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On tertions and dual numbers

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To Mladen Vassilev-Missana for his 70th anniversary

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Abstract: In a previous author's paper [1], the mathematical object called "tertion" was discussed. Some operations over tertions were introduced and their properties were studied. There, it was showed that the complex numbers and quaternions can be represented by tertions. Here, we show that the dual numbers also are representable by tertions. The concept of a "0-quaternion" is introduced and its representation by tertions is given. Ideas for future research are described.
Keywords: Dual number, Quaternion, Tertion, 0-Quaternion.
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1 Introduction

The *dual numbers* were introduced in 1873 by William Clifford in [4] (see also [3, 5–7]). They have the form $a + b\varepsilon$, where a, b are real numbers, and ε is an object that satisfies the equality $\varepsilon^2 = 0$ while $\varepsilon \neq 0$. For two dual numbers, the following equalities are valid:

$$(a + b\varepsilon) + (c + \varepsilon) = (a + c) + (b + d)\varepsilon,$$
$$(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc)\varepsilon,$$

and for the real number α:

$$\alpha(a+b\boldsymbol{\varepsilon}) = \alpha a + \alpha b\boldsymbol{\varepsilon}.$$

In a series of papers, collected in [2], the author introduced the concept of tertion and discussed its properties and various operations over tertions. In [1, 2], he gave tertion representations of the complex numbers.

In the present paper, we first introduce a new operation ($\circ_{11}$) over A- and V-tertions and after this, we give A- and V-tertion representations of the dual numbers.

2 Operation $\circ_{11}$ over A-tertions

In [1, 2] A- and V-tertions are defined. They are mathematical object with the forms $\left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash$ and $\left\backslash \begin{array}{ccc} b & & c \\ & a & \end{array} \right/$, respectively, over which ten operations ($\circ_1, \ldots, \circ_{10}$) are defined.

Here, for the first time we introduce the operation $\circ_{11}$ over A-tertions:

$$\left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash \circ_{11} \left/ \begin{array}{ccc} & d & \\ e & & f \end{array} \right\backslash = \left/ \begin{array}{ccc} & ad+bf+ce & \\ ae+bd & & af+cd \end{array} \right\backslash .$$

Let everywhere below, as in [1, 2],

$$E = \left/ \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \right\backslash , \quad I = \left/ \begin{array}{ccc} & 0 & \\ 1 & & 0 \end{array} \right\backslash , \quad J = \left/ \begin{array}{ccc} & 0 & \\ 0 & & 1 \end{array} \right\backslash , \quad O = \left/ \begin{array}{ccc} & 0 & \\ 0 & & 0 \end{array} \right\backslash .$$

Obviously, the set

$$A_2 = \left\{ \left/ \begin{array}{ccc} & \alpha & \\ \beta & & \gamma \end{array} \right\backslash \mid \alpha, \beta, \gamma \in \mathcal{R} \right\}$$

is closed with respect to operation "$\circ_{11}$", where here and hereafter $\mathcal{R}$ denotes the set of real numbers. As it is mentioned in [2], index "2" of the set A_2 shows that the elements of this set are tertions of the present form, i.e., triangles with two levels. In [2] tertions with three and n levels are discussed and for them, their sets are denoted as $A_3, \ldots, A_n$ (the same is for the sets $V_2, V_3, \ldots, V_n$).

The left and right identity element of A_2 are both $\left/ \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \right\backslash$, because

$$\left/ \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \right\backslash \circ_{11} \left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash = \left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash = \left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash \circ_{11} \left/ \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \right\backslash .$$

Also,

$$\left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash \circ_{11} \left/ \begin{array}{ccc} & 0 & \\ 1 & & 0 \end{array} \right\backslash = \left/ \begin{array}{ccc} & c & \\ a & & 0 \end{array} \right\backslash = \left/ \begin{array}{ccc} & 0 & \\ 1 & & 0 \end{array} \right\backslash \circ_{11} \left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash ,$$

$$\left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash \circ_{11} \left/ \begin{array}{ccc} & 0 & \\ 0 & & 1 \end{array} \right\backslash = \left/ \begin{array}{ccc} & b & \\ 0 & & a \end{array} \right\backslash = \left/ \begin{array}{ccc} & 0 & \\ 0 & & 1 \end{array} \right\backslash \circ_{11} \left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash .$$

