n-1}}{5^{2n-1}} + \frac{3^{2n-1}}{7^{2n-1}} - \frac{3^{2n-1}}{9^{2n-1} + \cdots}}}$$

Upon adding and subtracting $-\frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{11^{2n-1}} + \cdots$ in the denominator we get:

$$= \cfrac{1}{3^{2n-1} - 1 + \cfrac{3^{2(2n-1)}\left[\frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{11^{2n-1}} + \cdots\right]}{(5^{2n-1} - 3^{2n-1})\left[\frac{1}{5^{2n-1}} + \begin{matrix}\left[-\frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \cdots\right] + \\ \left[\frac{1}{7^{2n-1}} - \frac{1}{9^{2n-1}} + \cdots\right]\end{matrix}\right] + \frac{3^{2n-1}}{7^{2n-1}} - \frac{3^{2n-1}}{9^{2n-1}} + \cdots}}$$

$$= \cfrac{1}{3^{2n-1} - 1 + \cfrac{3^{2(2n-1)}}{5^{2n-1} - 3^{2n-1} + \cfrac{5^{2(2n-1)}\left[\frac{1}{7^{2n-1}} - \frac{1}{9^{2n-1}} + \frac{1}{11^{2n-1}}\right]}{1 - \frac{5^{2n-1}}{7^{2n-1}} + \frac{5^{2n-1}}{9^{2n-1}} - \frac{5^{2n-1}}{11^{2n-1}} + \cdots}}}$$

Continuing this process infinitely we get the following continued fraction:

$$\frac{4^{2n-1}(2n-2)!}{\pi^{2n-1}\frac{d^{2n-2}}{dx^{2n-2}}\cot x} - 1 = \cfrac{1}{3^{2n-1} - 1 + \cfrac{3^{2(2n-1)}}{5^{2n-1} - 3^{2n-1} + \cfrac{5^{2(2n-1)}}{7^{2n-1} - 5^{2n-1} + \cdots}}} \tag{22}$$

Thus by rearranging the terms of the above equation we arrive at a continued fraction for π^{2n-1} and hence the theorem. □

Theorem 5. *For any natural number $k \in \mathbb{N}$, we have*

$$\pi^2 = n^2 \sin^2 \frac{\pi}{n}\left(1 + 2\sum_{k\in\mathbb{N}} \frac{1 + k^2n^2}{(1 - k^2n^2)^2}\right). \tag{23}$$

Proof. Differentiating Equation (5) with respect to x we get

$$\frac{1}{2}\csc x - \frac{1}{2x^2} = \frac{x^2 + \pi^2}{(x^2 - \pi^2)^2} + \frac{x^2 + 4\pi^2}{(x^2 - 4\pi^2)^2} + \frac{x^2 + 9\pi^2}{(x^2 - 9\pi^2)^2} + \cdots \tag{24}$$

By substituting $x = \frac{\pi}{n}$, where $n(> 1) \in \mathbb{N}$, we get

$$\frac{1}{2}\csc^2 \frac{\pi}{n} - \frac{n^2}{2\pi^2} = \frac{\frac{\pi^2}{n^2} + \pi^2}{\left(\frac{\pi^2}{n^2} - \pi^2\right)^2} + \frac{\frac{\pi^2}{n^2} + 4\pi^2}{\left(\frac{\pi^2}{n^2} - 4\pi^2\right)^2} + \frac{\frac{\pi^2}{n^2} + 9\pi^2}{\left(\frac{\pi^2}{n^2} - 9\pi^2\right)^2} + \cdots$$

$$\Rightarrow \frac{\pi^2}{n^2}\left(\frac{1}{2}\csc^2\frac{\pi}{n} - \frac{n^2}{2\pi^2}\right) = \frac{1+n^2}{(1-n^2)^2} + \frac{1+4n^2}{(1-4n^2)^2} + \frac{1+9n^2}{(1-9n^2)^2} + \cdots$$
$$\Rightarrow \frac{\pi^2}{2n^2}\csc^2\frac{\pi}{n} - \frac{1}{2} = \sum_{k\in\mathbb{N}} \frac{1+k^2n^2}{(1-k^2n^2)^2}$$
$$\Rightarrow \pi^2 = n^2\sin^2\frac{\pi}{n}\left(1 + 2\sum_{k\in\mathbb{N}} \frac{1+k^2n^2}{(1-k^2n^2)^2}\right).$$

The convergence of $\pi^2 = n^2\sin^2\frac{\pi}{n}\left(1+2\sum_{k\in\mathbb{N}}\frac{1+k^2n^2}{(1-k^2n^2)^2}\right)$ is shown in Figure 3. □

