On unitary Zumkeller numbers

Bhabesh Das

Department of Mathematics, Arya Vidyapeeth College (Autonomous)
Guwahati – 16, Assam, India
e-mail: mtbdas99@gmail.com

Received: 31 March 2023
Accepted: 12 July 2024

Abstract: It is well known that if \( n \) is a Zumkeller number, then the positive divisors of \( n \) can be partitioned into two disjoint subsets of equal sum. Similarly for unitary Zumkeller number \( n \), the unitary divisors of \( n \) can be partitioned into two disjoint subsets of equal sum. In this article, we have derived some results related to unitary Zumkeller number, unitary half-Zumkeller number and also presented some numerical examples.

Keyword: Zumkeller number, Unitary Zumkeller number, Divisor function, Unitary divisor function.

2020 Mathematics Subject Classification: 11A25.

1 Introduction

The well-known classical perfect numbers are the solution of the functional equation \( \sigma(n) = 2n \), where the divisor function \( \sigma(n) \) denotes the sum of all positive divisors of \( n \). So far 51 (till May, 2024) such even perfect numbers are discovered [2]. All even perfect numbers [4] are of the form \( n = 2^{p-1}(2^p - 1) \), where \( p \) and \( 2^p - 1 \) are primes. The prime of the form \( 2^p - 1 \) is called Mersenne prime. There is no example of an odd perfect number. Using the notion of classical perfect numbers, a generalized notion of perfect numbers had been developed in recent years.

Zumkeller numbers are one of the generalizations of the classical perfect numbers. A positive integer \( n \) is said to be a Zumkeller number if the positive divisors of \( n \) can be partitioned into two
disjoint subsets $A$ and $B$ such that $\theta(A) = \theta(B)$, where $\theta(D)$ denotes the sum of all elements of the set $D$. For the Zumkeller number $n$, $\theta(A) + \theta(B) = \sigma(n)$. In 2003, the idea of Zumkeller number was first introduced by Zumkeller, [5]. Some examples of Zumkeller numbers are 6, 12, 20, 24, 28, 30, 40, etc. A positive integer $n$ is said to be a half-Zumkeller number [3] if the proper positive divisors of $n$ can be partitioned into two disjoint non-empty subsets of equal sum. For more results on Zumkeller and half-Zumkeller number, see [3].

Unitary perfect numbers [4] are also another generalization of perfect numbers. If $n$ is a unitary perfect number, then $\sigma^*(n) = 2n$, where the unitary divisor function $\sigma^*(n)$ denotes the sum of all unitary divisors of $n$. A positive integer $d$ is a unitary divisor of a positive integer $n$ if $d \mid n$ and $\gcd(d, \frac{n}{d}) = 1$. If $n > 1$ has the prime factorization $n = p_1^{u_1}p_2^{u_2} \cdots p_r^{u_r}$, then $d$ is a unitary divisor of $n$ if and only if $d = p_1^{v_1}p_2^{v_2} \cdots p_r^{v_r}$, where $u_i = 0$ or $u_i = \alpha_i$ for every $i \in \{1, 2, 3, \ldots, r\}$. For example, the unitary divisors of 18 are 1, 2, 9 and 18. Note that $\sigma^*(n)$ is a multiplicative function, i.e., if $\gcd(m,n) = 1$, then $\sigma^*(mn) = \sigma^*(n)\sigma^*(m)$.

The following is a standard well-known result for unitary divisor function $\sigma^*$.

**Lemma 1.1.** If $n = p_1^{u_1}p_2^{u_2} \cdots p_r^{u_r}$ is the prime factorization of the number $n > 1$, then

$$\sigma^*(n) = \prod_{i=1}^r (1 + p_i^{u_i})$$

and

$$\frac{\sigma^*(n)}{n} = \prod_{i=1}^r \left(1 + \frac{p_i^{u_i}}{p_i^{u_i}}\right) \leq \prod_{i=1}^r \left(1 + \frac{1}{p_i}\right).$$

## 2 Unitary Zumkeller numbers

**Definition 2.1.** A positive integer $n$ is said to be a unitary Zumkeller number [6] if the unitary positive divisors of $n$ can be partitioned into two disjoint subsets $A$ and $B$ such that $\theta(A) = \theta(B)$, where $\theta(D)$ denotes the sum of all elements of the set $D$. Following are some examples of unitary Zumkeller numbers, [6]:

$$6, 30, 42, 60, 66, 70, 78, 90, 102, 114, 138, 150, 174, 186, \ldots$$

The following proposition gives necessary conditions for a unitary Zumkeller number.