Operation $\circ_{11}$ is commutative, because

$$\left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \end{array} \right\backslash \circ_{11} \left/ \begin{array}{cc} \multicolumn{2}{c}{d} \\ e & f \end{array} \right\backslash = \left/ \begin{array}{cc} \multicolumn{2}{c}{ad+bf+ce} \\ ae+bd & af+cd \end{array} \right\backslash$$

$$= \left/ \begin{array}{cc} \multicolumn{2}{c}{d} \\ e & f \end{array} \right\backslash \circ_{11} \left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \end{array} \right\backslash ,$$

but it is not associative. It is distributive with respect to operation " $+$ ", because

$$\left(\left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \end{array} \right\backslash + \left/ \begin{array}{cc} \multicolumn{2}{c}{d} \\ e & f \end{array} \right\backslash \right) \circ_{11} \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash$$

$$= \left/ \begin{array}{cc} \multicolumn{2}{c}{a+d} \\ b+e & c+f \end{array} \right\backslash \circ_{11} \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash$$

$$= \left/ \begin{array}{cc} \multicolumn{2}{c}{ag+dg+bi+ei+ch+fh} \\ ah+dh+bg+eg & ai+di+cg+fg \end{array} \right\backslash$$

$$= \left/ \begin{array}{cc} \multicolumn{2}{c}{ag+bi+ch} \\ ah+bg & ai+cg \end{array} \right\backslash + \left/ \begin{array}{cc} \multicolumn{2}{c}{dg+ei+fh} \\ dh+eg & di+fg \end{array} \right\backslash$$

$$= \left(\left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \end{array} \right\backslash \circ_{11} \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash \right) + \left(\left/ \begin{array}{cc} \multicolumn{2}{c}{d} \\ e & f \end{array} \right\backslash \circ_{11} \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash \right).$$

Let $a \neq 0$. The solutions of the equation

$$\left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \end{array} \right\backslash \circ_{11} \left/ \begin{array}{cc} \multicolumn{2}{c}{x} \\ y & z \end{array} \right\backslash = \left/ \begin{array}{cc} \multicolumn{2}{c}{p} \\ q & r \end{array} \right\backslash$$

are equivalent to the solutions of the following linear system

$$\left\{ \begin{array}{llll} ax & +cy & +bz & = p \\ bx & +ay & & = q \\ cx & & +az & = r \end{array} \right. .$$

Let

$$\Delta = \begin{vmatrix} a & c & b \\ b & a & 0 \\ c & 0 & a \end{vmatrix} = a^3 - 2abc \neq 0,$$

i.e.,

$$\left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \end{array} \right\backslash \in A_{2,\circ_{11}} \equiv \left\{ \left/ \begin{array}{cc} \multicolumn{2}{c}{\alpha} \\ \beta & \gamma \end{array} \right\backslash | \alpha, \beta, \gamma \in \mathcal{R} \ \& \ \alpha(\alpha^2 - 2\beta\gamma) \neq 0 \right\},$$

then

$$\left/ \begin{array}{cc} \multicolumn{2}{c}{x} \\ y & z \end{array} \right\backslash = \left/ \begin{array}{cc} \multicolumn{2}{c}{\frac{a^2p-acq-abr}{\Delta}} \\ \frac{-abp+a^2q-bcq+b^2r}{\Delta} & \frac{-acp+c^2q+a^2r-bcr}{\Delta} \end{array} \right\backslash$$

and, in particular,

$$\Big/\begin{matrix} & a & \\ b & & c\end{matrix}\Big\backslash \circ_{11} \Big/\begin{matrix} & \frac{a^2}{\Delta} & \\ \frac{-ab}{\Delta} & & \frac{-ac}{\Delta}\end{matrix}\Big\backslash = \Big/\begin{matrix} & 1 & \\ 0 & & 0\end{matrix}\Big\backslash ,$$

$$\Big/\begin{matrix} & a & \\ b & & c\end{matrix}\Big\backslash \circ_{11} \Big/\begin{matrix} & \frac{-ac}{\Delta} & \\ \frac{a^2-bc}{\Delta} & & \frac{c^2}{\Delta}\end{matrix}\Big\backslash = \Big/\begin{matrix} & 0 & \\ 1 & & 0\end{matrix}\Big\backslash ,$$

$$\Big/\begin{matrix} & a & \\ b & & c\end{matrix}\Big\backslash \circ_{11} \Big/\begin{matrix} & \frac{-ab}{\Delta} & \\ \frac{b^2}{\Delta} & & \frac{a^2-bc}{\Delta}\end{matrix}\Big\backslash = \Big/\begin{matrix} & 0 & \\ 0 & & 1\end{matrix}\Big\backslash .$$

