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On the distribution of powerful and r-free lattice points

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Abstract: Let $1 < c < 2$. For $m, n \in \mathbb{N}$, a lattice point (m, n) is powerful if and only if $\gcd(m, n)$ is a powerful number, where $\gcd(*, *)$ is the greatest common divisor function. In this paper, we count the number of the ordered pairs (m, n), $m, n \leq x$ such that the lattice point $(\lfloor m^c \rfloor, \lfloor n^c \rfloor)$ is powerful. Moreover, we study r-free lattice points analogues of powerful lattice points.
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1 Introduction and results

Let r be a fixed integer ≥ 2. We say that a positive integer n is powerful (or r-full) if for any prime $p \mid n$ we have that $p^r \mid n$. Particularly, 2-full and 3-full numbers are called square-full and cube-full, respectively. Let $G(r)$ denote the set of all powerful numbers and let $f_r(n)$ be the characteristic function of $G(r)$. For $\Re(s) > 1/r$ we have

$$F_r(s) = \sum_{n=1}^{\infty} f_r(n)\, n^{-s} = \prod_p \left(1 + \frac{p^{-rs}}{1 - p^{-s}}\right) \tag{1}$$

(see [10, p. 33]). In 2022 Shunqi Ma [13] introduced the notion of r-full lattice points in $\mathbb{Z}^2$. Namely, a non-zero lattice point (m, n) is r-full if and only if $\gcd(m, n)$ is an r-full number, where $\gcd(*, *)$ is the greatest common divisor function. In [13], he showed that, for $x \geq 2$, we have

$$S_r(x) = x^2 \prod_p \left(1 - p^{-2} + p^{-2r}\right) + O(x \log^2 x),$$

where $S_r(x)$ denotes the number of r-full lattice points in the square area $[1, x] \times [1, x]$. Moreover, Shunqi Ma studied r-full lattice points in $\mathbb{N}^2$ from the viewpoint of random walks. For $0 < \alpha < 1$, an α-random walk is defined by

$$P_{i+1} = P_i + \begin{cases} (1, 0), & \text{with probability } \alpha, \\ (0, 1), & \text{with probability } 1 - \alpha, \end{cases}$$

for $i = 0, 1, 2, \ldots$, where $P_i = (x_i, y_i)$ is the coordinate of the α-random walker at the i-th step and $P_0 = (0, 0)$. For an α-random walk, define a sequence of random variables $\{W_i\}_{i \in \mathbb{N}}$ by

$$W_i = \begin{cases} 1, & \text{if } P_i \text{ is } r\text{-full}, \\ 0, & \text{otherwise}. \end{cases}$$

Shunqi Ma gave the density of r-full lattice points on a path of α-random walker. He showed that, for any $\alpha \in (0, 1)$, we have

$$\lim_{n \to \infty} \bar{S}_{r,\alpha}(n) = \prod_p \left(1 - p^{-2} + p^{-2r}\right),$$

where

$$\bar{S}_{r,\alpha}(n) = \frac{1}{n} \sum_{i=1}^{n} W_i.$$

The Piatetski-Shapiro sequence of parameter c is defined by

$$\mathbb{N}^c = \{\lfloor n^c \rfloor\}_{n \in \mathbb{N}} \qquad (c > 1, c \notin \mathbb{N}),$$

where $\lfloor z \rfloor$ is the integer part of $z \in \mathbb{R}$. The Piatetski-Shapiro sequence was introduced by Piatetski-Shapiro [14] to study prime numbers in a sequence of the form $\lfloor f(n) \rfloor$, where $f(n)$ is a polynomial.

