Multiplicative Sombor index of trees

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Abstract: For a graph Ω, the multiplicative Sombor index is defined as

\[ \prod_{SO}(\Omega) = \prod_{ab \in E(\Omega)} \sqrt{d_{\Omega}^2(a) + d_{\Omega}^2(b)}, \]

where \( d_{\Omega}(a) \) is the degree of vertex \( a \). Liu [Liu, H. (2022). Discrete Mathematics Letters, 9, 80–85] showed that, when \( T \) is a tree of order \( n \), \( \prod_{SO}(T) \geq \prod_{SO}(P_n) = 5(\sqrt{8})^{n-3} \). We improved this result and show that, if \( T \) is a tree of order \( n \) with maximum degree \( D \), then

\[ \prod_{SO}(T) \geq \begin{cases} (5(D^2 + 4)) \frac{8^{\frac{n-2D-1}{2}}}{2} & \text{if } D \leq \frac{n-1}{2}, \\ (D^2 + 1)^{\frac{2D+1-n}{2}} (5(D^2 + 4)) \frac{n-D-1}{2} & \text{if } D > \frac{n-1}{2}. \end{cases} \]

Also, we show that equality holds if and only if \( T \) is a spider whose all legs have length less than three or all legs have length more than one.
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1 Introduction

Consider a simple graph $\Omega = (\mathcal{V}(\Omega), \mathcal{E}(\Omega))$, where $\mathcal{V} = \mathcal{V}(\Omega)$ and $\mathcal{E} = \mathcal{E}(\Omega)$ are its vertex and edge set of $\Omega$, respectively. The integer $n = n(\Omega) = |\mathcal{V}|$ is the order $\Omega$. The open neighborhood of a vertex $a$ in the graph $\Omega$ is the set $N_\Omega(a) = \{b \in \mathcal{V}(\Omega) : ab \in \mathcal{E}(\Omega)\}$. The degree of a vertex $a$ in $\Omega$ is the cardinality of its open neighborhood. The maximum degree is denoted by $D$.

Recently, some variants of vertex-degree-based indices such as multiplicative Zagreb indices [9, 20], irregularity [2, 14, 22], Lanzhou index [7, 19], entire Zagreb indices [1, 13] have been introduced.

In 2021, a new degree-based topological index was put forward by Gutman [10], referred to as the Sombor index. Its definition for a graph $\Omega$ is

$$SO(\Omega) = \sum_{ab \in E(\Omega)} \sqrt{d^2_\Omega(a) + d^2_\Omega(b)}.$$ 

For more information see [3–6, 8, 11, 15–18, 21].

Recently Liu [12] defined the multiplicative version of the Sombor index. The multiplicative Sombor index is defined as:

$$\prod_{SO}(\Omega) = \prod_{ab \in E(\Omega)} \sqrt{d^2_\Omega(a) + d^2_\Omega(b)}.$$ 

In this paper, we establish some new lower bounds on the multiplicative Sombor index and determine the extremal trees attaining these bounds.

2 Trees

A leaf of a tree $T$ is a vertex of degree 1. A tree with a vertex recognized as the root is called a rooted tree. If $a$ is a non-root vertex of a tree, the vertex adjacent to $a$ on the path joining $a$ and the root vertex is known as the parent of $a$. Throughout this paper, let $T_{n,D}$ be the set of trees of order $n$ and maximum degree $D$.

Lemma 2.1. Let $T \in T_{n,D}$ and $T$ have non-root vertices of degrees greater than or equal to three. Then, there is $T' \in T_{n,D}$ such that $\prod_{SO}(T') < \prod_{SO}(T)$.

Proof. Assume that $T$ denotes a rooted tree with root $x$ such that $d_T(x) = D$ and $N_T(x) = \{x_1, x_2, \ldots, x_D\}$. Let $y$ be a vertex with maximum distance from $x$ among all the non-root vertices of $T$ of degrees greater than or equal to three, and let $d_T(y) = \kappa \geq 3$. Assume that $N_T(y) = \{y_1, y_2, \ldots, y_{\kappa-1}, y_\kappa\}$ where $y_\kappa$ is the parent of $y$. By our assumption, all vertices adjacent to $y$ except of $z$ are of degree one or two in $T$. We have the following cases.

