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# On certain relations among the generating functions for certain quadratic forms

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**Abstract:** The object of this article is to establish the relation between the generating function of the quadratic form $2m^2 + 2mn + 3n^2$ and the generating functions for the quadratic forms $m^2+mn+n^2$, $m^2+mn+2n^2$, $m^2+mn+4n^2$ and $2m^2+mn+2n^2$. In the process, we deduce certain interesting theta function identities.

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## 1 Introduction

For $\omega = e^{\frac{2\pi i}{3}}$ and $q = e^{2\pi i\tau}$ with $Im(\tau) > 0$, let

$$a_3(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$b(q) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2},$$

and

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{\left(m+\frac{1}{3}\right)^2+\left(m+\frac{1}{3}\right)\left(n+\frac{1}{3}\right)+\left(n+\frac{1}{3}\right)^2}.$$

J. M. Borwein and P. B. Borwein [6] proved that

$$a_3^3(q) = b^3(q) + c^3(q). \tag{1}$$

The above identity is the cubic analogue of fourth power of the following theta function identity:

$$\varphi^4(q) = \varphi^4(-q) + 16q\psi^4(q^2), \tag{2}$$

where $\varphi(q)$ and $\psi(q)$ as defined in (17) and (18) respectively below. The Identity (2) is due to Jacobi and Ramanujan also recorded (2) in Chapter 16 of his second notebook [11, p. 198, Entry 25(vii)]. This identity plays an important role in connecting Ramanujan's theta functions and the modular equations. In Chapter 33 of [5, p. 96] one can find the alternative proof of (1) using the theory of Ramanujan's theta functions and how these functions $a_3(q)$, $b(q)$ and $c(q)$ have been employed in obtaining the cubic theory of theta functions of Ramanujan.

In 1871, Lorenz [10] proved that

**Theorem 1.1.** *If* $a_3(q) = \sum\limits_{n=0}^{\infty} Q(n)q^n$, *then*

$$Q(n) = 6[d_{1,3}(n) - d_{2,3}(n)], \tag{3}$$

*where* $Q(n)$ *represents the number of integer solutions of a positive integer* $n$ *in the equation* $n = x_1^2 + x_1x_2 + x_2^2$ *and* $d_{k,l}(n)$ *denotes the number of divisors of* $n$ *which are congruent to* $k \pmod{l}$.

Many mathematicians including Liuoville and Ramanujan worked on this topic and obtained formulae analogous to $a_3^k(q)$ for many integral values of $k$. For details on the work of $a_3(q)$ one may refer to the books written by K. S. Williams [13, p. 224] and S. Cooper [7, p. 171]. Similar works can be found in the literature [7, 13] on the following functions like:

$$a_7(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2},$$

$$a_{11}(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+3n^2},$$

$$a_{15}(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+4n^2},$$

and

$$a_{15}^*(q) = \sum_{m,n=-\infty}^{\infty} q^{2m^2+mn+2n^2}.$$

Recently S. Cooper and D. Ye [8], proved that

$$a_3^2(q) + 5a_3^2(q^5) = 3a_{15}^2(q) + 3a_{15}^{*2}(q). \tag{4}$$

T. Anusha et.al [3] proved that

$$a_3^2(q) + a_3(q)a_3(q^7) + 7a_3^2(q^7) = 9a_7(q)a_7(q^3) \tag{5}$$

and

$$7a_3^2(q^7) - a_3^2(q) = 9a_7^2(q^3) - 3a_7^2(q). \tag{6}$$

Motivated by the above articles [3, 8], in this article, we consider

$$a_5(q) = \sum_{m,n=-\infty}^{\infty} q^{2m^2+2mn+3n^2}, \tag{7}$$

and we obtain the following relations between the generating function $a_5(q)$ and the generating functions $a_3(q)$, $a_7(q)$, $a_{15}(a)$, and $a_{15}^*(q)$:

**Theorem 1.2.** *We have*

$$\begin{aligned} 9a_5^2(q) - 27a_5^2(q^3) = a_3^2(q) - 4a_3^2(q^4) + 5a_3^2(q^5) - 20a_3^2(q^{20}) \\ - 12qf_2^2f_{10}^2 + 36q^3f_6^2f_{30}^2, \end{aligned} \tag{8}$$

$$\begin{aligned} 3a_5^2(q) - 9a_5^2(q^3) = a_{15}^2(q) + a_{15}^{*2}(q) - 4(a_{15}^2(q^4) + a_{15}^{*2}(q^4)) \\ - 4qf_2^2f_{10}^2 + 12q^3f_6^2f_{30}^2, \end{aligned} \tag{9}$$

$$\begin{aligned} 3a_5^2(q) - 21a_5^2(q^7) = a_7^2(q) + 5a_7^2(q^5) - 4a_7^2(q^4) - 20a_7^2(q^{20}) \\ - 4qf_2^2f_{10}^2 + 28q^7f_{14}^2f_{70}^2, \end{aligned} \tag{10}$$

*and*

$$\begin{aligned} a_5(q)a_5(q^3) + a_5(-q)a_5(-q^3) = \frac{2}{3}[a_3(q^2)a_3(q^{10}) + 2a_3(q^4)a_3(q^{20})] \\ + 8q^4f_4f_{12}f_{20}f_{60}. \end{aligned} \tag{11}$$

*where $f_n$ and $P_n$ as defined in* (19) *and* (21) *below respectively.*

In the process of proving the above relations, we also obtain the following interesting theta function identities:

**Theorem 1.3.** *We have*

$$\varphi^2(q)\varphi^2(q^5) = \frac{1}{6}\left(\varphi^4(q) + 5\varphi^4(q^5)\right) + \frac{8}{3}qf_2^2f_{10}^2, \tag{12}$$

$$q^3\psi^2(q^2)\psi^2(q^{10}) = \frac{1}{6}\left(q\psi^4(q^2) + 5q^5\psi^4(q^{10})\right) - \frac{1}{6}qf_2^2f_{10}^2, \tag{13}$$

$$a_5^2(q) = \frac{1}{6}\left(\varphi^4(q) + 5\varphi^4(q^5)\right) - \frac{4}{3}qf_2^2f_{10}^2, \tag{14}$$

*and*

$$\varphi^2(q)\varphi^2(q^5) = a_5^2(q) + 4qf_2^2f_{10}^2. \tag{15}$$

**Theorem 1.4.** *We have*

$$\begin{aligned} q^3\psi(q)\psi(q^3)\psi(q^5)\psi(q^{15}) =& \frac{1}{4}\left(\varphi(q)\varphi(q^3)\varphi(q^5)\varphi(q^{15}) - \varphi(-q^2)\varphi(-q^6)\varphi(-q^{10})\varphi(-q^{30})\right) \\ & - \frac{1}{2}\left(qf_1f_3f_5f_{15} + 2q^2f_2f_6f_{10}f_{30} + 4q^4f_4f_{12}f_{20}f_{60}\right). \end{aligned} \tag{16}$$

The identity (12) was given by Alaca, Alaca and Williams [2]; both (12) and (13) are given as Exercise 3 in Chapter 10 of Cooper's book [7], and (13) was proved by Kang [9]. The identities (14) and (15) are proved by Williams [14].

In Section 2, we prove the Theorem 1.2, which is the main object of this article. Also, in the process, we give a different proof of the Identity (14) and we prove Theorem 1.4.

We close this section by recalling some of the definitions and results which are required to prove our main results. Let $q = e^{2\pi i\tau}$ with $Im(\tau) > 0$. As usual, for any complex number $a$, we define

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n).$$

In Chapter 16 of his second notebook [11, p. 197], Ramanujan defined the following theta functions:

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{17}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{18}$$

$$f(-q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty$$

and

$$\chi(q) = (-q;q^2)_\infty.$$

For convenience, we set

$$f_n := f(-q^n) = (q^n;q^n)_\infty, \tag{19}$$

for any positive integer $n$. It is easy to see that

$$\chi(-q) = \frac{f_1}{f_2} \text{ and } \chi(q) = \frac{f_2^2}{f_1f_4}. \tag{20}$$

The Eisenstein series of weight 2, $P_n$, is defined by

$$P_n = P(q^n) := 1 - 24\sum_{k=1}^{\infty}\frac{kq^{nk}}{1 - q^{nk}}. \tag{21}$$

For convenience, we set $h(q) = \frac{1}{3}\left(P_1 - 2P_2 + 4P_4\right)$.

