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Multiplicative Sombor index of trees

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Abstract: For a graph Ω, the multiplicative Sombor index is defined as

$$\prod_{SO}(\Omega) = \prod_{ab\in\mathcal{E}(\Omega)} \sqrt{d_\Omega^2(a) + d_\Omega^2(b)},$$

where $d_\Omega(a)$ is the degree of vertex a. Liu [Liu, H. (2022). *Discrete Mathematics Letters*, 9, 80–85] showed that, when $\mathcal{T}$ is a tree of order n, $\prod_{SO}(\mathcal{T}) \geqslant \prod_{SO}(P_n) = 5(\sqrt{8})^{n-3}$. We improved this result and show that, if $\mathcal{T}$ is a tree of order n with maximum degree $\mathcal{D}$, then

$$\prod_{SO}(\mathcal{T}) \geqslant \begin{cases} (5(\mathcal{D}^2+4))^{\frac{\mathcal{D}}{2}} 8^{\frac{n-2\mathcal{D}-1}{2}} & \text{if } \mathcal{D} \leqslant \frac{n-1}{2}, \\ (\mathcal{D}^2+1)^{\frac{2\mathcal{D}+1-n}{2}} (5(\mathcal{D}^2+4))^{\frac{n-\mathcal{D}-1}{2}} & \text{if } \mathcal{D} > \frac{n-1}{2}. \end{cases}$$

Also, we show that equality holds if and only if $\mathcal{T}$ is a spider whose all legs have length less than three or all legs have length more than one.

Keywords: Sombor index, Multiplicative Sombor index, Trees.
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1 Introduction

Consider a simple graph $\Omega = (\mathcal{V}(\Omega), \mathcal{E}(\Omega))$, where $\mathcal{V} = \mathcal{V}(\Omega)$ and $\mathcal{E} = \mathcal{E}(\Omega)$ are its vertex and edge set of Ω, respectively. The integer $n = n(\Omega) = |\mathcal{V}|$ is the order Ω. The *open neighborhood* of a vertex a in the graph Ω is the set $N_\Omega(a) = \{b \in \mathcal{V}(\Omega) : ab \in \mathcal{E}(\Omega)\}$. The *degree* of a vertex a in Ω is the cardinality of its open neighborhood. The *maximum degree* is denoted by $\mathcal{D}$.

Recently, some variants of vertex-degree-based indices such as multiplicative Zagreb indices [9, 20], irregularity [2, 14, 22], Lanzhou index [7, 19], entire Zagreb indices [1, 13] have been introduced.

In 2021, a new degree-based topological index was put forward by Gutman [10], referred to as the *Sombor index*. Its definition for a graph Ω is

$$SO(\Omega) = \sum_{ab\in\mathcal{E}(\Omega)} \sqrt{d_\Omega^2(a) + d_\Omega^2(b)}.$$

For more information see [3–6, 8, 11, 15–18, 21].

Recently Liu [12] defined the multiplicative version of the Sombor index. The *multiplicative Sombor index* is defined as:

$$\prod_{SO}(\Omega) = \prod_{ab\in\mathcal{E}(\Omega)} \sqrt{d_\Omega^2(a) + d_\Omega^2(b)}.$$

In this paper, we establish some new lower bounds on the multiplicative Sombor index and determine the extremal trees attaining these bounds.

2 Trees

A *leaf* of a tree $\mathcal{T}$ is a vertex of degree 1. A tree with a vertex recognized as the root is called a *rooted tree*. If a is a non-root vertex of a tree, the vertex adjacent to a on the path joining a and the root vertex is known as the parent of a. Throughout this paper, let $\mathcal{T}_{n,\mathcal{D}}$ be the set of trees of order n and maximum degree $\mathcal{D}$.

