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On unitary Zumkeller numbers

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Abstract: It is well known that if *n* is a Zumkeller number, then the positive divisors of *n* can be partitioned into two disjoint subsets of equal sum. Similarly for unitary Zumkeller number *n*, the unitary divisors of n can be partitioned into two disjoint subsets of equal sum. In this article, we have derived some results related to unitary Zumkeller number, unitary half-Zumkeller number and also presented some numerical examples.
Keyword: Zumkeller number, Unitary Zumkeller number, Divisor function, Unitary divisor function.
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1 Introduction

The well-known classical perfect numbers are the solution of the functional equation $\sigma(n) = 2n$, where the divisor function $\sigma(n)$ denotes the sum of all positive divisors of *n*. So far 51 (till May, 2024) such even perfect numbers are discovered [2]. All even perfect numbers [4] are of the form $n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are primes. The prime of the form $2^p - 1$ is called Mersenne prime. There is no example of an odd perfect number. Using the notion of classical perfect numbers, a generalized notion of perfect numbers had been developed in recent years.

Zumkeller numbers are one of the generalizations of the classical perfect numbers. A positive integer *n* is said to be a Zumkeller number if the positive divisors of *n* can be partitioned into two disjoint subsets A and B such that $\theta(A)=\theta(B)$, where $\theta(D)$ denotes the sum of all elements of the set D. For the Zumkeller number n, $\theta(A)+\theta(B)=\sigma(n)$. In 2003, the idea of Zumkeller number was first introduced by Zumkeller, [5]. Some examples of Zumkeller numbers are 6, 12, 20, 24, 28, 30, 40, etc. A positive integer n is said to be a half-Zumkeller number [3] if the proper positive divisors of n can be partitioned into two disjoint non-empty subsets of equal sum. For more results on Zumkeller and half-Zumkeller number, see [3].

Unitary perfect numbers [4] are also another generalization of perfect numbers. If n is a unitary perfect number, then $\sigma^*(n)=2n$, where the unitary divisor function $\sigma^*(n)$ denotes the sum of all unitary divisors of n. A positive integer d is a unitary divisor of a positive integer n if $d \mid n$ and $\gcd(d,\frac{n}{d})=1$. If $n > 1$ has the prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, then d is a unitary divisor of n if and only if $d = p_1^{u_1} p_2^{u_2} \cdots p_r^{u_r}$, where $u_i = 0$ or $u_i = \alpha_i$ for every $i \in \{1,2,3,\dots,r\}$. For example, the unitary divisors of 18 are 1, 2, 9 and 18. Note that $\sigma^*(n)$ is a multiplicative function, i.e., if gcd(m,n) = 1, then $\sigma^*(nm)=\sigma^*(n)\sigma^*(m)$.

The following is a standard well-known result for unitary divisor function σ^*.

Lemma 1.1. *If* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ *is the prime factorization of the number* $n > 1$, *then*

$$\sigma^*(n)=\prod_{i=1}^{r}(1+p_i^{\alpha_i}) \text{ and } \frac{\sigma^*(n)}{n}=\prod_{i=1}^{r}\frac{(1+p_i^{\alpha_i})}{p_i^{\alpha_i}} \le \prod_{i=1}^{r}\left(\frac{1+p_i}{p_i}\right).$$

2 Unitary Zumkeller numbers

Definition 2.1. A positive integer n is said to be a unitary Zumkeller number [6] if the unitary positive divisors of n can be partitioned into two disjoint subsets A and B such that $\theta(A)=\theta(B)$, where $\theta(D)$ denotes the sum of all elements of the set D. Following are some examples of unitary Zumkeller numbers, [6]:

$$6, 30, 42, 60, 66, 70, 78, 90, 102, 114, 138, 150, 174, 186, ...$$

The following proposition gives necessary conditions for a unitary Zumkeller number.

Proposition 2.1. *If n is a unitary Zumkeller number, then*

(i) $\sigma^*(n)$ *is even.*

(ii) $\sigma^*(n) \ge 2n$. *If* $\sigma^*(n)=2n$, *then n is a unitary perfect number, i.e., unitary perfect numbers are unitary Zumkeller numbers.*

The following proposition gives a necessary and sufficient condition for n to be a unitary Zumkeller number.