Due to the commutativity of operation "$\circ_{11}$", the solutions of

$$\Big/\begin{matrix} & x & \\ y & & z\end{matrix}\Big\backslash \circ_{11} \Big/\begin{matrix} & a & \\ b & & c\end{matrix}\Big\backslash = \Big/\begin{matrix} & p & \\ q & & r\end{matrix}\Big\backslash$$

are the same.

We also see that

$$\Big/\begin{matrix} & a & \\ b & & c\end{matrix}\Big\backslash^{(2,\circ_{11})} = \Big/\begin{matrix} & a^2+2bc & \\ 2ab & & 2ac\end{matrix}\Big\backslash$$

and in particular

$$\Big/\begin{matrix} & a & \\ b & & 0\end{matrix}\Big\backslash^{(2,\circ_{11})} = \Big/\begin{matrix} & a^2 & \\ 2ab & & 0\end{matrix}\Big\backslash = a \Big/\begin{matrix} & a & \\ 2b & & 0\end{matrix}\Big\backslash ,$$

$$\Big/\begin{matrix} & a & \\ 0 & & c\end{matrix}\Big\backslash^{(2,\circ_{11})} = \Big/\begin{matrix} & a^2 & \\ 0 & & 2ac\end{matrix}\Big\backslash = a \Big/\begin{matrix} & a & \\ 0 & & 2c\end{matrix}\Big\backslash .$$

Proposition. *For each natural number n and every two real numbers a, b:*

$$\Big/\begin{matrix} & a & \\ b & & 0\end{matrix}\Big\backslash^{(n,\circ_{11})} = a^{n-1} \Big/\begin{matrix} & a & \\ nb & & 0\end{matrix}\Big\backslash ,$$

$$\Big/\begin{matrix} & a & \\ 0 & & c\end{matrix}\Big\backslash^{(n,\circ_{11})} = a^{n-1} \Big/\begin{matrix} & a & \\ 0 & & nc\end{matrix}\Big\backslash .$$

Proof. For $n = 1$ the assertion is obvious. Let us assume that the two equalities are valid for some natural number n. Then

$$\begin{aligned}
\Big/\begin{matrix} & a & \\ b & & 0\end{matrix}\Big\backslash^{(n+1,\circ_{11})} &= \Big/\begin{matrix} & a & \\ b & & 0\end{matrix}\Big\backslash^{(n,\circ_{11})} \circ_{11} \Big/\begin{matrix} & a & \\ b & & 0\end{matrix}\Big\backslash \\
&= a^{n-1} \Big/\begin{matrix} & a & \\ nb & & 0\end{matrix}\Big\backslash \circ_{11} \Big/\begin{matrix} & a & \\ b & & 0\end{matrix}\Big\backslash \\
&= a^{n-1} \Big/\begin{matrix} & a^2 & \\ ab+nab & & 0\end{matrix}\Big\backslash \\
&= a^{n-1} \Big/\begin{matrix} & a^2 & \\ (n+1)ab & & 0\end{matrix}\Big\backslash \\
&= a^{n} \Big/\begin{matrix} & a & \\ (n+1)b & & 0\end{matrix}\Big\backslash .
\end{aligned}$$

The second equality is checked in the same manner. This completes the proof. □

It is interesting to mention the following equalities

$$\begin{aligned}
\diagup\begin{array}{ccc} & a & \\ b & & 0 \end{array}\diagdown \circ_{11} \left(\diagup\begin{array}{ccc} & c & \\ d & & 0 \end{array}\diagdown \circ_{11} \diagup\begin{array}{ccc} & e & \\ f & & 0 \end{array}\diagdown \right) &= \diagup\begin{array}{ccc} & a & \\ b & & 0 \end{array}\diagdown \circ_{11} \diagup\begin{array}{ccc} & ce & \\ cf+de & & 0 \end{array}\diagdown \\
&= \diagup\begin{array}{ccc} & ace & \\ acf+ade+bce & & 0 \end{array}\diagdown \\
&= \diagup\begin{array}{ccc} & ac & \\ ad+bc & & 0 \end{array}\diagdown \circ_{11} \diagup\begin{array}{ccc} & e & \\ f & & 0 \end{array}\diagdown \\
&= \left(\diagup\begin{array}{ccc} & a & \\ b & & 0 \end{array}\diagdown \circ_{11} \diagup\begin{array}{ccc} & c & \\ d & & 0 \end{array}\diagdown \right) \circ_{11} \diagup\begin{array}{ccc} & e & \\ f & & 0 \end{array}\diagdown
\end{aligned},$$