Lemma 2.1. *Let $\mathcal{T} \in \mathcal{T}_{n,\mathcal{D}}$ and $\mathcal{T}$ have non-root vertices of degrees greater than or equal to three. Then, there is $\mathcal{T}' \in \mathcal{T}_{n,\mathcal{D}}$ such that $\prod_{SO}(\mathcal{T}') < \prod_{SO}(\mathcal{T})$.*

Proof. Assume that $\mathcal{T}$ denotes a rooted tree with root x such that $d_\mathcal{T}(x) = \mathcal{D}$ and $N_\mathcal{T}(x) = \{x_1, x_2, \ldots, x_\mathcal{D}\}$. Let y be a vertex with maximum distance from x among all the non-root vertices of $\mathcal{T}$ of degrees greater than or equal to three, and let $d_\mathcal{T}(y) = \kappa \geq 3$. Assume that $N_T(y) = \{y_1, y_2, \ldots, y_{\kappa-1}, y_\kappa\}$ where y_κ is the parent of y. By our assumption, all vertices adjacent to y except of z are of degree one or two in $\mathcal{T}$. We have the following cases.

Case 1. At least two vertices adjacent to y in $\mathcal{T}$ are leaves.
We can assume that y_1 and y_2 are leaves and $\mathcal{T}'$ is the tree derived from $\mathcal{T} - \{y_1\}$ by adding the path y_1y_2. Assume that

$$Q = \mathcal{E}(\mathcal{T}) - \{yy_1, yy_2, \ldots, yy_{\kappa-1}, yy_\kappa\},$$

and

$$Q' = \mathcal{E}(\mathcal{T}') - \{y_1y_2, yy_2, \ldots, yy_{\kappa-1}, yy_\kappa\}.$$

We can see that

$$\beta = \prod_{ab\notin Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} = \prod_{ab\notin Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)}.$$

Then

$$\begin{aligned}
\prod_{SO}(\mathcal{T}) - \prod_{SO}(\mathcal{T}') &= \prod_{ab\notin Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} \prod_{ab\in Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} \\
&\quad - \prod_{ab\notin Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)} \prod_{ab\in Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)} \\
&= \beta\sqrt{d^2_{\mathcal{T}}(y) + d^2_{\mathcal{T}}(y_1)}\sqrt{d^2_{\mathcal{T}}(y) + d^2_{\mathcal{T}}(y_2)} \prod_{i=3}^{\kappa} \sqrt{d^2_{\mathcal{T}}(y) + d^2_{\mathcal{T}}(y_i)} \\
&\quad - \beta\sqrt{d^2_{\mathcal{T}'}(y) + d^2_{\mathcal{T}'}(y_2)}\sqrt{d^2_{\mathcal{T}'}(y_1) + d^2_{\mathcal{T}'}(y_2)} \prod_{i=3}^{\kappa} \sqrt{d^2_{\mathcal{T}'}(y) + d^2_{\mathcal{T}'}(y_i)} \\
&= \beta(\kappa^2 + 1) \prod_{i=3}^{\kappa} \sqrt{\kappa^2 + d^2_{\mathcal{T}}(y_i)} \\
&\quad - \beta\sqrt{5(\kappa-1)^2 + 20} \prod_{i=3}^{\kappa} \sqrt{(\kappa-1)^2 + d^2_{\mathcal{T}}(y_i)} \\
&\geqslant \beta(\kappa^2 + 1 - \sqrt{5(\kappa-1)^2 + 20}) \prod_{i=3}^{\kappa} \sqrt{(\kappa-1)^2 + d^2_{\mathcal{T}}(y_i)} \\
&> 0.
\end{aligned}$$

Case 2. Exactly one vertex adjacent to y in $\mathcal{T}$ is a leaf.
We can assume that y_1 is a leaf adjacent to y and $yu_1u_2 \ldots u_k$ is a path in $\mathcal{T}$ for $k \geq 2$ and $y_2 = u_1$. Let $\mathcal{T}'$ be the tree derived from $\mathcal{T} - \{y_1\}$ by adding the path u_ky_1. Suppose

$$Q = \mathcal{E}(\mathcal{T}) - \{yy_1, yy_2, \ldots, yy_{\kappa-1}, yy_\kappa, u_{k-1}u_k\},$$

and

$$Q' = \mathcal{E}(\mathcal{T}') - \{y_1u_k, yy_2, \ldots, yy_{\kappa-1}, yy_\kappa, u_{k-1}u_k\}.$$