Proposition 2.2. *The integer n is unitary Zumkeller if and only if* $\frac{\sigma^*(n)}{2}-n$ *is a sum of some distinct proper positive unitary divisors of n.*

Proposition 2.3. *If n is a unitary Zumkeller number and p is a prime with* $\gcd(n,p) = 1$, *then* np^{α} *is a unitary Zumkeller number for any positive integer* α.

Proof. Let $\{A,B\}$ be a unitary Zumkeller partition of *n*, then $\{A\cup p^{\alpha}A, B\cup p^{\alpha}B\}$ is a unitary Zumkeller partition of np^{α}. □

Example 2.1. $A=\{1,2,5,9,10,18,45\}$ and $B=\{90\}$ are unitary Zumkeller partitions of 90. Since gcd(90, 7) = 1, so $A\cup p^{\alpha}A$ = $\{1,2,5,9,10,18,45,49,98,245,441,490,882,2205\}$ and $B\cup p^{\alpha}B=\{90,4410\}$ are unitary Zumkeller partitions of $4410=90\times 7^2$.

From the Proposition 2.3, we have the following corollary.

Corollary 2.1. *If the integer n is unitary Zumkeller and w is relatively prime to n, then nw is a unitary Zumkeller number.*

The following proposition follows from Lemma 1.1 which was mentioned earlier before the beginning of Section 2 and Proposition 2.1.

Proposition 2.4. *If* $n=\prod_{i=1}^{r} p_i^{\alpha_i}$ *is the prime factorization of the unitary Zumkeller number n, then*

$$2\le \prod_{i=1}^{r}\frac{p_i+1}{p_i}.$$

Proposition 2.5. *There is no unitary Zumkeller number of the form* $n=p^{\alpha}$, *where p is prime and* $\alpha\ge 1$.

Proposition 2.6. *The only unitary Zumkeller number n of the form* $p_1^{\alpha_1}p_2^{\alpha_2}$ *is* 6, *where* p_i *are distinct primes and* $\alpha_i\ge 1$, $i=1,2$.

Proof. Let $n=p_1^{\alpha_1}p_2^{\alpha_2}$ be a unitary Zumkeller number. Without loss of generality, let $p_1<p_2$. By Proposition 2.1, $\sigma^*(n)\ge 2n$. That is, $\sigma^*(p_1^{\alpha_1}p_2^{\alpha_2})\ge 2p_1^{\alpha_1}p_2^{\alpha_2}$.

This leads us to $p_1^{\alpha_1}(p_2^{\alpha_2}-1)\le p_2^{\alpha_2}+1$. This inequality can be written as

$$p_1^{\alpha_1}\le 1+\frac{2}{(p_2^{\alpha_2}-1)}.$$

$1\le\alpha_1$ and since $2\le p_1<p_2$, we see that $3\le p_2$. But, if $3<p_2$ or $1<\alpha_2$, then

$$1+\frac{2}{(p_2^{\alpha_2}-1)}<2.$$

Hence $p_2=3$ and $\alpha_2=1$. Going back to the same inequality, we get that

$$p_1^{\alpha_1}\le 1+\frac{2}{(3-1)}=2.$$

This gives us that $p_1=2$ and $\alpha_2=1$. This completes the proof. □

The above proposition tells us in particular that the only unitary Zumkeller number of the form $2^{\alpha} p_1^{\alpha_1}$ is 6. It would be interesting to investigate other unitary Zumkeller numbers of the form $2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. We provide one such result.

Proposition 2.7. *Let* $n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$ *be a unitary Zumkeller number, where* $3 \leq p_1 < p_2 < p_3 < \cdots < p_r$ *and* α_i *are positive integers, then*

(i) if $p_1 = 3$*, then* $p_i \geq 5$*, where* $2 \leq i \leq r$.

(ii) if $r = 2$ *and* $p_1 = 5$*, then* $p_2 = 7$.