**Proposition 2.1.** If $n$ is a unitary Zumkeller number, then

(i) $\sigma^*(n)$ is even.

(ii) $\sigma^*(n) \geq 2n$. If $\sigma^*(n) = 2n$, then $n$ is a unitary perfect number, i.e., unitary perfect numbers are unitary Zumkeller numbers.

The following proposition gives a necessary and sufficient condition for $n$ to be a unitary Zumkeller number.

**Proposition 2.2.** The integer $n$ is unitary Zumkeller if and only if $\frac{\sigma^*(n)}{2} - n$ is a sum of some distinct proper positive unitary divisors of $n$. 437
Proposition 2.3. If $n$ is a unitary Zumkeller number and $p$ is a prime with $\gcd(n, p) = 1$, then $np^\alpha$ is a unitary Zumkeller number for any positive integer $\alpha$.

Proof. Let $\{A, B\}$ be a unitary Zumkeller partition of $n$, then $\{A \cup p^\alpha A, B \cup p^\alpha B\}$ is a unitary Zumkeller partition of $np^\alpha$. \qed

Example 2.1. $A = \{1,2,5,9,10,18,45\}$ and $B = \{90\}$ are unitary Zumkeller partitions of 90. Since $\gcd(90, 7) = 1$, so $A \cup p^\alpha A = \{1,2,5,9,10,18,45,49,98,245,441,490,882,2205\}$ and $B \cup p^\alpha B = \{90,4410\}$ are unitary Zumkeller partitions of $4410 = 90 \times 7^2$.

From the Proposition 2.3, we have the following corollary.

Corollary 2.1. If the integer $n$ is unitary Zumkeller and $w$ is relatively prime to $n$, then $nw$ is a unitary Zumkeller number.

The following proposition follows from Lemma 1.1 which was mentioned earlier before the beginning of Section 2 and Proposition 2.1.

Proposition 2.4. If $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ is the prime factorization of the unitary Zumkeller number $n$, then

$$2 \leq \prod_{i=1}^{r} \frac{p_i + 1}{p_i}.$$

Proposition 2.5. There is no unitary Zumkeller number of the form $n = p^\alpha$, where $p$ is prime and $\alpha \geq 1$.

Proposition 2.6. The only unitary Zumkeller number $n$ of the form $p_1^{\alpha_1} p_2^{\alpha_2}$ is 6, where $p_i$ are distinct primes and $\alpha_i \geq 1$, $i = 1, 2$.

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2}$ be a unitary Zumkeller number. Without loss of generality, let $p_1 < p_2$. By Proposition 2.1, $\sigma^*(n) \geq 2n$. That is, $\sigma^*(p_1^{\alpha_1} p_2^{\alpha_2}) \geq 2p_1^{\alpha_1} p_2^{\alpha_2}$.

This leads to $p_1^{\alpha_1} (p_2^{\alpha_2} - 1) \leq p_2^{\alpha_2} + 1$. This inequality can be written as

$$\frac{p_1^{\alpha_1}}{1} \leq 1 + \frac{2}{(p_2^{\alpha_2} - 1)}.$$

$1 \leq \alpha_1$ and since $2 \leq p_1 < p_2$, we see that $3 \leq p_2$. But, if $3 < p_2$ or $1 < \alpha_2$, then

$$1 + \frac{2}{(p_2^{\alpha_2} - 1)} < 2.$$

Hence $p_2 = 3$ and $\alpha_2 = 1$. Going back to the same inequality, we get that

$$p_1^{\alpha_1} \leq 1 + \frac{2}{(3-1)} = 2.$$

This gives us that $p_1 = 2$ and $\alpha_2 = 1$. This completes the proof. \qed
The above proposition tells us in particular that the only unitary Zumkeller number of the form $2^a p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is 6. It would be interesting to investigate other unitary Zumkeller numbers of the form $2^a p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. We provide one such result.

**Proposition 2.7.** Let $n = 2^a p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_r^{a_r}$ be a unitary Zumkeller number, where $3 \leq p_1 < p_2 < p_3 < \cdots < p_r$ and $a_i$ are positive integers, then

(i) if $p_1 = 3$, then $p_i \geq 5$, where $2 \leq i \leq r$.

(ii) if $r = 2$ and $p_1 = 5$, then $p_2 = 7$.