It is easy to see that

$$\beta = \prod_{ab\notin Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} = \prod_{ab\notin Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)}.$$

Then

$$\begin{aligned}
\prod_{SO}(\mathcal{T}) - \prod_{SO}(\mathcal{T}') &= \prod_{ab\notin Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} \prod_{ab\in Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} \\
&\quad - \prod_{ab\notin Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)} \prod_{ab\in Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)} \\
&= \beta\sqrt{d^2_{\mathcal{T}}(y) + d^2_{\mathcal{T}}(y_1)}\sqrt{d^2_{\mathcal{T}}(u_k) + d^2_{\mathcal{T}}(u_{k-1})} \prod_{i=2}^{\kappa} \sqrt{d^2_{\mathcal{T}}(y) + d^2_{\mathcal{T}}(y_i)} \\
&\quad - \beta\sqrt{d^2_{\mathcal{T}'}(y_1) + d^2_{\mathcal{T}'}(u_k)}\sqrt{d^2_{\mathcal{T}'}(u_k) + d^2_{\mathcal{T}'}(u_{k-1})} \prod_{i=2}^{\kappa} \sqrt{d^2_{\mathcal{T}'}(y) + d^2_{\mathcal{T}'}(y_i)} \\
&= \beta\sqrt{5(\kappa^2+1)} \prod_{i=2}^{\kappa} \sqrt{\kappa^2 + d^2_{\mathcal{T}}(y_i)} - \beta\sqrt{40} \prod_{i=2}^{\kappa} \sqrt{(\kappa-1)^2 + d^2_{\mathcal{T}}(y_i)} \\
&\geqslant \beta(\sqrt{5(\kappa^2+1)} - \sqrt{40}) \prod_{i=2}^{\kappa} \sqrt{(\kappa-1)^2 + d^2_{\mathcal{T}}(y_i)} \\
&> 0.
\end{aligned}$$

Case 3. None of the vertices adjacent to y are leaves.
Let $yu_1u_2\dots u_t$, $uv_1v_2\dots v_s$, $(t, s \geq 2)$ be two paths in $\mathcal{T}$ with $y_1 = u_1$ and $y_2 = v_1$ and let $\mathcal{T}'$ be the tree derived from $\mathcal{T} - \{y_1\}$ by adding the path v_sy_1. Assume that

$$Q = \mathcal{E}(\mathcal{T}) - \{yy_1, yy_2, \dots, yy_{\kappa-1}, yy_\kappa, v_{s-1}v_s\},$$

and

$$Q' = \mathcal{E}(\mathcal{T}') - \{y_1v_s, yy_2, \dots, yy_{\kappa-1}, yy_\kappa, v_{s-1}v_s\}.$$

We can see that

$$\beta = \prod_{ab\notin Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} = \prod_{ab\notin Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)}.$$