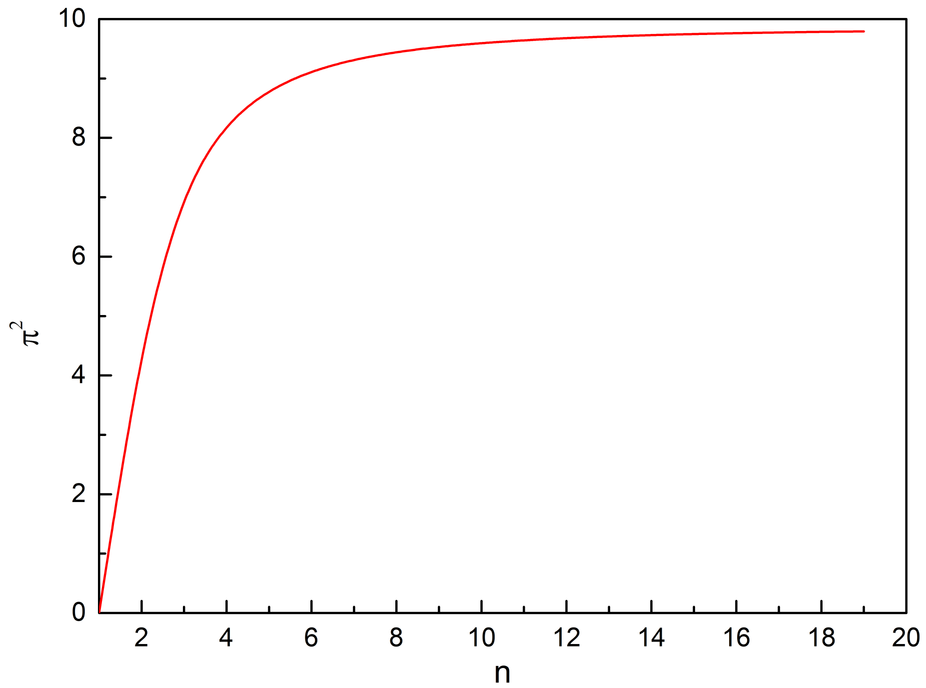

Figure 3. The graph showing the convergence of $n^2\sin^2\frac{\pi}{n}\left(1+2\sum_{k\in\mathbb{N}}\frac{1+k^2n^2}{(1-k^2n^2)^2}\right)$

Observation 5. *If $n = i$ is substituted in Equation (23), then*

$$\pi^2 = i\sinh\pi\left(1 + 2\sum_{k\in\mathbb{N}} \frac{1-k^2}{(1+k^2)^2}\right). \tag{25}$$

Observation 6. *For any natural number $k \in \mathbb{N}$ and $n = 2$, we have*

$$\frac{\pi}{2} = \sqrt{1 + 2\sum_{k\in\mathbb{N}} \frac{1+(2k)^2}{(1-(2k)^2)^2}}. \tag{26}$$

Lemma 3. *For any real number γ the following product holds true:*

$$\frac{\sinh\gamma\cdot\sin\gamma}{\gamma^2} = \prod_{n\in\mathbb{N}}\left[1 - \frac{\gamma^4}{n^4\pi^4}\right] \tag{27}$$

Proof. According to the expansion of $\sin x$, we have

$$\frac{\sin x}{x} = \prod_{n\in\mathbb{N}} \left(1 - \frac{x^2}{n^2\pi^2}\right) \tag{28}$$

Upon substituting $x = i\gamma$ in the above equation, we obtain $\frac{\sinh\gamma}{\gamma} = \prod_{n\in\mathbb{N}}\left(1 + \frac{\gamma^2}{n^2\pi^2}\right)$. Now, by substituting $x = \gamma$ in Equation (28) we get $\frac{\sin\gamma}{\gamma} = \prod_{n\in\mathbb{N}}\left(1 - \frac{\gamma^2}{n^2\pi^2}\right)$. By multiplying these two expressions we get:

$$\frac{\sinh\gamma\cdot\sin\gamma}{\gamma^2} = \prod_{n\in\mathbb{N}}\left(1 - \frac{\gamma^2}{n^2\pi^2}\right)\cdot\prod_{n\in\mathbb{N}}\left(1 + \frac{\gamma^2}{n^2\pi^2}\right) = \prod_{n\in\mathbb{N}}\left(1 - \frac{\gamma^4}{n^4\pi^4}\right). \tag{29}$$