Proof. If $n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is a unitary Zumkeller number, then from Proposition 2.4 it is evident that

$$2 \leq \frac{3}{2}\left(\frac{p_1+1}{p_1}\right)\left(\frac{p_2+1}{p_2}\right)\cdots\left(\frac{p_r+1}{p_r}\right).$$

(i) Any prime p, $1+\frac{1}{p} > 1$. If $p_1 = 3$, then from the above inequality we can write

$$2 \leq \frac{3}{2}.\frac{4}{3}\left(1+\frac{1}{p_2}\right)\left(1+\frac{1}{p_3}\right)\cdots\left(1+\frac{1}{p_r}\right) \leq 2\left(1+\frac{1}{p_2}\right)\left(1+\frac{1}{p_3}\right)\cdots\left(1+\frac{1}{p_r}\right).$$

The last inequality is true for any odd prime $p_i \geq 5$, where $2 \leq i \leq r$.

(ii) If $p_1 = 5$ and $p_2 \neq 7$, then $2 \leq \frac{3}{2}.\frac{6}{5}.\frac{12}{11} = 1.93636636 < 2$, which is a contradiction. Therefore, we must have $p_2 = 7$. □

Moreover, from this proposition we have the following result.

Corollary 2.2. *Let* p_i, $1 \leq i \leq r$*, be distinct primes and* $\alpha \geq 1$*. If* $n = 2^{\alpha} p_1 p_2 \cdots p_r$ *is a unitary Zumkeller number, then* $p_1 = 3$ *or* $p_1 = 5$ *and* $p_2 = 7$.

3 Unitary half-Zumkeller numbers

Definition 3.1. A positive integer n is said to be a unitary half-Zumkeller number [7] if the proper unitary positive divisors of n can be partitioned into two disjoint non-empty subsets C and D such that $\theta(C) = \theta(D)$.

The numbers 6, 12, 20, 30, 42, 56, 60, 66, 70, 72, 78, 84, 90, 120 are some examples of unitary half-Zumkeller number [7]. Unitary half-Zumkeller numbers may not be unitary Zumkeller number. For n = 120, $\sigma^*(n) = 216 < 2n$, so 120 is not a unitary Zumkeller number, but the proper unitary divisors of 120 can be partitioned into two disjoint sets $C = \{8, 40\}$ and $D = \{1, 3, 5, 15, 24\}$ of equal sums, i.e., 120 is a half-Zumkeller number.

Following is a necessary and sufficient condition for *n* to be a unitary half-Zumkeller number.

Proposition 3.1. *A positive integer n is a unitary half-Zumkeller number if and only if*

$$\frac{\sigma^*(n)-n}{2}$$

is a sum of some distinct proper positive unitary divisors of n.

Proposition 3.2. *If m and n are unitary half-Zumkeller numbers with* gcd(m, n) = 1, *then mn is a unitary half-Zumkeller number.*

Proposition 3.3. *If $n = 2^\alpha p$ is a unitary half-Zumkeller number, then p must be a Mersenne prime or Fermat prime.*

Proof. The proper unitary divisors of the number $n = 2^\alpha p$ are 1, 2^α and p. Let C and D be two partitions of n. Then $C = \{2^\alpha\}$ and $D = \{1, p\}$ or $C = \{1, 2^\alpha\}$ and $D = \{p\}$.

Case 1. If $C = \{2^\alpha\}$ and $D = \{1, p\}$, then $2^\alpha = 1 + p \Rightarrow p = 2^\alpha - 1$. The prime number of the form $p = 2^\alpha - 1$ is called Mersenne prime, where α must be a prime number.

Case 2. If $C = \{1, 2^\alpha\}$ and $D = \{p\}$, then $2^\alpha + 1 = p$. The prime number of the form $p = 2^\alpha + 1$ is called Fermat prime, where α must be a power of 2. □

From the above proposition, we have the following corollary.