The study of the distribution of arithmetical functions on Piatetski-Shapiro is studied by many authors; see, for example, [1–9, 11–13, 15–20] and the references contained therein. Drawing inspiration from this fact, we shall consider these problems on the two-dimensional lattice $\mathbb{N}^c \times \mathbb{N}^c$ instead of $\mathbb{Z}^2$, where the sequence $\mathbb{N}^c := \{\lfloor n^c \rfloor\}_{n\in\mathbb{N}},\ (c > 1, c \notin \mathbb{N},)$. First, for $1 < c < 2$, we let

$$S_r^c(x) := \sum_{\substack{m,n\le x \\ \gcd(\lfloor m^c\rfloor, \lfloor n^c\rfloor) \text{ is } r\text{-full}}} 1.$$

We expect that,

$$\lim_{x\to\infty} \frac{S_r^c(x)}{x^2} = \lim_{x\to\infty} \frac{S_r(x)}{x^2} = \prod_p \left(1 - p^{-2} + p^{-2r}\right).$$

We prove the following theorem.

Theorem 1.1. *For $x \ge 1$, we have*

$$S_r^c(x) = x^2 \prod_p \left(1 - p^{-2} + p^{-2r}\right) + \begin{cases} O\left(x^{(c+4)/3}\right), & \text{for } 1 < c \le \frac{5}{4}, \\ O\left(x^{c+1/2}\right), & \text{for } \frac{5}{4} < c < \frac{3}{2}. \end{cases}$$

Moreover, we shall study r-free lattice points analogues of r-full lattice points. A positive integer n is called r-free whenever it is not divisible by the r-th power of any prime. As usual, 2-free and 3-free integers are called square-free and cube-free, respectively. Let $q_r(n)$ be the characteristic function of the set of r-free numbers, and for $\Re(s) > 1$

$$Q_r(s) = \sum_{n=1}^{\infty} q_r(n)\, n^{-s} = \frac{\zeta(s)}{\zeta(rs)}, \tag{2}$$

(see [10, p. 32]). A non-zero lattice point (m, n) is r-free if and only if $\gcd(m, n)$ is an r-free number. We obtain the following theorem.

Theorem 1.2. *For $x \ge 1$, we have*

$$\sum_{\substack{m,n\, \le\, x \\ \gcd(\lfloor m^c\rfloor, \lfloor n^c\rfloor) \text{ is } r\text{-free}}} 1 = \frac{1}{\zeta(2r)} x^2 + \begin{cases} O\left(x^{(c+4)/3}\right), & \text{for } 1 < c \le \frac{5}{4}, \\ O\left(x^{c+1/2} \log x\right), & \text{for } \frac{5}{4} < c < \frac{3}{2}. \end{cases}$$

Furthermore, we consider these problem over the different sequences. We prove the following theorem, which are generalized results of Theorems 1.1 and 1.2.

Theorem 1.3. *Let $1 < c_1 \le c_2 < 3/2$. For $x \ge 1$, we have*

$$\sum_{\substack{m,n\, \le\, x \\ \gcd(\lfloor m^{c_1}\rfloor, \lfloor n^{c_2}\rfloor) \text{ is powerful}}} 1 = x^2 \prod_p \left(1 - p^{-2} + p^{-2r}\right) + \begin{cases} O\left(x^{(c_2+4)/3}\right), & \text{for } 1 < c_1 \le \frac{5}{4}, \\ O\left(x^{1/2+(2c_1+c_2)/3}\right), & \text{for } \frac{5}{4} < c_1 < \frac{3}{2}. \end{cases}$$

Theorem 1.4. *Let $1 < c_1 \le c_2 < 3/2$. For $x \ge 1$, we have*

$$\sum_{\substack{m,n\, \le\, x \\ \gcd(\lfloor m^{c_1}\rfloor, \lfloor n^{c_2}\rfloor) \text{ is } r\text{-free}}} 1 = \frac{1}{\zeta(2r)} x^2 + \begin{cases} O\left(x^{(c_2+4)/3}\right), & \text{for } 1 < c_1 \le \frac{5}{4}, \\ O\left(x^{1/2+(2c_1+c_2)/3} \log x\right), & \text{for } \frac{5}{4} < c_1 < \frac{3}{2}. \end{cases}$$

2 Lemmas

The main ingredient in the following proof is a good estimation for the number of integer n up to x such that $\lfloor n^c \rfloor$ belongs to an arithmetic progression. Deshouillers [4] proved the following lemma.