This completes the proof. □

Lemma 4. *For any natural number n the following π series holds true:*

$$\frac{\pi}{2n}\left[\coth\frac{\pi}{n} + \cot\frac{\pi}{n}\right] = 1 - \sum_{k\in\mathbb{N}}\frac{1}{k^2n^2-1} + \sum_{k\in\mathbb{N}}\frac{1}{k^2n^2+1}. \tag{30}$$

Proof. With the help of the Lemma 4 we can see that:

$$\begin{aligned}\ln\left(\frac{\sinh\gamma\cdot\sin\gamma}{\gamma^2}\right) = \ln\left(1-\frac{\gamma^2}{\pi^2}\right) + \ln\left(1+\frac{\gamma^2}{\pi^2}\right) + \ln\left(1-\frac{\gamma^2}{4\pi^2}\right) + \\ \ln\left(1+\frac{\gamma^2}{4\pi^2}\right) + \ln\left(1-\frac{\gamma^2}{9\pi^2}\right) + \cdots\end{aligned} \tag{31}$$

by differentiating the above equation with respect to γ, we arrive at:

$$\begin{aligned}\coth\gamma + \cot\gamma - \frac{2}{\gamma} &= -\frac{2\gamma}{\pi^2-\gamma^2} + \frac{2\gamma}{\pi^2+\gamma^2} - \frac{2\gamma}{4\pi^2-\gamma^2} + \frac{2\gamma}{4\pi^2+\gamma^2} - \cdots \\ \frac{\coth\gamma+\cot\gamma}{2\gamma} &= \frac{1}{\gamma^2} - \frac{1}{\pi^2-\gamma^2} + \frac{1}{\pi^2+\gamma^2} - \frac{1}{4\pi^2-\gamma^2} + \frac{1}{4\pi^2+\gamma^2} + \cdots\end{aligned} \tag{32}$$

Replace γ by $\frac{\pi}{n}$, where $n\ (>1) \in \mathbb{N}$, then,

$$\begin{aligned}\frac{\coth\frac{\pi}{n} + \cot\frac{\pi}{n}}{2\frac{\pi}{n}} &= \frac{n^2}{\pi^2} - \frac{n^2}{n^2\pi^2-\pi^2} + \frac{n^2}{n^2\pi^2+\pi^2} - \frac{n^2}{4n^2\pi^2+\pi^2} + \frac{n^2}{4n^2\pi^2-\pi^2} - \cdots \\ \frac{\pi}{2n}\left[\coth\frac{\pi}{n} + \cot\frac{\pi}{n}\right] &= 1 - \frac{1}{n^2-1} + \frac{1}{n^2+1} - \frac{1}{4n^2-1} + \frac{1}{4n^2+1} - \cdots\end{aligned} \tag{33}$$

Hence the proof. □

Theorem 6. *For any natural number n the continued fraction for π can be expressed as*

$$\pi = \cfrac{2n}{\cfrac{\coth\frac{\pi}{n} + \cot\frac{\pi}{n}}{1 + \cfrac{1}{n^2 - 2 + \cfrac{(n^2-1)^2}{2 + \cfrac{(n^2+1)^2}{3n^2 - 2 + \cfrac{(4n^2-1)^2}{2+\cdots}}}}}}. \tag{34}$$

Proof. In Lemma 4, an expression is derived for $\frac{\pi}{2n}\left[\coth\frac{\pi}{n}+\cot\frac{\pi}{n}\right]$ which is further utilized to arrive at the continued fraction for π as stated in the statement. Consider

$$\frac{2n}{\pi\left[\coth\frac{\pi}{n}+\cot\frac{\pi}{n}\right]}-1=\frac{1}{\dfrac{\pi\left[\coth\frac{\pi}{n}+\cot\frac{\pi}{n}\right]}{2n}}-1=\frac{1-\dfrac{\pi\left[\coth\frac{\pi}{n}+\cot\frac{\pi}{n}\right]}{2n}}{\dfrac{\pi\left[\coth\frac{\pi}{n}+\cot\frac{\pi}{n}\right]}{2n}} \tag{35}$$

Now by implementing the expression obtained in Lemma 4 we get:

$$\frac{1-\dfrac{\pi\left[\coth\frac{\pi}{n}+\cot\frac{\pi}{n}\right]}{2n}}{\dfrac{\pi\left[\coth\frac{\pi}{n}+\cot\frac{\pi}{n}\right]}{2n}}=\frac{1-1+\dfrac{1}{n^2-1}-\dfrac{1}{n^2+1}+\dfrac{1}{4n^2-1}-\dfrac{1}{4n^2+1}+\cdots}{1-\dfrac{1}{n^2-1}+\dfrac{1}{n^2+1}-\dfrac{1}{4n^2-1}+\dfrac{1}{4n^2+1}-\cdots} \tag{36}$$