For $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{N}$ and $x_1, x_2, y_1, y_2 \in \mathbb{Z}$, we define

$$\begin{aligned} &N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3; n) = \\ &\mathrm{card}\{(x_1, y_1, x_2, y_2) | n = \alpha_1 x_1^2 + \alpha_2 x_1 y_1 + \alpha_3 y_1^2 + \beta_1 x_2^2 + \beta_2 x_2 y_2 + \beta_3 y_2^2\}. \end{aligned} \tag{22}$$

Alaca [1] has deduced the following identities using the theory of modular forms:

$$\begin{aligned} \varphi^2(q)\varphi^2(q^{15}) = &\frac{-1}{12}[h(q) - 3h(q^3) + 5h(q^5) - 15h(q^{15})] \\ &+ \frac{2}{3}(qf_1f_3f_5f_{15} - 2q^2f_2f_6f_{10}f_{30} \\ &+ 4q^4f_4f_{12}f_{20}f_{60} + 4qf_3f_5f_6f_{10} + 8qf_6f_{10}f_{12}f_{20}), \end{aligned} \tag{23}$$

$$\begin{aligned} \varphi^2(q^3)\varphi^2(q^5) = &\frac{-1}{12}[h(q) - 3h(q^3) + 5h(q^5) - 15h(q^{15})] \\ &- \frac{2}{3}(5qf_1f_3f_5f_{15} + 14q^2f_2f_6f_{10}f_{30} + 20q^4f_4f_{12}f_{20}f_{60} \\ &- 4qf_3f_5f_6f_{10} - 8q^2f_6f_{10}f_{12}f_{20}), \end{aligned} \tag{24}$$

and

$$\begin{aligned} \varphi(q)\varphi(q^3)\varphi(q^5)\varphi(q^{15}) = &\frac{-1}{16}[h(q) + 3h(q^3) - 5h(q^5) - 15h(q^{15})] + \frac{3}{2}qf_1f_3f_5f_{15} \\ &+ q^2f_2f_6f_{10}f_{30} + 6q^4f_4f_{12}f_{20}f_{60}. \end{aligned} \tag{25}$$

Ramanujan [11, p. 247, Entry 11 (xiv)] recorded a modular equation of degree $15$, which is equivalent to the following theta function identity:

$$\begin{aligned} 4q^3\psi(q)\psi(q^3)\psi(q^5)\psi(q^{15}) = &\varphi(q)\varphi(q^3)\varphi(q^5)\varphi(q^{15}) \\ &- \varphi(-q^2)\varphi(-q^6)\varphi(-q^{10})\varphi(-q^{30}) \\ &- 2qf(q)f(q^3)f(q^5)f(q^{15}). \end{aligned} \tag{26}$$

A proof of (26) can be found in [4, p. 395]. Again, from [11, p. 245, Entry 9(v, vi)], we have

$$\varphi(q)\varphi(q^{15}) - \varphi(q^3)\varphi(q^5) = 2qf_2f_{30}\chi(q^3)\chi(q^5) \tag{27}$$

and

$$\varphi(q)\varphi(q^{15}) + \varphi(q^3)\varphi(q^5) = 2f_6f_{10}\chi(q)\chi(q^{15}). \tag{28}$$

From [12, pp. 46, 47], we find

$$-P_1 + 4P_4 = 3\varphi^4(q) \tag{29}$$

and

$$-P_1 + 2P_2 = \varphi^4(q) + 16q\psi^4(q^2). \tag{30}$$

In [7] and [8], we find that

$$a_3^2(q) = \frac{1}{2}\left(-P_1 + 3P_3\right), \tag{31}$$

$$a_7^2(q) = \frac{1}{2}\left(-P_1 + 7P_7\right), \tag{32}$$

$$a_3(q)a_3(q^5) = \frac{1}{16}\left(-P_1 - 3P_3 + 5P_5 + 15P_{15}\right) + \frac{9}{2}qf_1f_3f_5f_{15}, \tag{33}$$

$$a_{15}^2(q) = \frac{1}{12}\left(-P_1 + 3P_3 - 5P_5 + 15P_{15}\right) + 2qf_1f_3f_5f_{15}, \tag{34}$$

and

$$a_{15}^{*2}(q) = \frac{1}{12}\left(-P_1 + 3P_3 - 5P_5 + 15P_{15}\right) - 2qf_1f_3f_5f_{15}. \tag{35}$$