Lemma 2.1. *For* $1 < c < 2$*, let* $x \in \mathbb{R}$ *and* $a, q \in \mathbb{Z}$ *such that* $0 \le a < q \le x^c$,

$$\sum_{\substack{n \le x \\ \lfloor n^c \rfloor \equiv a \pmod q}} 1 = \frac{x}{q} + O\Big(\min\Big(\frac{x^c}{q}, \frac{x^{(c+1)/3}}{q^{1/3}}\Big)\Big).$$

To prove Theorems 1.1–1.4, we need the following lemmas.

Lemma 2.2. *For* $1 < c < 2$*, we have*

$$\sum_{\substack{m,n \le x \\ \gcd(\lfloor m^c \rfloor, \lfloor n^c \rfloor) = d}} 1 = \frac{1}{d^2\zeta(2)}x^2 + O\Big(\frac{x^{(c+4)/3}}{d^{4/3}}\Big) + O\Big(\frac{x^{c+1/2}}{d}\Big).$$

Proof. Let $x > 1$, we have

$$A_c(d;x) := \sum_{\substack{m,n \le x \\ \gcd(\lfloor m^c \rfloor, \lfloor n^c \rfloor) = d}} 1.$$

We have

$$\begin{aligned} A_c(d;x) = \sum_{\substack{m,n \le x \\ d \mid \lfloor m^c \rfloor,\ d \mid \lfloor n^c \rfloor \\ \gcd(\lfloor \frac{m^c}{d} \rfloor, \lfloor \frac{n^c}{d} \rfloor) = 1}} 1 &= \sum_{\substack{m,n \le x \\ d \mid \lfloor m^c \rfloor,\ d \mid \lfloor n^c \rfloor}} \sum_{r \mid \gcd(\lfloor \frac{m^c}{d} \rfloor, \lfloor \frac{n^c}{d} \rfloor)} \mu(r) \\ &= \sum_{\substack{m,n \le x \\ d \mid \lfloor m^c \rfloor,\ d \mid \lfloor n^c \rfloor}} \sum_{\substack{r \mid \lfloor \frac{m^c}{d} \rfloor \\ r \mid \frac{\lfloor n^c \rfloor}{d}}} \mu(r) \\ &= \sum_{r \le \frac{x^c}{d}} \mu(r) \sum_{\substack{m,n \le x \\ \lfloor m^c \rfloor \equiv 0 \pmod{rd} \\ \lfloor n^c \rfloor \equiv 0 \pmod{rd}}} 1 \end{aligned}$$

In view of Lemma 2.1, we have

$$\begin{aligned} A_c(d;x) &= \sum_{r \le \frac{x^c}{d}} \mu(r) \Big(\frac{x}{rd} + O\Big(\min\Big\{\frac{x^{(c+1)/3}}{(rd)^{1/3}}, \frac{x^c}{rd}\Big\}\Big)\Big)^2 \\ &= \sum_{r \le x^{c-1/2}d^{-1}} \mu(r)\Big(\frac{x}{rd} + O\Big(\frac{x^{(c+1)/3}}{(rd)^{1/3}}\Big)\Big)^2 + \sum_{x^{c-1/2}d^{-1} < r \le \frac{x^c}{d}} \mu(r)\Big(\frac{x}{rd} + O\Big(\frac{x^c}{rd}\Big)\Big)^2 \\ &= \sum_{r \le x^{c-1/2}d^{-1}} \mu(r)\Big(\frac{x^2}{r^2d^2} + O\Big(\frac{x^{(c+4)/3}}{(rd)^{4/3}}\Big) + O\Big(\frac{x^{(2c+2)/3}}{(rd)^{2/3}}\Big)\Big) \\ &\quad + \sum_{x^{c-1/2}d^{-1} < r \le \frac{x^c}{d}} \mu(r)\Big(\frac{x^2}{r^2d^2} + O\Big(\frac{x^{c+1}}{r^2d^2}\Big) + O\Big(\frac{x^{2c}}{r^2d^2}\Big)\Big) \end{aligned}$$