**Proof.** If $n = 2^a p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is a unitary Zumkeller number, then from Proposition 2.4 it is evident that

$$2 \leq \frac{3}{2} \left( \frac{p_1+1}{p_1} \right) \left( \frac{p_2+1}{p_2} \right) \cdots \left( \frac{p_r+1}{p_r} \right).$$

(i) Any prime $p$, $1 + \frac{1}{p} > 1$. If $p_1 = 3$, then from the above inequality we can write

$$2 \leq \frac{3}{2} \left( 1 + \frac{1}{p_1} \right) \left( 1 + \frac{1}{p_2} \right) \cdots \left( 1 + \frac{1}{p_r} \right) \leq 2 \left( 1 + \frac{1}{p_2} \right) \left( 1 + \frac{1}{p_3} \right) \cdots \left( 1 + \frac{1}{p_r} \right).$$

The last inequality is true for any odd prime $p_i \geq 5$, where $2 \leq i \leq r$.

(ii) If $p_1 = 5$ and $p_2 \neq 7$, then $2 \leq \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{12}{11} = 1.93636636 < 2$, which is a contradiction.

Therefore, we must have $p_2 = 7$. \(\square\)

Moreover, from this proposition we have the following result.

**Corollary 2.2.** Let $p_i$, $1 \leq i \leq r$, be distinct primes and $a \geq 1$. If $n = 2^a p_1 p_2 \cdots p_r$ is a unitary Zumkeller number, then $p_1 = 3$ or $p_1 = 5$ and $p_2 = 7$.

## 3 Unitary half-Zumkeller numbers

**Definition 3.1.** A positive integer $n$ is said to be a unitary half-Zumkeller number [7] if the proper unitary positive divisors of $n$ can be partitioned into two disjoint non-empty subsets $C$ and $D$ such that $\theta(C) = \theta(D)$.

The numbers 6, 12, 20, 30, 42, 56, 60, 66, 70, 72, 78, 84, 90, 120 are some examples of unitary half-Zumkeller number [7]. Unitary half-Zumkeller numbers may not be unitary Zumkeller number. For $n = 120$, $\sigma(n) = 216 < 2n$, so 120 is not a unitary Zumkeller number, but the proper unitary divisors of 120 can be partitioned into two disjoint sets $C = \{8, 40\}$ and $D = \{1, 3, 5, 15, 24\}$ of equal sums, i.e., 120 is a half-Zumkeller number.

Following is a necessary and sufficient condition for $n$ to be a unitary half-Zumkeller number.
Proposition 3.1. A positive integer \( n \) is a unitary half-Zumkeller number if and only if

\[
\frac{\sigma^*(n)-n}{2}
\]

is a sum of some distinct proper positive unitary divisors of \( n \).

Proposition 3.2. If \( m \) and \( n \) are unitary half-Zumkeller numbers with \( \gcd(m, n) = 1 \), then \( mn \) is a unitary half-Zumkeller number.

Proposition 3.3. If \( n = 2^a p \) is a unitary half-Zumkeller number, then \( p \) must be a Mersenne prime or Fermat prime.

Proof. The proper unitary divisors of the number \( n = 2^a p \) are 1, \( 2^a \) and \( p \). Let \( C \) and \( D \) be two partitions of \( n \). Then \( C = \{2^a\} \) and \( D = \{1, p\} \) or \( C = \{1, 2^a\} \) and \( D = \{p\} \).

Case 1. If \( C = \{2^a\} \) and \( D = \{1, p\} \), then \( 2^a = 1 + p \Rightarrow p = 2^a - 1 \). The prime number of the form \( p = 2^a - 1 \) is called Mersenne prime, where \( \alpha \) must be a prime number.

Case 2. If \( C = \{1, 2^a\} \) and \( D = \{p\} \), then \( 2^a + 1 = p \). The prime number of the form \( p = 2^a + 1 \) is called Fermat prime, where \( \alpha \) must be a power of 2. \( \square \)

From the above proposition, we have the following corollary.

Corollary 3.1. If \( n = 2^{p-1}(2^p - 1) \) is a perfect number, where \( 2^p - 1 \) is Mersenne prime, then \( 2n \) is a unitary half-Zumkeller number.

Proposition 3.4. If \( n = p_1^{\alpha_1} p_2^{\alpha_2} \) with \( p_1 < p_2 \) is a unitary half-Zumkeller number, then \( n = 72 \), i.e., \( p_1 = 2, \alpha_1 = 3, p_2 = 3, \alpha_2 = 2 \).