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Case 1. At least two vertices adjacent to $y$ in $\mathcal{T}$ are leaves.
We can assume that $y_1$ and $y_2$ are leaves and $\mathcal{T}'$ is the tree derived from $\mathcal{T} - \{y_1\}$ by adding the path $y_1 y_2$. Assume that

$$Q = \mathcal{E}(\mathcal{T}) - \{yy_1, yy_2, \ldots, yy_{\kappa-1}, y y_{\kappa}\},$$

and

$$Q' = \mathcal{E}(\mathcal{T}') - \{y_1 y_2, yy_2, \ldots, yy_{\kappa-1}, y y_{\kappa}\}.$$

We can see that

$$\beta = \prod_{a b \in Q} \sqrt{d_{\mathcal{T}}^2(a) + d_{\mathcal{T}}^2(b)} = \prod_{a b \in Q'} \sqrt{d_{\mathcal{T}'}^2(a) + d_{\mathcal{T}'}^2(b)}.$$

Then

$$\prod_{SO} (\mathcal{T}) - \prod_{SO} (\mathcal{T}') = \prod_{a b \in Q} \sqrt{d_{\mathcal{T}}^2(a) + d_{\mathcal{T}}^2(b)} \prod_{a b \in Q'} \sqrt{d_{\mathcal{T}'}^2(a) + d_{\mathcal{T}'}^2(b)}$$

$$- \prod_{a b \in Q'} \sqrt{d_{\mathcal{T}'}^2(a) + d_{\mathcal{T}'}^2(b)} \prod_{a b \in Q} \sqrt{d_{\mathcal{T}}^2(a) + d_{\mathcal{T}}^2(b)}$$

$$= \beta \sqrt{d_{\mathcal{T}}^2(y) + d_{\mathcal{T}'}^2(y_1)} \sqrt{d_{\mathcal{T}}^2(y) + d_{\mathcal{T}'}^2(y_2)} \prod_{i=3}^{\kappa} \sqrt{d_{\mathcal{T}}^2(y) + d_{\mathcal{T}'}^2(y_i)}$$

$$- \beta \sqrt{d_{\mathcal{T}'}^2(y) + d_{\mathcal{T}'}^2(y_2)} \sqrt{d_{\mathcal{T}'}^2(y_1) + d_{\mathcal{T}'}^2(y_2)} \prod_{i=3}^{\kappa} \sqrt{d_{\mathcal{T}}^2(y) + d_{\mathcal{T}'}^2(y_i)}$$

$$= \beta (\kappa^2 + 1) \prod_{i=3}^{\kappa} \sqrt{\kappa^2 + d_{\mathcal{T}'}^2(y_i)}$$

$$- \beta \sqrt{5(\kappa - 1)^2 + 20} \prod_{i=3}^{\kappa} \sqrt{(\kappa - 1)^2 + d_{\mathcal{T}'}^2(y_i)}$$

$$\geq \beta (\kappa^2 + 1 - \sqrt{5(\kappa - 1)^2 + 20}) \prod_{i=3}^{\kappa} \sqrt{(\kappa - 1)^2 + d_{\mathcal{T}'}^2(y_i)}$$

$$> 0.$$

Case 2. Exactly one vertex adjacent to $y$ in $\mathcal{T}$ is a leaf.
We can assume that $y_1$ is a leaf adjacent to $y$ and $yu_1 u_2 \ldots u_k$ is a path in $\mathcal{T}$ for $k \geq 2$ and $y_2 = u_1$.
Let $\mathcal{T}'$ be the tree derived from $\mathcal{T} - \{y_1\}$ by adding the path $u_k y_1$. Suppose

$$Q = \mathcal{E}(\mathcal{T}) - \{yy_1, yy_2, \ldots, yy_{\kappa-1}, y y_{\kappa}, u_k y_1\},$$

and

$$Q' = \mathcal{E}(\mathcal{T}') - \{y_1 u_k, yy_2, \ldots, yy_{\kappa-1}, y y_{\kappa}, u_k - 1 u_k\}.$$