i.e., in this case the operation $\circ_{11}$ is associative. Similarly,

$$\diagdown\begin{array}{ccc} b & & 0 \\ & a & \end{array}\diagup \circ_{11} \left(\diagdown\begin{array}{ccc} d & & 0 \\ & c & \end{array}\diagup \circ_{11} \diagdown\begin{array}{ccc} f & & 0 \\ & e & \end{array}\diagup \right) = \left(\diagdown\begin{array}{ccc} b & & 0 \\ & a & \end{array}\diagup \circ_{11} \diagdown\begin{array}{ccc} d & & 0 \\ & c & \end{array}\diagup \right) \circ_{11} \diagdown\begin{array}{ccc} f & & 0 \\ & e & \end{array}\diagup .$$

In addition,

$$\diagup\begin{array}{ccc} & 0 & \\ b & & c \end{array}\diagdown \circ_{11} \diagup\begin{array}{ccc} & 0 & \\ c & & b \end{array}\diagdown = \diagup\begin{array}{ccc} & 2bc & \\ 0 & & 0 \end{array}\diagdown ,$$

$$\diagdown\begin{array}{ccc} b & & c \\ & 0 & \end{array}\diagup \circ_{11} \diagdown\begin{array}{ccc} c & & b \\ & 0 & \end{array}\diagup = \diagdown\begin{array}{ccc} 0 & & 0 \\ & 2bc & \end{array}\diagup .$$

Finally, we note that the following table is valid for operation "$\circ_{11}$":

$\circ_{11}$	E	I	J
E	E	I	J
I	I	O	E
J	J	E	O

.

3 Operation $\circ_{11}$ over V-tertions

As it is discussed in [1, 2], the object having the form

$$\diagdown\begin{array}{ccc} b & & c \\ & a & \end{array}\diagup$$

is called "V-tertion". For it, we can define all operations defined over the A-tertions (in particular, operation "$\circ_{11}$"), but now, over V-tertions.

We will only mention the definition of operation "$\circ_{11}$" now over V-tertions that are elements of set

$$V_2 = \left\{ \diagdown\begin{array}{ccc} \beta & & \gamma \\ & \alpha & \end{array}\diagup \,|\alpha, \beta, \gamma, \in \mathcal{R} \right\}.$$

The operation is as follows:

$$\diagdown\begin{array}{ccc} b & & c \\ & a & \end{array}\diagup \circ_{11} \diagdown\begin{array}{ccc} e & & f \\ & d & \end{array}\diagup = \diagdown\begin{array}{ccc} ae+bd & & af+cd \\ & ad+bf+ce & \end{array}\diagup .$$

Let everywhere below, as defined in [1, 2],

$$\overline{E} = \diagdown\begin{array}{ccc} 0 & & 0 \\ & 1 & \end{array}\diagup, \quad \overline{I} = \diagdown\begin{array}{ccc} 1 & & 0 \\ & 0 & \end{array}\diagup, \quad \overline{J} = \diagdown\begin{array}{ccc} 0 & & 1 \\ & 0 & \end{array}\diagup, \quad \overline{O} = \diagdown\begin{array}{ccc} 0 & & 0 \\ & 0 & \end{array}\diagup .$$

4 On the representations of the dual numbers by *A*- and *V*-tertions

Let us juxtapose the A-tertion $\diagup\begin{array}{ccc} & a & \\ b & & 0 \end{array}\diagdown$ to the dual number $a + b\boldsymbol{\varepsilon}$. Then, we can immediately see that the dual number equalities mentioned in the Introduction have the following respective A-tertion forms:

$$\diagup\begin{array}{ccc} & a & \\ b & & 0 \end{array}\diagdown + \diagup\begin{array}{ccc} & c & \\ d & & 0 \end{array}\diagdown = \diagup\begin{array}{ccc} & a+c & \\ b+d & & 0 \end{array}\diagdown ,$$