Then

$$\begin{aligned}
\prod_{SO}(\mathcal{T}) - \prod_{SO}(\mathcal{T}') &= \prod_{ab\notin Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} \prod_{ab\in Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} \\
&\quad - \prod_{ab\notin Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)} \prod_{ab\in Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)} \\
&= \beta\sqrt{d^2_{\mathcal{T}}(y) + d^2_{\mathcal{T}}(y_1)}\sqrt{d^2_{\mathcal{T}}(v_s) + d^2_{\mathcal{T}}(v_{s-1})} \prod_{i=2}^{\kappa} \sqrt{d^2_{\mathcal{T}}(y) + d^2_{\mathcal{T}}(y_i)} \\
&\quad - \beta\sqrt{d^2_{\mathcal{T}'}(y_1) + d^2_{\mathcal{T}'}(v_s)}\sqrt{d^2_{\mathcal{T}'}(v_s) + d^2_{\mathcal{T}'}(v_{s-1})} \prod_{i=2}^{\kappa} \sqrt{d^2_{\mathcal{T}'}(y) + d^2_{\mathcal{T}'}(y_i)} \\
&= \beta\sqrt{5(\kappa^2+4)} \prod_{i=2}^{\kappa} \sqrt{\kappa^2 + d^2_{\mathcal{T}}(y_i)} - 8\beta \prod_{i=2}^{\kappa} \sqrt{(\kappa-1)^2 + d^2_{\mathcal{T}}(y_i)} \\
&\geqslant \beta(\sqrt{5(\kappa^2+4)} - 8) \prod_{i=2}^{\kappa} \sqrt{(\kappa-1)^2 + d^2_{\mathcal{T}}(y_i)} \\
&> 0.
\end{aligned}$$

This completes the proof. □

A *spider* is a tree that has no more than one vertex of degree greater than 2. Such a vertex is known as the *center* of the spider. A path connecting the center of a spider to one of its pendent vertices is called a *leg* of the spider. For example, a star with n vertices, S_n, is a spider with $n-1$ legs, each of length 1.

Proposition 2.1. *Let $\mathcal{T}$ be a spider of order n and $\mathcal{D} \geq 3$ legs. Assume that $\mathcal{T}$ contains a leg with length 1 and another leg with length greater than 2. Then there is a spider $\mathcal{T}'$ of order n and $\mathcal{D}$ legs with $\prod_{SO}(\mathcal{T}') < \prod_{SO}(\mathcal{T})$.*

Proof. Let x be the center of $\mathcal{T}$ and $N_{\mathcal{T}}(x) = \{x_1, \ldots, x_{\mathcal{D}}\}$. Root $\mathcal{T}$ at x. We may assume that $d(x_1) = 1$ and let $xy_1y_2\ldots y_t$, $t \geq 3$ be the longest leg of $\mathcal{T}$ such that $y_1 = x_2$. Assume that $\mathcal{T}'$ is the tree derived from $\mathcal{T} - \{y_ty_{t-1}\}$ by adding the path x_1y_t. Assume that

$$Q = \mathcal{E}(\mathcal{T}) - \{xx_1, y_{t-1}y_{t-2}, y_ty_{t-1}\},$$

and

$$Q' = \mathcal{E}(\mathcal{T}') - \{xx_1, y_{t-1}y_{t-2}, y_tx_1\}.$$

We can see that

$$\beta = \prod_{ab \notin Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} = \prod_{ab \notin Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)}.$$

Then

$$\begin{aligned}
\prod_{SO}(\mathcal{T}) - \prod_{SO}(\mathcal{T}') &= \prod_{ab \notin Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} \prod_{ab \in Q} \sqrt{d^2_{\mathcal{T}}(a) + d^2_{\mathcal{T}}(b)} \\
&\quad - \prod_{ab \notin Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)} \prod_{ab \in Q'} \sqrt{d^2_{\mathcal{T}'}(a) + d^2_{\mathcal{T}'}(b)} \\
&= \beta\sqrt{d^2_{\mathcal{T}}(x) + d^2_{\mathcal{T}}(x_1)}\sqrt{d^2_{\mathcal{T}}(y_t) + d^2_{\mathcal{T}}(y_{t-1})}\sqrt{d^2_{\mathcal{T}}(y_{t-1}) + d^2_{\mathcal{T}}(y_{t-2})} \\
&\quad - \beta\sqrt{d^2_{\mathcal{T}'}(x) + d^2_{\mathcal{T}'}(x_1)}\sqrt{d^2_{\mathcal{T}'}(y_t) + d^2_{\mathcal{T}'}(x_1)}[\sqrt{d^2_{\mathcal{T}'}(y_{t-1}) + d^2_{\mathcal{T}'}(y_{t-2})}] \\
&= 2\beta\sqrt{10(\mathcal{D}^2+1)} - 5\beta\sqrt{\mathcal{D}^2+4} \\
&> 0.
\end{aligned}$$