$$= \frac{x^2}{d^2} \sum_{r \le x^c d^{-1}} \frac{\mu(r)}{r^2} + O\Big(\frac{x^{(c+4)/3}}{d^{4/3}} \sum_{r \le x^{c-1/2} d^{-1}} \frac{1}{r^{4/3}}\Big) + O\Big(\frac{x^{(2c+2)/3}}{d^{2/3}} \sum_{r \le x^{c-1/2} d^{-1}} \frac{1}{r^{2/3}}\Big)$$

$$+ O\Big(\frac{x^{c+1}}{d^2} \sum_{x^{c-1/2} d^{-1} < r \le \frac{x^c}{d}} \frac{1}{r^2}\Big) + O\Big(\frac{x^{2c}}{d^2} \sum_{x^{c-1/2} d^{-1} < r \le \frac{x^c}{d}} \frac{1}{r^2}\Big)$$

$$= \frac{x^2}{d^2 \zeta(2)} + O\Big(\frac{x^{(c+4)/3}}{d^{4/3}}\Big) + O\Big(\frac{x^{c+1/2}}{d}\Big).$$

This proves Lemma 2.2. □

Next Lemma is a generalized result of Lemma 2.2.

Lemma 2.3. *For* $1 < c_1 \le c_2 < 2$, *we have*

$$\sum_{\substack{m,n \le x \\ \gcd(\lfloor m^{c_1} \rfloor, \lfloor n^{c_2} \rfloor) = d}} 1 = \frac{1}{d^2 \zeta(2)} x^2 + O\Big(\frac{x^{(c_2+4)/3}}{d^{4/3}}\Big) + O\Big(\frac{x^{1/2+(2c_1+c_2)/3}}{d}\Big).$$

Proof. Let $x > 1$, we have

$$A_{c_1,c_2}(d;x) := \sum_{\substack{m,n \le x \\ \gcd(\lfloor m^{c_1} \rfloor, \lfloor n^{c_2} \rfloor) = d}} 1.$$

We have

$$A_{c_1,c_2}(d;x) = \sum_{\substack{m,n \le x \\ d | \lfloor m^{c_1} \rfloor,\ d | \lfloor n^{c_2} \rfloor \\ \gcd(\lfloor \frac{m^{c_1}}{d} \rfloor, \lfloor \frac{n^{c_2}}{d} \rfloor) = 1}} 1 = \sum_{\substack{m,n \le x \\ d | \lfloor m^{c_1} \rfloor,\ d | \lfloor n^{c_2} \rfloor}} \sum_{r | \gcd(\lfloor \frac{m^{c_1}}{d} \rfloor, \frac{\lfloor n^{c_2} \rfloor}{d})} \mu(r)$$

$$= \sum_{\substack{m,n \le x \\ d | \lfloor m^{c_1} \rfloor,\ d | \lfloor n^{c_2} \rfloor}} \sum_{\substack{r | \lfloor \frac{m^{c_1} \rfloor}{d} \\ r | \frac{\lfloor n^{c_2} \rfloor}{d}}} \mu(r)$$

$$= \sum_{r \le \frac{x^{c_1}}{d}} \mu(r) \sum_{\substack{m,n \le x \\ \lfloor m^{c_1} \rfloor \equiv 0 \pmod{rd} \\ \lfloor n^{c_2} \rfloor \equiv 0 \pmod{rd}}} 1.$$

In view of Lemma 2.1, we have

$$A_{c_1,c_2}(d;x) = \sum_{r \le x^{c_1} d^{-1}} \mu(r) \Big(\frac{x}{rd} + O\Big(\min\Big\{\frac{x^{(c_1+1)/3}}{(rd)^{1/3}}, \frac{x^{c_1}}{rd}\Big\}\Big)\Big)\Big(\frac{x}{rd} + O\Big(\min\Big\{\frac{x^{(c_2+1)/3}}{(rd)^{1/3}}, \frac{x^{c_2}}{rd}\Big\}\Big)\Big).$$