Now, add and subtract $-\frac{n^2-1-1}{n^2+1}+\frac{n^2-1-1}{4n^2+1}+\cdots$ in the denominator of the right-hand side of (36) we get:

$$\frac{\dfrac{1}{n^2-1}-\dfrac{1}{n^2+1}+\dfrac{1}{4n^2-1}-\dfrac{1}{4n^2+1}+\cdots}{\left[\begin{array}{l}\dfrac{n^2-1-1}{n^2-1}+\left(-\dfrac{n^2-1-1}{n^2+1}+\dfrac{n^2-1-1}{4n^2-1}+\cdots\right)+\\ \left(\dfrac{n^2-2}{n^2+1}-\dfrac{n^2-2}{n^2+1}+\cdots\right)+\dfrac{1}{n^2+1}-\dfrac{1}{4n^2-1}+\cdots\end{array}\right]} \tag{37}$$

By dividing the numerator and denominator by $\frac{1}{n^2-1}-\frac{1}{n^2+1}+\frac{1}{4n^2+1}-\frac{1}{4n^2-1}+\cdots$ Thus the above equation further extends to:

$$=\frac{1}{n^2-2+\dfrac{(n^2-1)\left(\dfrac{1}{n^2+1}-\dfrac{1}{4n^2-1}+\cdots\right)}{1-\dfrac{n^2-1}{n^2+1}+\dfrac{n^2-1}{4n^2-1}-\cdots}}$$

$$=\frac{1}{n^2-2+\dfrac{(n^2-1)^2\left(\dfrac{1}{n^2+1}-\dfrac{1}{4n^2-1}+\cdots\right)}{\left[\begin{array}{l}\dfrac{2}{n^2+1}+\left(-\dfrac{2}{4n^2-1}+\dfrac{2}{4n^2+1}+\cdots\right)+\\ \left(\dfrac{2}{4n^2-1}-\dfrac{2}{4n^2+1}+\cdots\right)+\dfrac{n^2-1}{4n^2-1}-\dfrac{n^2-1}{4n^2+1}+\cdots\end{array}\right]}}$$

$$= \cfrac{1}{n^2-2+\cfrac{(n^2-1)^2}{2+\cfrac{(n^2+1)^2\left(\frac{1}{4n^2-1}-\frac{1}{4n^2+1}+\frac{1}{9n^2-1}-\frac{1}{9n^2+1}+\cdots\right)}{1-\frac{n^2+1}{4n^2-1}+\frac{n^2+1}{4n^2+1}+\cdots}}}$$

Continuing this process further, we arrive at the following expression:

$$\frac{2n}{\pi\left[\coth\frac{\pi}{n}+\cot\frac{\pi}{n}\right]}-1=\cfrac{1}{n^2-2+\cfrac{(n^2-1)^2}{2+\cfrac{(n^2+1)^2}{3n^2-2+\cfrac{4n^2-1}{2+\cdots}}}}.$$

Thus by rearranging the terms, we prove the theorem. □

Theorem 7. *For $r\in\mathbb{R}$ and $n\in\mathbb{N}$, the complex sine product and Basel's problems are related as follows:*

$$\ln\left(\frac{\sinh r}{r}\right)=\sum_{n\in\mathbb{N}}\frac{2^{2n}}{(2n)(2n)!}B_{2n}r^{2n}. \tag{38}$$

Furthermore, the hyperbolic cotangent function is related to the Basel constants as follows:

$$-\frac{1}{2r}+\frac{\coth r}{2}=\sum_{n\in\mathbb{N}}\frac{2^{2n}}{2(2n)!}B_{2n}r^{2n-1}. \tag{39}$$

Proof. From the $\pi-$series of the sine function we have:

$$\frac{\sin x}{x}=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\cdots . \tag{40}$$