$$\diagup\begin{array}{ccc} & a & \\ b & & 0 \end{array}\diagdown \circ_{11} \diagup\begin{array}{ccc} & c & \\ d & & 0 \end{array}\diagdown = \diagup\begin{array}{ccc} & ac & \\ ad+bc & & 0 \end{array}\diagdown = \diagup\begin{array}{ccc} & c & \\ d & & 0 \end{array}\diagdown \circ_{11} \diagup\begin{array}{ccc} & a & \\ b & & 0 \end{array}\diagdown ,$$

$$\alpha \diagup\begin{array}{ccc} & a & \\ b & & 0 \end{array}\diagdown = \diagup\begin{array}{ccc} & \alpha a & \\ \alpha b & & 0 \end{array}\diagdown .$$

For the representation of the fourth basic dual number equality

$$(a + b\boldsymbol{\varepsilon})^{-1} = \frac{1}{a} - \frac{b}{a^2}\boldsymbol{\varepsilon},$$

using the equality

$$\diagup\begin{array}{ccc} & a & \\ b & & c \end{array}\diagdown \circ_{11} \diagup\begin{array}{ccc} & \frac{a}{a^2-2bc} & \\ \frac{-b}{a^2-2bc} & & \frac{-c}{a^2-2bc} \end{array}\diagdown = \diagup\begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array}\diagdown ,$$

we can define

$$\diagup\begin{array}{ccc} & a & \\ b & & c \end{array}\diagdown^{-1} = \diagup\begin{array}{ccc} & \frac{a}{a^2-2bc} & \\ \frac{-b}{a^2-2bc} & & \frac{-c}{a^2-2bc} \end{array}\diagdown .$$

Therefore, its representation is

$$\diagup\begin{array}{ccc} & a & \\ b & & 0 \end{array}\diagdown^{-1} = \diagup\begin{array}{ccc} & \frac{1}{a} & \\ \frac{-b}{a^2} & & 0 \end{array}\diagdown .$$

By analogy, we can represent the dual number $a+b\varepsilon$ with the A-tertion $\bigg/\!\begin{array}{ccc} & a & \\ 0 & & b \end{array}\!\bigg\backslash$. Then, the above formulas obtain, respectively, the forms

$$\bigg/\!\begin{array}{ccc} & a & \\ 0 & & b \end{array}\!\bigg\backslash + \bigg/\!\begin{array}{ccc} & c & \\ 0 & & d \end{array}\!\bigg\backslash = \bigg/\!\begin{array}{ccc} & a+c & \\ 0 & & b+d \end{array}\!\bigg\backslash,$$

$$\bigg/\!\begin{array}{ccc} & a & \\ 0 & & b \end{array}\!\bigg\backslash \circ_{11} \bigg/\!\begin{array}{ccc} & c & \\ 0 & & d \end{array}\!\bigg\backslash = \bigg/\!\begin{array}{ccc} & ac & \\ 0 & & ad+bc \end{array}\!\bigg\backslash,$$

$$\alpha \bigg/\!\begin{array}{ccc} & a & \\ 0 & & b \end{array}\!\bigg\backslash = \bigg/\!\begin{array}{ccc} & \alpha a & \\ 0 & & \alpha b \end{array}\!\bigg\backslash,$$

$$\bigg/\!\begin{array}{ccc} & a & \\ 0 & & b \end{array}\!\bigg\backslash^{-1} = \bigg/\!\begin{array}{ccc} & \frac{1}{a} & \\ 0 & & \frac{-b}{a^2} \end{array}\!\bigg\backslash.$$

Therefore, each dual number can be represented by an A-tertion. The opposite is not valid, since the A- (and V-) tertions are composed of three components each while the dual numbers, as well as the complex numbers, are composed of just two.

With respect to V_2-tertions, we will mention that a V_2-tertion $\bigg\backslash\!\begin{array}{ccc} b & & 0 \\ & a & \end{array}\!\bigg/$ can be juxtaposed to each dual number $a+b\varepsilon$. Therefore, the above expressions for A-tertions will obtain the forms

$$\bigg\backslash\!\begin{array}{ccc} b & & 0 \\ & a & \end{array}\!\bigg/ + \bigg\backslash\!\begin{array}{ccc} d & & 0 \\ & c & \end{array}\!\bigg/ = \bigg\backslash\!\begin{array}{ccc} b+d & & 0 \\ & a+c & \end{array}\!\bigg/,$$