It is easy to see that

$$\beta = \prod_{a b \in Q} \sqrt{d_{\mathcal{T}}^2(a) + d_{\mathcal{T}}^2(b)} = \prod_{a b \in Q'} \sqrt{d_{\mathcal{T}'}^2(a) + d_{\mathcal{T}'}^2(b)}.$$
Then

\[
\prod_{ab \in Q} (\mathcal{T}) - \prod_{ab \in Q} (\mathcal{T}') = \prod_{ab \in Q} \sqrt{d_{\mathcal{T}}^2(a) + d_{\mathcal{T}}^2(b)} \prod_{ab \in Q'} \sqrt{d_{\mathcal{T}'}^2(a) + d_{\mathcal{T}'}^2(b)} \\
- \prod_{ab \in Q'} \sqrt{d_{\mathcal{T}'}^2(a) + d_{\mathcal{T}'}^2(b)} \prod_{ab \in Q'} \sqrt{d_{\mathcal{T}'}^2(a) + d_{\mathcal{T}'}^2(b)}
\]

\[
= \beta \sqrt{d_{\mathcal{T}}^2(y) + d_{\mathcal{T}}^2(y_1)} \sqrt{d_{\mathcal{T}}^2(u_k) + d_{\mathcal{T}}^2(u_{k-1})} \prod_{i=2}^{\kappa} \sqrt{d_{\mathcal{T}}^2(y) + d_{\mathcal{T}}^2(y_i)}

- \beta \sqrt{d_{\mathcal{T}'}^2(y_1) + d_{\mathcal{T}'}^2(u_k)} \sqrt{d_{\mathcal{T}'}^2(u_k) + d_{\mathcal{T}'}^2(u_{k-1})} \prod_{i=2}^{\kappa} \sqrt{d_{\mathcal{T}'}^2(y) + d_{\mathcal{T}'}^2(y_i)}
\]

\[
= \beta \sqrt{5(\kappa^2 + 1)} \prod_{i=2}^{\kappa} \sqrt{\kappa^2 + d_{\mathcal{T}}^2(y_i)} - \beta \sqrt{40} \prod_{i=2}^{\kappa} \sqrt{(\kappa - 1)^2 + d_{\mathcal{T}'}^2(y_i)}
\]

\[
\geq \beta (\sqrt{5(\kappa^2 + 1)} - \sqrt{40}) \prod_{i=2}^{\kappa} \sqrt{(\kappa - 1)^2 + d_{\mathcal{T}'}^2(y_i)}
\]

> 0.

**Case 3.** None of the vertices adjacent to \( y \) are leaves.

Let \( yu_1u_2 \ldots u_t, wv_1v_2 \ldots v_s \ (t, s \geq 2) \) be two paths in \( \mathcal{T} \) with \( y_1 = u_1 \) and \( y_2 = v_1 \) and let \( \mathcal{T}' \) be the tree derived from \( \mathcal{T} - \{y_1\} \) by adding the path \( v_s y_1 \). Assume that

\[
Q = \mathcal{E}(\mathcal{T}) - \{yy_1, yy_2, \ldots, yy_{\kappa-1}, yy_\kappa, v_{s-1}v_s\},
\]

and

\[
Q' = \mathcal{E}(\mathcal{T}') - \{y_1v_s, y_2y_\kappa, \ldots, yy_{\kappa-1}, yy_\kappa, v_{s-1}v_s\}.
\]

We can see that

\[
\beta = \prod_{ab \in Q} \sqrt{d_{\mathcal{T}}^2(a) + d_{\mathcal{T}}^2(b)} = \prod_{ab \in Q'} \sqrt{d_{\mathcal{T}'}^2(a) + d_{\mathcal{T}'}^2(b)}.
\]