□

Theorem 2.1. *Let $\mathcal{T} \in \mathcal{T}_{n,\mathcal{D}}$. Then*

$$\prod_{SO}(\mathcal{T}) \geqslant \begin{cases} (5(\mathcal{D}^2+4))^{\frac{\mathcal{D}}{2}} 8^{\frac{n-2\mathcal{D}-1}{2}} & \text{if } \mathcal{D} \leqslant \frac{n-1}{2}, \\ (\mathcal{D}^2+1)^{\frac{2\mathcal{D}+1-n}{2}} (5(\mathcal{D}^2+4))^{\frac{n-\mathcal{D}-1}{2}} & \text{if } \mathcal{D} > \frac{n-1}{2}. \end{cases}$$

The equality holds if and only if Ω is a spider whose all legs have length less than three or all legs have length more than one.

Proof. Let $\mathcal{T}' \in \mathcal{T}_{n,\mathcal{D}}$ such that $\prod_{SO}(\mathcal{T}') \leqslant \prod_{SO}(\mathcal{T})$ for each $\mathcal{T} \in \mathcal{T}_{n,\mathcal{D}}$. Choose a vertex x of $\mathcal{T}'$ with degree $\mathcal{D}$ as the root of $\mathcal{T}'$. If $\mathcal{D} = 2$, then $\mathcal{T}$ is a path of order n and $\prod_{SO}(P_n) = 5(\sqrt{8})^{n-3}$. Let $\mathcal{D} \geqslant 3$. By the choice of $\mathcal{T}'$, it can be deduced from Lemma 2.1 that $\mathcal{T}'$ is a spider with center x. By Proposition 2.1 and the selection of $\mathcal{T}'$, all legs of $\mathcal{T}'$ either have length less than three or have length more than one. We let first all legs of $\mathcal{T}'$ have length more than 1. Clearly, $\mathcal{D} \leqslant \frac{n-1}{2}$. Then

$$\prod_{SO}(\mathcal{T}') = (5(\mathcal{D}^2+4))^{\frac{\mathcal{D}}{2}}\, 8^{\frac{n-2\mathcal{D}-1}{2}}.$$

Next we assume that all legs of $\mathcal{T}'$ have length less than three. Considering the previous case, it might be assumed that $\mathcal{T}'$ has a leg of length 1. The number of leaves adjacent to x is $2\mathcal{D}+1-n$, and thus

$$\prod_{SO}(\mathcal{T}') = (\mathcal{D}^2+1)^{\frac{2\mathcal{D}+1-n}{2}}(5(\mathcal{D}^2+4))^{\frac{n-\mathcal{D}-1}{2}}. \qquad \square$$

The following observation is immediately achieved from the definitions of multiplicative Sombor index.

Observation 2.1. *Let Ω be a graph and $e \notin \mathcal{E}(\Omega)$. Then*

$$\prod_{SO}(\Omega + e) > \prod_{SO}(\Omega).$$

Applying Theorem 2.1 and Observation 2.1, we yield the next result.

Corollary 2.1. *If Ω is a graph of order n with maximum degree $\mathcal{D}$, then*

$$\prod_{SO}(\Omega) \geqslant \begin{cases} (5(\mathcal{D}^2+4))^{\frac{\mathcal{D}}{2}} 8^{\frac{n-2\mathcal{D}-1}{2}} & \text{if } \mathcal{D} \leqslant \frac{n-1}{2}, \\ (\mathcal{D}^2+1)^{\frac{2\mathcal{D}+1-n}{2}}(5(\mathcal{D}^2+4))^{\frac{n-\mathcal{D}-1}{2}} & \text{if } \mathcal{D} > \frac{n-1}{2}. \end{cases}$$

The equality holds if and only if Ω is a spider whose all legs have length less than three or all legs have length more than one.

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