$$\bigg\backslash\!\begin{array}{ccc} b & & 0 \\ & a & \end{array}\!\bigg/ \circ_{11} \bigg\backslash\!\begin{array}{ccc} d & & 0 \\ & c & \end{array}\!\bigg/ = \bigg\backslash\!\begin{array}{ccc} ad+bc & & 0 \\ & ac & \end{array}\!\bigg/,$$

$$\alpha \bigg\backslash\!\begin{array}{ccc} b & & 0 \\ & a & \end{array}\!\bigg/ = \bigg\backslash\!\begin{array}{ccc} \alpha b & & 0 \\ & \alpha a & \end{array}\!\bigg/,$$

$$\bigg\backslash\!\begin{array}{ccc} b & & 0 \\ & a & \end{array}\!\bigg/^{-1} = \bigg\backslash\!\begin{array}{ccc} \frac{-b}{a^2} & & 0 \\ & \frac{1}{a} & \end{array}\!\bigg/.$$

Again, as above, we can represent the dual number $a+b\varepsilon$ with the V-tertion $\bigg\backslash\!\begin{array}{ccc} 0 & & b \\ & a & \end{array}\!\bigg/$ and the respective formulas are the following:

$$\bigg\backslash\!\begin{array}{ccc} 0 & & b \\ & a & \end{array}\!\bigg/ + \bigg\backslash\!\begin{array}{ccc} 0 & & d \\ & c & \end{array}\!\bigg/ = \bigg\backslash\!\begin{array}{ccc} 0 & & b+d \\ & a+c & \end{array}\!\bigg/,$$

$$\bigg\backslash\!\begin{array}{ccc} 0 & & b \\ & a & \end{array}\!\bigg/ \circ_{11} \bigg\backslash\!\begin{array}{ccc} 0 & & d \\ & c & \end{array}\!\bigg/ = \bigg\backslash\!\begin{array}{ccc} 0 & & ad+bc \\ & ac & \end{array}\!\bigg/,$$

$$\alpha \bigg\backslash\!\begin{array}{ccc} 0 & & b \\ & a & \end{array}\!\bigg/ = \bigg\backslash\!\begin{array}{ccc} 0 & & \alpha b \\ & \alpha a & \end{array}\!\bigg/,$$

$$\bigg\backslash\!\begin{array}{ccc} 0 & & b \\ & a & \end{array}\!\bigg/^{-1} = \bigg\backslash\!\begin{array}{ccc} 0 & & \frac{-b}{a^2} \\ & \frac{1}{a} & \end{array}\!\bigg/.$$

5 Dual quaternions and their representations by A- and V-tertions

As we mentioned in [1, 2], when we have an A-tertion and a V-tertion, we can construct the new object

$$\left\langle \begin{array}{c} a \\ b \quad c \\ d \end{array} \right\rangle ,$$

which in practice is the well-known object "quaternion". It can be represented by A- and V-tertions, e.g., as follows

$$\Big/ \begin{array}{c} a \\ b \quad x \end{array} \Big\backslash \;*_1\; \Big\backslash \begin{array}{c} x \quad c \\ d \end{array} \Big/ \;=\; \left\langle \begin{array}{c} a \\ b \quad c \\ d \end{array} \right\rangle,$$

$$\Big/ \begin{array}{c} a \\ b \quad c \end{array} \Big\backslash \;*_2\; \Big\backslash \begin{array}{c} b \quad c \\ d \end{array} \Big/ \;=\; \left\langle \begin{array}{c} a \\ b \quad c \\ d \end{array} \right\rangle,$$

$$\Big/ \begin{array}{c} a \\ b \quad c \end{array} \Big\backslash \;*_3\; \Big\backslash \begin{array}{c} d \quad e \\ f \end{array} \Big/ \;=\; \left\langle \begin{array}{c} a(d+e) \\ be \qquad\qquad cd \\ (b+c)f \end{array} \right\rangle.$$

Let

$$E^* = \left\langle \begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \end{array} \right\rangle,\; I^* = \left\langle \begin{array}{c} 0 \\ 1 \quad 0 \\ 0 \end{array} \right\rangle,\; J^* = \left\langle \begin{array}{c} 0 \\ 0 \quad 1 \\ 0 \end{array} \right\rangle,\; K^* = \left\langle \begin{array}{c} 0 \\ 0 \quad 0 \\ 1 \end{array} \right\rangle,\; O^* = \left\langle \begin{array}{c} 0 \\ 0 \quad 0 \\ 0 \end{array} \right\rangle.$$