Then

\[
\prod_{ab \in Q} (\mathcal{T}) - \prod_{ab \in Q} (\mathcal{T}') = \prod_{ab \in Q} \sqrt{d_{\mathcal{T}}^2(a) + d_{\mathcal{T}}^2(b)} \prod_{ab \in Q'} \sqrt{d_{\mathcal{T}'}^2(a) + d_{\mathcal{T}'}^2(b)} \\
- \prod_{ab \in Q'} \sqrt{d_{\mathcal{T}'}^2(a) + d_{\mathcal{T}'}^2(b)} \prod_{ab \in Q'} \sqrt{d_{\mathcal{T}'}^2(a) + d_{\mathcal{T}'}^2(b)}
\]

\[
= \beta \sqrt{d_{\mathcal{T}}^2(y) + d_{\mathcal{T}}^2(y_1)} \sqrt{d_{\mathcal{T}}^2(v_s) + d_{\mathcal{T}}^2(v_{s-1})} \prod_{i=2}^{\kappa} \sqrt{d_{\mathcal{T}}^2(y) + d_{\mathcal{T}}^2(y_i)}

- \beta \sqrt{d_{\mathcal{T}'}^2(y_1) + d_{\mathcal{T}'}^2(v_s)} \sqrt{d_{\mathcal{T}'}^2(v_s) + d_{\mathcal{T}'}^2(v_{s-1})} \prod_{i=2}^{\kappa} \sqrt{d_{\mathcal{T}'}^2(y) + d_{\mathcal{T}'}^2(y_i)}
\]

\[
= \beta \sqrt{5(\kappa^2 + 4)} \prod_{i=2}^{\kappa} \sqrt{\kappa^2 + d_{\mathcal{T}}^2(y_i)} - 8\beta \prod_{i=2}^{\kappa} \sqrt{(\kappa - 1)^2 + d_{\mathcal{T}'}^2(y_i)}
\]

\[
\geq \beta (\sqrt{5(\kappa^2 + 4)} - 8) \prod_{i=2}^{\kappa} \sqrt{(\kappa - 1)^2 + d_{\mathcal{T}'}^2(y_i)}
\]

> 0.

This completes the proof. \( \square \)
A spider is a tree that has no more than one vertex of degree greater than 2. Such a vertex is known as the center of the spider. A path connecting the center of a spider to one of its pendant vertices is called a leg of the spider. For example, a star with \( n \) vertices, \( S_n \), is a spider with \( n - 1 \) legs, each of length 1.

**Proposition 2.1.** Let \( T \) be a spider of order \( n \) and \( D \geq 3 \) legs. Assume that \( T \) contains a leg with length 1 and another leg with length greater than 2. Then there is a spider \( T' \) of order \( n \) and \( D \) legs with \( \prod_{SO}(T') < \prod_{SO}(T) \).

**Proof.** Let \( x \) be the center of \( T \) and \( N_T(x) = \{ x_1, \ldots, x_D \} \). Root \( T \) at \( x \). We may assume that \( d(x_1) = 1 \) and let \( xy_1y_2 \ldots y_t, t \geq 3 \) be the longest leg of \( T \) such that \( y_1 = x_2 \). Assume that \( T' \) is the tree derived from \( T - \{ y_{t-1} \} \) by adding the path \( x_1 y_t \). Assume that

\[
Q = \mathcal{E}(T) - \{ xx_1, y_{t-1}y_{t-2}, y_{t}y_{t-1} \},
\]

and

\[
Q' = \mathcal{E}(T') - \{ xx_1, y_{t-1}y_{t-2}, y_{t}x_1 \}.
\]

We can see that

\[
\beta = \prod_{ab \notin Q} \sqrt{d_T^2(a) + d_T^2(b)} = \prod_{ab \notin Q'} \sqrt{d_T^2(a) + d_T^2(b)}.
\]