Proof. If \( n = p_1^{\alpha_1} p_2^{\alpha_2} \) is an odd and unitary half-Zumkeller number, then \( \sigma^*(n) - n \) must be even. Since the difference between two odd numbers is always even, so \( \sigma^*(n) \) must be odd. But \( \sigma^*(n) = (1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2}) \), which is an even integer. Therefore, \( \sigma^*(n) - n \) cannot be even if \( n = p_1^{\alpha_1} p_2^{\alpha_2} \) is odd. Thus, there does not exist any odd unitary half-Zumkeller number of the form \( n = p_1^{\alpha_1} p_2^{\alpha_2} \). This implies that \( p_1 = 2 \) and \( \alpha_i \geq 1 \).

Let \( n = 2^{a_i} p^{a_2} \). Using Proposition 3.3, we assume without loss of generality that \( \alpha_2 \geq 2 \). Now the unitary divisors of \( n \) are \( 1, 2^{a_i}, p^{a_2} \) and \( 2^{a_i} p^{a_2} \). For \( n \) to be unitary half-Zumkeller, either \( 1 + 2^{a_i} = p^{a_2} \) or \( 1 + p^{a_2} = 2^{a_i} \).

Case 1. \( 1 + 2^{a_i} = p^{a_2} \) and \( \alpha_2 = 2m \). In this case, \( 2^{a_i} = p^{2m} - 1 \). Hence \( 2^{a_i} = (p^m - 1)(p^m + 1) \). This implies that \( p^m + 1 = 2^a \) and \( p^m - 1 = 2^b \) for some positive integers \( a \) and \( b \). Hence, \( 2 = 2^a - 2^b \). This implies that \( a = 2 \) and \( b = 1 \). This in turn implies that \( p = 3 \) and \( m = 1 \). This further implies that \( \alpha_2 = 3 \).
Case 2. \(1 + 2^{q_1} = p^{\alpha_2}\) and \(\alpha_2 = 2m + 1\). Here \(2^{q_1} + 1 = p^{2m+1}\). In this case, \(2^{q_1} = p^{(2m+1)} - 1\) and hence \(2^{q_1} = (p - 1)(p^{2m} + p^{2m-1} + \cdots + p + 1)\). But \((p^{2m} + p^{2m-1} + \cdots + p + 1)\) is odd for any odd prime \(p\). Therefore, \(2^{q_1} = (p - 1)(p^{2m} + p^{2m-1} + \cdots + p + 1)\) is impossible.

Case 3. \(1 + p^{q_2} = 2^{q_1}\) and \(\alpha_2 = 2m + 1\). Here \(2^{q_1} = p^{(2m+1)} + 1 = (p + 1)(p^{2m} - p^{2m-1} + \cdots + 1)\). But \((p^{2m} - p^{2m-1} + \cdots + 1)\) is odd for any odd prime \(p\). Hence, \(2^{q_1} = (p + 1)(p^{2m} - p^{2m-1} + \cdots + 1)\) is impossible.

Case 4. \(1 + p^{q_2} = 2^{q_1}\) and \(\alpha_2 = 2m\). Here \(2^{q_1} = p^{2m} + 1\). Clearly, \(\alpha_i \neq 1, 2\). Since \(p\) is an odd prime, \(p\) is of the form \(4t \pm 1\) for some positive integer \(t\). Then, \(p^{2m} \equiv 1 \pmod{4}\). Hence, \(p^{2m} + 1 \equiv 2 \pmod{4}\). But for \(\alpha_i \geq 3\), \(2^{q_i} \equiv 0 \pmod{4}\). Hence \(2^{q_i} = p^{2m} + 1\) is impossible.

Proposition 3.5. There does not exist any odd unitary half-Zumkeller number.

We leave the following problem for other researchers to solve.

Conjecture 3.1. If \(n\) is even and Unitary Zumkeller number, then \(n\) is unitary half-Zumkeller number.

4 Conclusion

We have studied unitary Zumkeller numbers and unitary half-Zumkeller numbers in this paper. Similarly, it would be interesting to study generalizations of Zumkeller numbers of other forms.

Acknowledgements

The author sincerely thanks the anonymous referees for their valuable feedback and suggestions, which helped in improving the overall presentation of the paper.

References


Zumkeller or integer-perfect numbers: Numbers $n$ whose divisors can be partitioned into two disjoint sets with equal sum. Available online at: https://oeis.org/A083207.

Unitary Zumkeller numbers: Numbers $k$ whose unitary divisors can be partitioned into two disjoint subsets whose sums are both $\sigma_1(k)/2$. Available online at: https://oeis.org/A290466.

Unitary half-Zumkeller numbers: Numbers $k$ whose unitary proper divisors can be partitioned into two disjoint sets whose sums are equal. Available online at: https://oeis.org/A290467.