Therefore,

$$\begin{aligned}
E^* &= E *_1 \overline{O} = E *_2 \overline{O} = E *_3 \overline{I} = E *_3 \overline{J} = E *_3 \overline{E} \\
I^* &= I *_1 \overline{O} = I *_2 \overline{I} = I *_3 \overline{J} \\
J^* &= O *_1 \overline{J} = J *_2 \overline{J} = J *_3 \overline{I} \\
K^* &= O *_1 \overline{E} = O *_2 \overline{E} = O *_3 \overline{E} = I *_3 \overline{E} = J *_3 \overline{E} \\
O^* &= O *_1 \overline{O} = J *_1 \overline{I} = O *_2 \overline{O} = O *_3 \overline{O}.
\end{aligned}$$

Now, we can define

$$\left\langle \begin{array}{c} a \\ b \quad c \\ d \end{array} \right\rangle + \left\langle \begin{array}{c} e \\ f \quad g \\ h \end{array} \right\rangle = \left\langle \begin{array}{c} a+e \\ b+f \quad c+g \\ d+h \end{array} \right\rangle .$$

In [1] we gave four interpretations of the concept of a quaternion, but in all of them $(E^*)^2$, $(I^*)^2, (J^*)^2, (K^*)^2 \neq O^*$.

Now, following the idea of Clifford, we will define the concept of a “0-quaternion”. Obviously, the name “dual quaternion” is more suitable, but it is reserved for another concept (see, e.g., [3, 5–7]). The 0-quaternion will be an object of the form

$$\left\langle \begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \right\rangle,$$

for which $(I^*)^2, (J^*)^2, (K^*)^2 = O^*$. For this aim, we can define the operation $\#$ between two 0-quaternions by:

$$\left\langle \begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \right\rangle \# \left\langle \begin{matrix} & e & \\ f & & g \\ & h & \end{matrix} \right\rangle = \left\langle \begin{matrix} & ae & \\ af+be+ch+dg & & ag+bh+ce+dfc \\ & ah+bg+cf+de & \end{matrix} \right\rangle$$

Then the following table is valid for the operation $\#$:

$\#$	E^*	I^*	J^*	K^*
E^*	E^*	I^*	J^*	K^*
I^*	I^*	O^*	${}^*K$	J^*
J^*	J^*	K^*	O^*	I^*
K^*	K^*	J^*	I^*	O^*

Therefore,

$$(I^*)^2 = (J^*)^2 = (K^*)^2 = O^*$$

$$I^*J^*K^* = J^*K^*I^* = K^*I^*J^* = I^*K^*J^* = K^*J^*I^* = J^*I^*K^* = O^*.$$

Now, we see that

$$\left\langle \begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \right\rangle = aE^* + bI^* + cJ^* + dK^*$$

and

$$\left\langle \begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \right\rangle \# \left\langle \begin{matrix} & e & \\ f & & g \\ & h & \end{matrix} \right\rangle = \begin{cases} a \left\langle \begin{matrix} & e & \\ f & & g \\ & h & \end{matrix} \right\rangle + b \left\langle \begin{matrix} & 0 & \\ e & & h \\ & g & \end{matrix} \right\rangle + c \left\langle \begin{matrix} & 0 & \\ h & & e \\ & f & \end{matrix} \right\rangle + d \left\langle \begin{matrix} & 0 & \\ g & & f \\ & e & \end{matrix} \right\rangle \\ \\ e \left\langle \begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \right\rangle + f \left\langle \begin{matrix} & 0 & \\ a & & d \\ & c & \end{matrix} \right\rangle + g \left\langle \begin{matrix} & 0 & \\ d & & a \\ & b & \end{matrix} \right\rangle + h \left\langle \begin{matrix} & 0 & \\ c & & b \\ & a & \end{matrix} \right\rangle \end{cases}.$$

6 Conclusion

Above, we showed first that the dual numbers, similarly to the complex numbers are representable by tertions, and second, we gave an example that quaternions with unit elements I^*, J^* and K^* so that $(I^*)^2 = (J^*)^2 = (K^*)^2 = O^*$ are representable by tertions. We called these objects 0-quaternions. In a next part of the present research, we will discuss the remaining forms of the 0-quaternions.

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