Then

\[
\prod_{SO}(T) - \prod_{SO}(T') = \prod_{ab \notin Q} \sqrt{d_T^2(a) + d_T^2(b)} \prod_{ab \in Q} \sqrt{d_T^2(a) + d_T^2(b)}
\]

\[
- \prod_{ab \notin Q'} \sqrt{d_T^2(a) + d_T^2(b)} \prod_{ab \in Q'} \sqrt{d_T^2(a) + d_T^2(b)}
\]

\[
= \beta \sqrt{d_T^2(x) + d_T^2(x_1)} \sqrt{d_T^2(y_t) + d_T^2(y_{t-1})} \sqrt{d_T^2(y_{t-1}) + d_T^2(y_{t-2})}
\]

\[
- \beta \sqrt{d_T^2(x) + d_T^2(x_1)} \sqrt{d_T^2(y_t) + d_T^2(x)} \sqrt{d_T^2(y_{t-1}) + d_T^2(y_{t-2})}
\]

\[
= 2\beta \sqrt{10(D^2 + 1) - 5\beta \sqrt{D^2 + 4}} > 0.
\]

**Theorem 2.1.** Let \( T \in \mathcal{T}_{n,D} \). Then

\[
\prod_{SO}(T) \geq \begin{cases} 
(5(D^2 + 4))^\frac{D}{2} \frac{n-2D-1}{2} & \text{if } D \leq \frac{n-1}{2}, \\
(D^2 + 1)^{\frac{D+1-n}{2}} (5(D^2 + 4))^\frac{n-D-1}{2} & \text{if } D > \frac{n-1}{2}.
\end{cases}
\]

The equality holds if and only if \( \Omega \) is a spider whose all legs have length less than three or all legs have length more than one.

**Proof.** Let \( T' \in \mathcal{T}_{n,D} \) such that \( \prod_{SO}(T') \leq \prod_{SO}(T) \) for each \( T \in \mathcal{T}_{n,D} \). Choose a vertex \( x \) of \( T' \) with degree \( D \) as the root of \( T' \). If \( D = 2 \), then \( T \) is a path of order \( n \) and \( \prod_{SO}(P_n) = 5(\sqrt{8})^{n-3} \).

Let \( D \geq 3 \). By the choice of \( T' \), it can be deduced from Lemma 2.1 that \( T' \) is a spider with center \( x \). By Proposition 2.1 and the selection of \( T' \), all legs of \( T' \) either have length less than

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three or have length more than one. We let first all legs of \( T' \) have length more than 1. Clearly, \( \mathcal{D} \leq \frac{n-1}{2} \). Then
\[
\prod_{SO} (T') = (5(D^2 + 4))^{\frac{\mathcal{D}}{2}} 8^{\frac{n-2\mathcal{D}-1}{2}}.
\]
Next we assume that all legs of \( T' \) have length less than three. Considering the previous case, it might be assumed that \( T' \) has a leg of length 1. The number of leaves adjacent to \( x \) is \( 2\mathcal{D} + 1 - n \), and thus
\[
\prod_{SO} (T') = (D^2 + 1)^{\frac{2\mathcal{D}+1-n}{2}} (5(D^2 + 4))^{\frac{n-\mathcal{D}-1}{2}}.
\]

The following observation is immediately achieved from the definitions of multiplicative Sombor index.

**Observation 2.1.** Let \( \Omega \) be a graph and \( e \notin \mathcal{E}(\Omega) \). Then
\[
\prod_{SO} (\Omega + e) > \prod_{SO} (\Omega).
\]

Applying Theorem 2.1 and Observation 2.1, we yield the next result.

**Corollary 2.1.** If \( \Omega \) is a graph of order \( n \) with maximum degree \( \mathcal{D} \), then
\[
\prod_{SO} (\Omega) \geq \begin{cases} 
(5(D^2 + 4))^{\frac{\mathcal{D}}{2}} 8^{\frac{n-2\mathcal{D}-1}{2}} & \text{if } \mathcal{D} \leq \frac{n-1}{2}, \\
(D^2 + 1)^{\frac{2\mathcal{D}+1-n}{2}} (5(D^2 + 4))^{\frac{n-\mathcal{D}-1}{2}} & \text{if } \mathcal{D} > \frac{n-1}{2}.
\end{cases}
\]
The equality holds if and only if \( \Omega \) is a spider whose all legs have length less than three or all legs have length more than one.

**References**