# 2 Main results

*Proof of* (14). From the definition of $a_5(q)$, we have

$$\begin{aligned} a_5(q) &= \sum_{m,n=-\infty}^{\infty} q^{2m^2+2mn+3n^2} \\ &= \sum_{m,n=-\infty}^{\infty} q^{2m^2+2m(2n)+3(2n)^2} + \sum_{m,n=-\infty}^{\infty} q^{2m^2+2m(2n+1)+3(2n+1)^2} \\ &= \sum_{m,n=-\infty}^{\infty} q^{2((m+n)^2+5n^2)} + \sum_{m,n=-\infty}^{\infty} q^{4\left(\frac{(m+n)^2+(m+n)+5(n^2+n)}{2}\right)+3} \end{aligned}$$

This implies that

$$a_5(q) = \varphi(q^2)\varphi(q^{10}) + 4q^3\psi(q^4)\psi(q^{20}). \tag{36}$$

On squaring the above equation and employing $\varphi(q)\psi(q^2) = \psi^2(q)$ repeatedly, we obtain

$$a_5^2(q) = \varphi^2(q^2)\varphi^2(q^{10}) + 16q^6\psi^2(q^4)\psi^2(q^{20}) + 8q^3\psi^2(q^2)\psi^2(q^{10}). \tag{37}$$

On employing (12) and (13) in the above equation, we obtain the required result. □

**Corollary 2.1.** *Let $n \in \mathbb{N}$ and $N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3; n)$ as defined in* (22). *Then*

$$N(2,2,3,2,2,3;n) = \frac{4}{3}\left(\gamma(n) - 4\gamma(n/4) + 5\gamma(n/5) - 20\gamma(n/20) - t_1(n)\right), \tag{38}$$

*where $\gamma(n) = \sum_{\substack{m|n \\ m>0}} m$ and $qf_2^2f_{10}^2 = \sum_{n=1}^{\infty} t_1(n)q^n$.*

*Proof.* On employing (29) in the identity (14) and then comparing the coefficients of $q^n$ in the resulting identity, we obtain (38). □

*Proof of* (15): From (12) and (14), we obtain (15). □

*Proof of* (16). From (23) and (24), it follows that

$$\begin{aligned} \varphi^2(-q)\varphi^2(-q^{15}) - \varphi^2(-q^3)\varphi^2(-q^5) = &- 4qf(q)f(q^3)f(q^5)f(q^{15}) \\ &+ 8q^2f_2f_6f_{10}f_{30} \\ &+ 8q^4f_4f_{12}f_{20}f_{60}. \end{aligned} \tag{39}$$

On multiplying the identity (27) by (28) and then changing $q$ to $-q$, we find that

$$\varphi^2(-q)\varphi^2(-q^{15}) - \varphi^2(-q^3)\varphi^2(-q^5) = -4qf_1f_3f_5f_{15}. \tag{40}$$

From (39) and (40), we have

$$qf(q)f(q^3)f(q^5)f(q^{15}) = qf_1f_3f_5f_{15} + 2q^2f_2f_6f_{10}f_{30} + 4q^4f_4f_{12}f_{20}f_{60}. \tag{41}$$

Employing the above identity in (26), we obtain (16). □

*Proof of* (8). From (12) and (29), we have

$$\begin{aligned} a_5^2(q) - 3a_5^2(q^3) = \frac{1}{18}[&-P_1 + 4P_4 - 5P_5 + 20P_{20} + 3P_3 - 12P_{12} \\ &+ 15P_{15} - 60P_{60}] - \frac{4}{3}qf_2^2f_{10}^2 + 4q^3f_6^2f_{30}^2. \end{aligned}$$

Applying (31) four times in the right side of the above with $q = q,\ q^4,\ q^5,\ q^{20}$, we obtain the required result. □

*Proof of* (9). Applying (4) in the right side of the (8) with $q = q,\ q^4$, we obtain the required result. □

*Proof of* (10). From (12) and (29), we have

$$\begin{aligned} a_5^2(q) - 7a_5^2(q^7) = \frac{1}{18}[&-P_1 + 4P_4 - 5P_5 + 20P_{20} + 7P_7 - 28P_{28} \\ &+ 35P_{35} - 140P_{140}] - \frac{4}{3}qf_2^2f_{10}^2 + \frac{28}{3}q^7f_{14}^2f_{70}^2. \end{aligned}$$