Since $1 < c_1 < c_2 < \frac{3}{2}$, the case $c_2 - \frac{1}{2} > c_1$ does not holds. Then we write

$$A_{c_1,c_2}(d;x) = \sum_{r \le x^{c_1-1/2} d^{-1}} \mu(r) \Big(\frac{x}{rd} + O\Big(\frac{x^{(c_1+1)/3}}{(rd)^{1/3}}\Big)\Big)\Big(\frac{x}{rd} + O\Big(\frac{x^{(c_2+1)/3}}{(rd)^{1/3}}\Big)\Big)$$

$$+ \sum_{x^{c_1-1/2} d^{-1} < r \le x^{c_2-1/2} d^{-1}} \mu(r) \Big(\frac{x}{rd} + O\Big(\frac{x^{c_1}}{rd}\Big)\Big)\Big(\frac{x}{rd} + O\Big(\frac{x^{(c_2+1)/3}}{(rd)^{1/3}}\Big)\Big)$$

$$+ \sum_{x^{c_2-1/2} d^{-1} < r \le x^{c_1} d^{-1}} \mu(r) \Big(\frac{x}{rd} + O\Big(\frac{x^{c_1}}{rd}\Big)\Big)\Big(\frac{x}{rd} + O\Big(\frac{x^{c_2}}{rd}\Big)\Big)$$

$$
\begin{aligned}
&= \frac{x^2}{d^2}\sum_{r\le x^{c_1}d^{-1}}\frac{\mu(r)}{r^2} + O\Big(\frac{x^{(c_2+4)/3}}{d^{4/3}}\sum_{r\le x^{c_1-1/2}d^{-1}}\frac{1}{r^{4/3}}\Big) + O\Big(\frac{x^{(c_1+c_2+2)/3}}{d^{2/3}}\sum_{r\le x^{c_1-1/2}d^{-1}}\frac{1}{r^{2/3}}\Big)\\
&\quad + O\Big(\frac{x^{c_1+1}}{d^2}\sum_{x^{c_1-1/2}d^{-1}<r\le x^{c_2-1/2}d^{-1}}\frac{1}{r^2}\Big) + O\Big(\frac{x^{(3c_1+c_2+1)/3}}{d^{4/3}}\sum_{x^{c_1-1/2}d^{-1}<r\le x^{c_2-1/2}d^{-1}}\frac{1}{r^{4/3}}\Big)\\
&\quad + O\Big(\frac{x^{c_1+c_2}}{d^2}\sum_{x^{c_2-1/2}d^{-1}<r\le x^{c_1}d^{-1}}\frac{1}{r^2}\Big)\\
&= \frac{x^2}{d^2\zeta(2)} + O\Big(\frac{x^{(c_2+4)/3}}{d^{4/3}}\Big) + O\Big(\frac{x^{(4c_1+2c_2+3)/6}}{d}\Big).
\end{aligned}
$$

This proves Lemma 2.3. □

3 Proofs

Proof of Theorem 1.1. Let $x > 1$, we have

$$
S_r^c(x) = \sum_{\substack{m,n\le x\\ \gcd(\lfloor m^c\rfloor,\lfloor n^c\rfloor)\ \text{is}\ r\text{-full}}} 1.
$$

It follows that

$$
S_r^c(x) = \sum_{d\,\le\, x^c} f_r(d)A_c(d;x).
$$

Thus,

$$
\begin{aligned}
S_r^c(x) &= \sum_{d\,\le\, x^c} f_r(d)\Big(\frac{x^2}{d^2\zeta(2)} + O\Big(\frac{x^{(c+4)/3}}{d^{4/3}}\Big) + O\Big(\frac{x^{c+1/2}}{d}\Big)\Big)\\
&= \frac{x^2}{\zeta(2)}\sum_{d\,\le\, x^c}\frac{f_r(d)}{d^2} + O\Big(x^{(c+4)/3}\sum_{d\,\le\, x^c}\frac{f_r(d)}{d^{4/3}}\Big) + O\Big(x^{c+1/2}\sum_{d\,\le\, x^c}\frac{f_r(d)}{d}\Big). \qquad (3)
\end{aligned}
$$