Corollary 3.1. *If $n = 2^{p-1}\left(2^p - 1\right)$ is a perfect number, where $2^p - 1$ is Mersenne prime, then* $2n$ *is a unitary half-Zumkeller number.*

Proposition 3.4. *If $n = p_1^{\alpha_1} p_2^{\alpha_2}$ with $p_1 < p_2$ is a unitary half-Zumkeller number, then n* = 72, *i.e., $p_1 = 2, \alpha_1 = 3, p_2 = 3, \alpha_2 = 2$.*

Proof. If $n = p_1^{\alpha_1} p_2^{\alpha_2}$ is an odd and unitary half-Zumkeller number, then $\sigma^*(n) - n$ must be even. Since the difference between two odd numbers is always even, so $\sigma^*(n)$ must be odd. But $\sigma^*(n) = (1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2})$, which is an even integer. Therefore, $\sigma^*(n) - n$ cannot be even if $n = p_1^{\alpha_1} p_2^{\alpha_2}$ is odd. Thus, there does not exist any odd unitary half-Zumkeller number of the form $n = p_1^{\alpha_1} p_2^{\alpha_2}$. This implies that $p_1 = 2$ and $\alpha_1 \geq 1$.

Let $n = 2^{\alpha_1} p^{\alpha_2}$. Using Proposition 3.3, we assume without loss of generality that $\alpha_2 \geq 2$. Now the unitary divisors of n are $1, 2^{\alpha_1}, p^{\alpha_2}$ and $2^{\alpha_1} p^{\alpha_2}$. For n to be unitary half-Zumkeller, either $1 + 2^{\alpha_1} = p^{\alpha_2}$ or $1 + p^{\alpha_2} = 2^{\alpha_1}$.

Case 1. $1 + 2^{\alpha_1} = p^{\alpha_2}$ and $\alpha_2 = 2m$. In this case, $2^{\alpha_1} = p^{2m} - 1$. Hence $2^{\alpha_1} = (p^m - 1)(p^m + 1)$. This implies that $(p^m + 1) = 2^a$ and $(p^m - 1) = 2^b$ for some positive integers a and b. Hence, $2 = 2^a - 2^b$. This implies that $a = 2$ and $b = 1$. This in turn implies that p = 3 and m = 1. This further implies that $\alpha_1 = 3$.

Case 2. $1+2^{\alpha_1}=p^{\alpha_2}$ and $\alpha_2=2m+1$. Here $2^{\alpha_1}+1=p^{2m+1}$. In this case, $2^{\alpha_1}=p^{(2m+1)}-1$ and hence $2^{\alpha_1}=(p-1)(p^{2m}+p^{(2m-1)}+\cdots+p+1)$. But $(p^{2m}+p^{(2m-1)}+\cdots+p+1)$ is odd for any odd prime p. Therefore, $2^{\alpha_1}=(p-1)(p^{2m}+p^{(2m-1)}+\cdots+p+1)$ is impossible.

Case 3. $1+p^{\alpha_2}=2^{\alpha_1}$ and $\alpha_2=2m+1$. Here $2^{\alpha_1}=p^{(2m+1)}+1=(p+1)(p^{2m}-p^{(2m-1)}+\cdots+1)$. But $(p^{2m}-p^{(2m-1)}+\cdots+1)$ is odd for any odd prime p. Hence, $2^{\alpha_1}=(p+1)(p^{2m}-p^{(2m-1)}+\cdots+1)$ is impossible.

Case 4. $1+p^{\alpha_2}=2^{\alpha_1}$ and $\alpha_2=2m$. Here $2^{\alpha_1}=p^{2m}+1$. Clearly, $\alpha_1\neq 1,2$. Since p is an odd prime, p is of the form $4t\pm 1$ for some positive integer t. Then, $p^{2m}\equiv 1 \pmod 4$. Hence, $p^{2m}+1\equiv 2 \pmod 4$. But for $\alpha_1\geq 3$, $2^{\alpha_1}\equiv 0 \pmod 4$. Hence $2^{\alpha_1}=p^{2m}+1$ is impossible. □

Proposition 3.5. *There does not exist any odd unitary half-Zumkeller number.*

We leave the following problem for other researchers to solve.

Conjecture 3.1. *If n is even and Unitary Zumkeller number, then n is unitary half-Zumkeller number.*

4 Conclusion

We have studied unitary Zumkeller numbers and unitary half-Zumkeller numbers in this paper. Similarly, it would be interesting to study generalizations of Zumkeller numbers of other forms.

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