Applying (32) four times in the right side of the above with $q = q,\ q^4,\ q^5,\ q^{20}$, we obtain the required result. □

*Proof of* (11). From (36), one can easily establish that

$$a_5(q) + a_5(-q) = 2\varphi(q^2)\varphi(q^{10}) \tag{42}$$

and

$$a_5(q) - a_5(-q) = 8q^3\psi(q^4)\psi(q^{20}). \tag{43}$$

Changing $q$ to $q^3$ for the above identities, we get

$$a_5(q^3) + a_5(-q^3) = 2\varphi(q^6)\varphi(q^{30}) \tag{44}$$

and

$$a_5(q^3) - a_5(-q^3) = 8q^9\psi(q^{12})\psi(q^{60}). \tag{45}$$

On multiplying equation (42) with (44) and equation (43) with (45), then adding the resulting identities, we obtain

$$\begin{aligned} a_5(q)a_5(q^3) + a_5(-q)a_5(-q^3) = 2[&\varphi(q^2)\varphi(q^{10})\varphi(q^6)\varphi(q^{30}) \\ &+ 16q^{12}\psi(q^4)\psi(q^{20})\psi(q^{12})\psi(q^{60})]. \end{aligned} \tag{46}$$

Employing (25) and (16) in the above, we find that

$$a_5(q)a_5(q^3) + a_5(-q)a_5(-q^3) = \frac{-1}{24}[P_2 + 2P_4 + 3P_6 + 6P_{12} - 5P_{10} \\ - 10P_{20} - 15P_{30} - 30P_{60}] \\ + 3q^2 f_2 f_6 f_{10} f_{30} - 2q^4 f_4 f_{12} f_{20} f_{60}. \tag{47}$$

Changing $q$ to $q^2$ and $q$ to $q^4$ in (33) respectively, we obtain

$$a_3(q^2)a_3(q^{10}) = \frac{1}{16}(-P_2 - 3P_6 + 5P_{10} + 15P_{30}) + \frac{9}{2}q^2 f_2 f_6 f_{10} f_{30} \tag{48}$$

and

$$a_3(q^4)a_3(q^{20}) = \frac{1}{16}(-P_4 - 3P_{12} + 5P_{20} + 15P_{60}) + \frac{9}{2}q^4 f_4 f_{12} f_{20} f_{60}. \tag{49}$$

Applying (48) and (49) in the right side of the equation (47), we obtain the required result. □

The following corollary follows from the identities (8)–(11).

**Corollary 2.2.** *Let* $n \in \mathbb{N}$ *and* $N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3; n)$ *as defined in* (22). *Then*

*i)* $9N(2,2,3,2,2,3;n) - 27N(6,6,9,6,6,9;n)$
$$= N(1,1,1,1,1,1;n) - 4N(4,4,4,4,4,4;n) \\ + 5N(5,5,5,5,5,5;n) - 20N(20,20,20,20,20,20;n) \\ - 12t_1(n) + 36t_2(n),$$

*ii)* $3N(2,2,3,2,2,3;n) - 9N(6,6,9,6,6,9;n)$
$$= N(1,1,4,1,1,4;n) + N(2,1,2,2,1,2;n) \\ - 4N(4,4,16,4,4,16;n) - 4N(8,4,8,8,4,8;n) - 4t_1(n) + 12t_2 n,$$

*iii)* $3N(2,2,3,2,2,3;n) - 21N(14,14,21,14,14,21;n)$
$$= N(1,1,2,1,1,2;n) + 5N(5,5,10,5,5,10;n) \\ - 4N(4,4,8,4,4,8;n) - 20N(20,20,40,20,20,40;n) \\ - 4t_1(n) + 28t_3(n),$$

*iv)* $N(2,2,3,6,6,9;n) + (-1)^{x_1+y_1} N(2,2,3,6,6,9;n)$
$$= \frac{2}{3}[N(2,2,2,10,10,10;n) + 5N(4,4,4,20,20,20;n)] + 8t_4(n),$$

*where* $q^i f_{2i} f_{10i} = \sum_{k=0}^{\infty} t_i(k) q^k$ *with* $i = 1, 3, 7$ *and* $q^4 f_4 f_{12} f_{20} f_{60} = \sum_{k=0}^{\infty} t_4(k) q^k$.

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