From the series (1) is absolutely convergent for $\Re(s) > 1/r$, then the second and the last sums in (3) are $O(1)$. Moreover,

$$
\begin{aligned}
\sum_{d\,\le\, x^c}\frac{f_r(d)}{d^2} &= \sum_{d=1}^{\infty}\frac{f_r(d)}{d^2} - \sum_{d>x^c}\frac{f_r(d)}{d^2}\\
&= \prod_p\Big(1+\frac{p^{-rs}}{1-p^{-s}}\Big) + O(x^{c/r-c}).
\end{aligned}
$$

Since $c/r - c + 2 < (c+4)/3$, we have

$$
\begin{aligned}
S_r^c(x) &= \frac{x^2}{\zeta(2)}\prod_p\Big(1+\frac{p^{-rs}}{1-p^{-s}}\Big) + O(x^{c/r-c+2}) + O\Big(x^{(c+4)/3}\Big) + O\Big(x^{c+1/2}\Big)\\
&= \frac{x^2}{\zeta(2)}\prod_p\Big(1+\frac{p^{-rs}}{1-p^{-s}}\Big) + O\Big(x^{(c+4)/3}\Big) + O\Big(x^{c+1/2}\Big).
\end{aligned}
$$

By the Euler product of $\zeta(2)$, we have

$$\prod_p \Big(1 - p^{-2} + p^{-2r}\Big)\zeta(2) = \prod_p \Big(1 - p^{-2} + p^{-2r}\Big)\Big(1 - p^{-2}\Big)^{-1} = \prod_p \Big(1 + \frac{p^{-2r}}{1 - p^{-2}}\Big).$$

This proves Theorem 1.1. □

Proof of Theorem 1.2. We write

$$\sum_{\substack{m,n\le x \\ \gcd(\lfloor m^c\rfloor,\lfloor n^c\rfloor) \text{ is } r\text{-free}}} 1 = \sum_{d\,\le\, x^c} q_r(d) A_c(d;x).$$

Thus,

$$\begin{aligned}\sum_{\substack{m,n\le x \\ \gcd(\lfloor m^c\rfloor,\lfloor n^c\rfloor) \text{ is } r\text{-free}}} 1 &= \sum_{d\,\le\, x^c} q_r(d)\Big(\frac{x^2}{d^2\zeta(2)} + O\Big(\frac{x^{(c+4)/3}}{d^{4/3}}\Big) + O\Big(\frac{x^{c+1/2}}{d}\Big)\Big) \\ &= \frac{x^2}{\zeta(2)}\sum_{d\,\le\, x^c}\frac{q_r(d)}{d^2} + O\Big(x^{(c+4)/3}\sum_{d\,\le\, x^c}\frac{q_r(d)}{d^{4/3}}\Big) + O\Big(x^{c+1/2}\sum_{d\,\le\, x^c}\frac{q_r(d)}{d}\Big).\end{aligned}$$

In view of (2), we have

$$\sum_{d\,\le\, x^c}\frac{q_r(d)}{d^2} = \sum_{d=1}^{\infty}\frac{q_r(d)}{d^2} - \sum_{d>x^c}\frac{q_r(d)}{d^2} = \frac{\zeta(2)}{\zeta(2r)} + O(x^{-c}),$$

$$\sum_{d\,\le\, x^c}\frac{q_r(d)}{d^{4/3}} = O(1), \text{ and } \sum_{d\,\le\, x^c}\frac{q_r(d)}{d} = O(\log x).$$

Thus,

$$\sum_{\substack{m,n\le x \\ \gcd(\lfloor m^c\rfloor,\lfloor n^c\rfloor) \text{ is } r\text{-free}}} 1 = \frac{1}{\zeta(2r)}x^2 + O\Big(x^{(c+4)/3}\Big) + O\Big(x^{c+1/2}\log x\Big).$$

This proves Theorem 1.2. □

Proof of Theorems 1.3 and 1.4. We prove Theorems 1.3 and 1.4 by using Lemma 2.3 and a similar proof as in the proof of Theorem 1.1 and 1.2. □

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