Upon substituting $x=ir$ for some $r\in\mathbb{R}$, we get the hyperbolic sine function as follows:

$$\frac{\sinh r}{r}=\left(1+\frac{r^2}{\pi^2}\right)\left(1+\frac{r^2}{4\pi^2}\right)\left(1+\frac{r^2}{9\pi^2}\right)\left(1+\frac{r^2}{16\pi^2}\right)\cdots \tag{41}$$

$$\Rightarrow\ln\left(\frac{\sinh r}{r}\right)=\ln\left(1+\frac{r^2}{\pi^2}\right)+\ln\left(1+\frac{r^2}{4n^2\pi^2}\right)+\ln\left(1+\frac{r^2}{9n^2\pi^2}\right)+\cdots \tag{42}$$

$$\Rightarrow\ln\left(\frac{\sinh r}{r}\right)=\frac{r^2}{\pi^2}\sum_{n\in\mathbb{N}}\frac{1}{n^2}+\frac{r^4}{2\pi^4}\sum_{n\in\mathbb{N}}\frac{1}{n^4}+\frac{r^6}{3\pi^6}\sum_{n\in\mathbb{N}}\frac{1}{n^6}+\frac{r^8}{4\pi^8}\sum_{n\in\mathbb{N}}\frac{1}{n^8}+\cdots, \tag{43}$$

where the terms $\sum_{n\in\mathbb{N}}\frac{1}{n^2}$, $\sum_{n\in\mathbb{N}}\frac{1}{n^4}$, $\sum_{n\in\mathbb{N}}\frac{1}{n^6}$, $\sum_{n\in\mathbb{N}}\frac{1}{n^8}$, ... are the Basel's problems discovered by Euler [15]. Therefore, the above equation can be reduced to

$$\ln\left(\frac{\sinh r}{r}\right)=\frac{r^2}{6}-\frac{r^4}{180}+\frac{r^6}{2833}-\cdots=\sum_{n\in\mathbb{N}}\frac{2^{2n}}{2n(2n)!}B_{2n}r^{2n}. \tag{44}$$

Observation 7. *For the case when* $r = \pi$ *and raising Equation* (44) *to the power of* e *on both sides, we get*

$$\frac{\sinh \pi}{\pi} = \frac{e^{\frac{\pi^2}{6}}}{(\sqrt{e})^{\frac{\pi^4}{90}}} \cdot \frac{\sqrt[3]{e^{\frac{\pi^6}{945}}}}{(\sqrt[4]{e})^{\frac{\pi^8}{9450}}} \cdot \frac{\sqrt[5]{e^{\frac{\pi^{10}}{93555}}}}{(\sqrt[6]{e})^{\frac{691\pi^{12}}{638512875}}} \tag{45}$$

We now eliminate the term n from the denominator of Equation (44) by differentiating it with respect to r and then we arrive at an expression having Basel constants.

$$\frac{d}{dr}\left[\ln\left(\frac{\sinh r}{r}\right)\right] = \frac{2r}{6} - \frac{4r^3}{2\cdot 90} + \frac{6r^5}{3\cdot 945} - \frac{8r^7}{4\cdot 9450} + \cdots,$$

$$\frac{r\cosh r - \sinh r}{2r\sinh r} = \frac{r}{6} - \frac{r^3}{90} + \frac{r^5}{945} - \frac{r^7}{9450} + \cdots,$$

where $6,\ 90,\ 945,\ 9450, \ldots$ are the Basel constants. Thus, we arrive at the following expression

$$\frac{r\cosh r - \sinh r}{2r\sinh r} = -\frac{1}{2r} + \frac{\coth r}{2} = \sum_{n\in\mathbb{N}} \frac{2^{2n}}{2(2n)!} B_{2n} r^{2n-1}.$$

Hence the theorem. □

Remark 1. *The continued fraction for* $\coth x$ *is as follows:*

$$1 - \frac{\coth x}{2} + \frac{1}{2x} = \cfrac{1}{1 + \cfrac{1}{\frac{6}{x} - 1 + \cfrac{\frac{6^2}{x^2}}{\frac{90}{x^3} - \frac{6}{x} + \cfrac{\frac{90^2}{x^6}}{\frac{945}{x^3} - \frac{90}{x^3} + \cdots}}}}.$$

3 Conclusion

In this paper, a few results related to the Euler sine product and continued fractions are derived. Also, some of the special cases were observed where the infinite series were expressed in terms of Basel constants. Furthermore, Brouncker's expression for π was extended to obtain the continued fraction for the odd powers of π in terms of $\cot x$ where $x \in \mathbb{R}$. Also, the convergence of the expression:

$$\frac{1}{2xk}\frac{d}{dx}\left(\frac{1}{2(k-1)x}\frac{d}{dx}\left(\cdots\left(\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{2x^2} - \frac{\cot x}{2x}\right)\right)\right)\right) = \sum_{n\in\mathbb{N}}\left[\frac{1}{(n\pi - x)(n\pi + x)}\right]^{k+1}$$

and

$$\sum_{k=0}^{\infty}\left(\frac{1}{k\pi + x}\right)^{2n-1} - \sum_{k\in\mathbb{N}}\left(\frac{1}{k\pi - x}\right)^{2n-1}$$

was visualized through the graphs.

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